THE ROLE OF SYMMETRY IN PHYSICS

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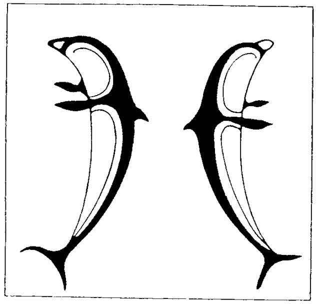
THE NOTION OF SYMMETRY

Symmetry from the Greek σύμμετρια (well-ordered, well-proportioned) was originally introduced to describe certain properties of artifacts (Polykleitos, Περι βελοποιϊκών, IV,2).
All ancient civilizations used this concept.

Decorative motif
(Sumerian, circa 2000 B.C.)
Translation symmetry



Tile found at the Megaron in Tyrins
(Late Helladic, circa 1200 B.C.)
Reflection symmetry

The language of symmetry is mathematics.
The Greeks developed mathematics (geometry), introduced the five regular polyhedra, tetrahedron, octahedron, cube, icosahedron and dodecahedron, and associated them with the constituents of the Universe: fire (tetra-), air (octa-), earth (cube), water (icosa-) and the Universe itself (penta-dodeca-hedron). (Plato, Timaeus, 55C)

The study of symmetry took another step forward during the Italian Renaissance. Regular polyhedra were complemented by more complex structures, the Archimedean polyhedra.

An Archimedean polyhedron with 26 bases (after Leonardo da Vinci) (From Luca Pacioli, *Divina Proportione*, Venice, 1509)

The symmetries of the regular bodies were described in detail (Piero della Francesca, *De quinque corporibus regularis*, 1482). A mathematical description was introduced (projective transformation), which, translated into modern mathematical language, is the theory of group transformation, or simply group theory.

Symmetry was assumed to be the fundamental law of Nature and is therefore above all (divina).

Luca Pacioli instructing Guidobaldo da Montefeltro in mathematics (geometry) (Oil painting by Jacopo de' Barbari, 1494)

Symmetry became so important that in 1595 Kepler stated: the planetary system (known at the time), Saturn, Jupiter, Mars, Earth, Venus and Mercury can be reduced to the motion of regular bodies.

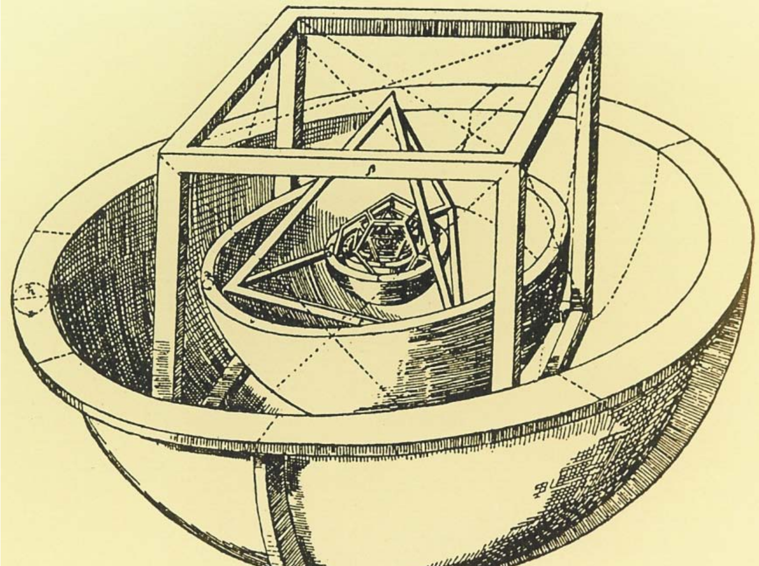


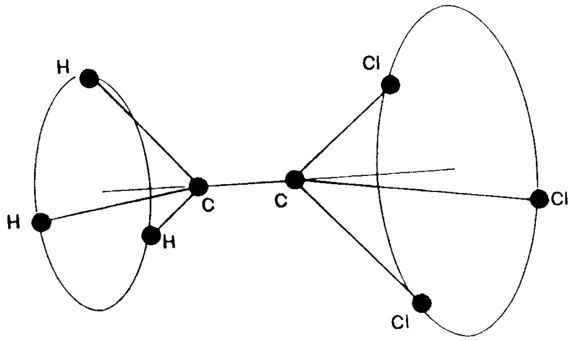
FIG. 46

(From Kepler, *Mysterium Cosmographicum*, Tübingen, 1595).

Kepler concluded his book with the sentence:
Credo spatioso numen in orbe
I believe in a geometric order of the Universe

SYMMETRY IN PHYSICS

Originally introduced to describe certain properties of constituents of matter: crystals, molecules,…



The molecule $H_3C_2Cl_3$ with
C_3 symmetry
(From M. Hamermesh, *Group theory and its application to physical problems*, 1962)

Symmetry acquired importance when it became apparent that the laws of Nature appear to possess symmetry properties (end of the 19th Century)

Maxwell equations are invariant under Lorentz transformations

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$t' = \frac{t - \left(\frac{v}{c^2}\right)x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The birth of the symmetry approach to physics!

Symmetry became the guiding principle in constructing theories of the Universe (Einstein, 1905).

At the same time, the mathematical language needed to describe symmetries was further developed with the introduction of Lie algebras and groups (Lie, 1880’s) and their classification (Cartan, 1890’s).

The birth of the group theory approach to physics!

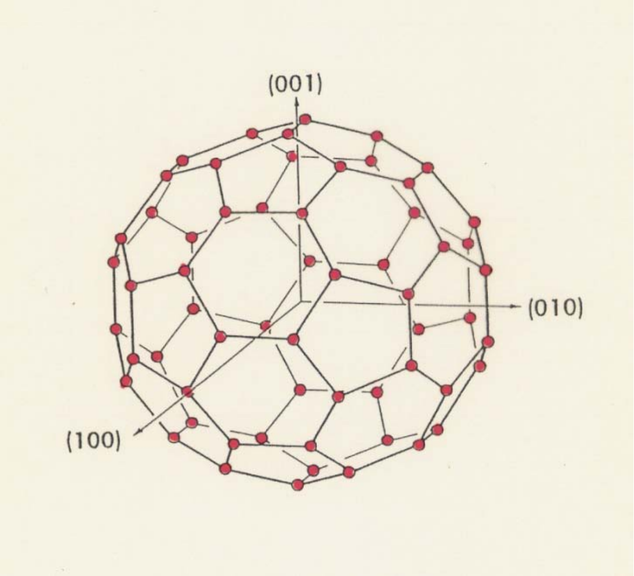
Group theory became one of the major tools for studying physics problems and their solution.

Symmetry and its language Group theory are used today in a variety of ways.

THE MANY WAYS OF SYMMETRY IN PHYSICS

1. Geometric symmetry

Describes the arrangement of constituent particles into a structure
Example: Atoms in a molecule.
Mathematical framework: Point groups



The molecule C_{60} with icosahedral I_h symmetry

(Curl, Kroto and Smalley, 1985)

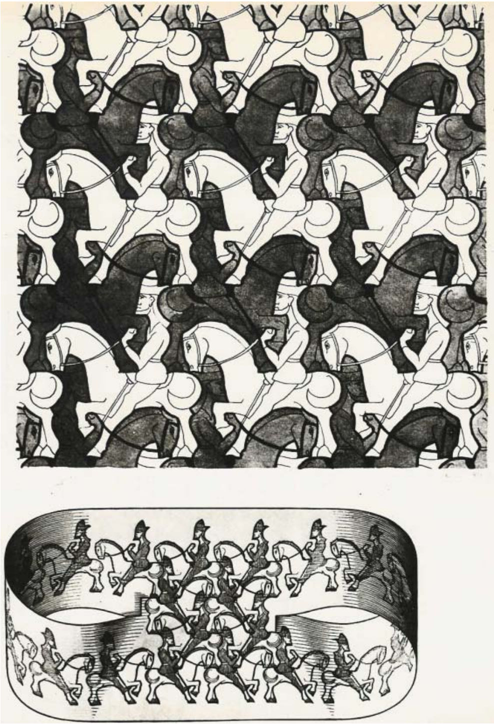
2. Permutation symmetry

Describes properties of systems of identical particles
Mathematical framework: Permutation group S_n

Became particularly important with the development of quantum mechanics (1920's)

$$\psi\left(1,2\right) = +\psi(2,1) \quad \text{Bosons}$$

$$\psi(1,2) = -\psi\left(2,1\right) \quad \text{Fermions}$$



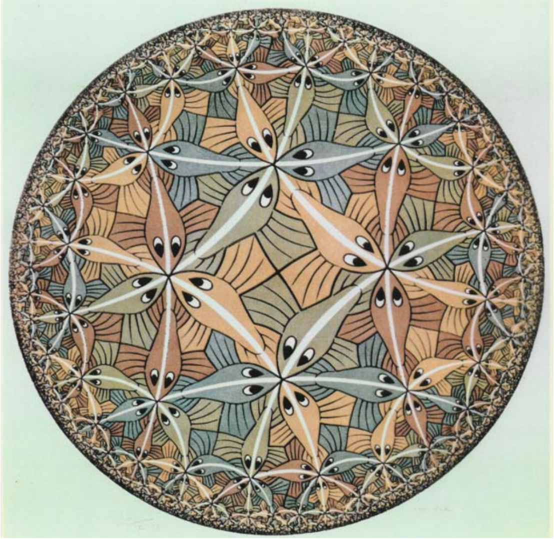
(From M.C. Escher, *Study of the regular division of the plane with horsemen*, 1946)

3. Space-time (or fundamental) symmetry

Fixes the form of the equations of motion.
Mathematical framework: Continuous Lie groups
Example: Free Dirac equation

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) = 0$$

The free Dirac equation is invariant under the group of Lorentz transformations, SO(3,1), in general under the Poincare' group, ISO(3,1)



(From M.C. Escher, *Circle Limit III*, 1959)

Tessellation of the hyperbolic Poincare' plane

All laws of Nature appear to be invariant under Lorentz transformations!

4. Gauge symmetry

Fixes the form of the interaction between particles and external fields. Fix the form of the equation satisfied by the fields.
Mathematical framework: Continuous Lie groups
Example: Dirac equation in an external electromagnetic field

$$\left[\gamma_\mu\left(i\partial_\mu - eA_\mu\right) - m\right]\psi(x) = 0$$

The laws of electrodynamics, Maxwell equations, are invariant under U(1) gauge transformations

$$A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$$

A major discovery of the 2nd part of the 20th Century has been that strong, weak and electromagnetic interactions all appear to be governed by gauge symmetries

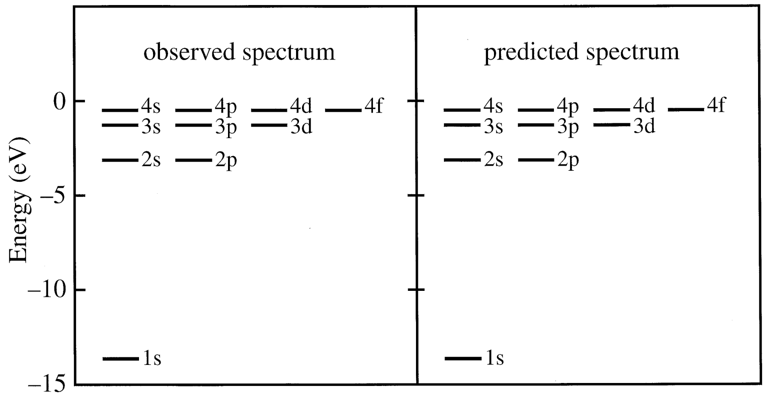
$$SU_c(3) \otimes SU_w(2) \otimes U(1)$$

5. Dynamic symmetry

Fixes the interaction between constituent particles and/or external fields.
Determines the spectral properties of quantum systems (patterns of energy levels).
Mathematical framework: Continuous Lie groups

Introduced implicitly by Pauli (1926) for the hydrogen atom.
The Hamiltonian with Coulomb interaction is invariant under a set of transformations, G, larger than rotations (Runge-Lenz transformations, SO(4)). It can be written in terms of Casimir operators of G.

$$H = \frac{p^2}{2m} - \frac{e^2}{r} = -\frac{A}{C_2\left(SO(4)\right)+1} \qquad E(n,\ell,m) = -\frac{A}{n^2}$$

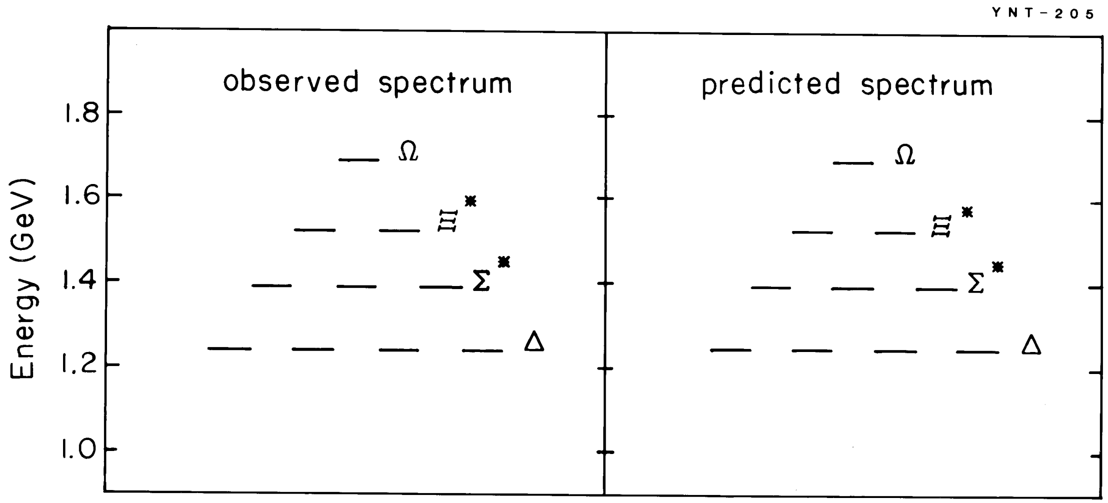


The spectrum of the hydrogen atom

Assumed an important role in physics with the introduction of flavor symmetry (Gell'Mann-Ne'eman, 1962), $SU_f(3)$

$$M = a + b\left[C_1(U(1))\right] + c\left[C_2\left(SU(2)\right) - \frac{1}{4}C_1^2\left(U(1)\right)\right]$$

$$M\left(Y, I, I_3\right) = a + bY + c\left[I(I+1) - \frac{1}{4}Y^2\right]$$



The spectrum of the baryon decuplet is shown as an example of dynamic symmetry in hadrons

A major discovery of the 2nd part of the 20th century has been that dynamic symmetries are pervasive in physics.

Dynamic symmetries have been found in all fields of physics from molecular to atomic, from nuclear to particle physics.

One of the examples (nuclear physics) and the model upon which it is constructed will now be discussed.

A second example (molecular physics) will be briefly mentioned.

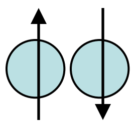
THE INTERACTING BOSON MODEL

In 1974, it was realized that collective properties of even-even nuclei can be described by assuming that nucleons bind together into correlated pairs with angular momentum J=0 and 2, similar to the Cooper pairs in the electron gas. This is due to the fact that the effective nucleon-nucleon interaction between identical particles is attractive for J=0 and J=2. This property is called pairing and plays a crucial role in many fields of physics. In the electron gas it appears as s-pairing and leads to the phenomenon of superconductivity (Bardeen, Cooper and Schriffer, 1956). In nuclei it appears as s- and d-pairing (Iachello, 1974).

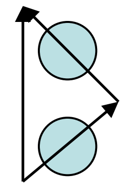
If pairs are treated as bosons, this observation leads to the INTERACTING BOSON MODEL ¶ of the nucleus (Arima and Iachello, 1976).

¶ For a review, see F. Iachello and A. Arima, *The Interacting Boson Model*, Cambridge University Press, Cambridge (1987).

Even-even nuclei composed of pairs of nucleons with J=0 and J=2 treated as bosons, s and d-bosons.



J=0



J=2

Building blocks $\qquad b_\alpha^\dagger(\alpha = 1,\ldots,6) \equiv s^\dagger, d_\mu^\dagger(\mu = 0, \pm 1, \pm 2)$

Span a six-dimensional space with algebraic structure $\quad g := U(6)$

Representations $\qquad \left[N\right] = \square\square\ldots\square$

Elements $\qquad G_{\alpha\beta} = b_\alpha^\dagger b_\beta \qquad (\alpha, \beta = 1,\ldots,6)$

Hamiltonian

$$H = E_0 + \sum_{\alpha\beta} \varepsilon_{\alpha\beta} G_{\alpha\beta} + \sum_{\alpha\beta\gamma\delta} u_{\alpha\beta\gamma\delta} G_{\alpha\beta} G_{\gamma\delta}$$

In general, diagonalized numerically in the basis [N].

Since H is expanded into elements of g, this algebra is called Spectrum Generating Algebra (SGA).

DYNAMIC SYMMETRY

The concept of dynamic symmetry introduced implicitly by Pauli in 1926 and Gell'Mann in 1962, was formalized in the 1970's ¶

Definition: Dynamic symmetry is a situation in which the Hamiltonian H describing the system does not contain all elements of g, but only the invariant Casimir operators of a chain of algebras

$$g \supset g' \supset g'' \supset \ldots$$

$$H = \alpha C(g) + \alpha' C(g') + \ldots$$

In this case, the eigenvalues can be obtained in explicit analytic form as

$$E = \alpha \langle C(g) \rangle + \alpha' \langle C(g') \rangle + \alpha'' \langle C(g'') \rangle + \ldots$$

All other properties of the system can be calculated in explicit form:
Exactly solvable problems
Simplicity in complexity program

¶ F. Iachello, in *Group Theoretical Methods in Physics*, Lecture Notes in Physics, Vol. 34, A. Bohm, ed., Lange Springer, Berlin (1979), pp. 420-429.

DYNAMIC SYMMETRIES OF THE INTERACTING BOSON MODEL

Obtained by breaking the algebra U(6) into its subalgebras.
Only three breakings containing SO(3) are possible:

$$U(6) \begin{cases} \nearrow \; U(5) \supset SO(5) \supset SO(3) \supset SO(2) & \leftarrow \text{(I)} \\ \rightarrow \; SU(3) \supset SO(3) \supset SO(2) & \leftarrow \text{(II)} \\ \searrow \; SO(6) \supset SO(5) \supset SO(3) \supset SO(2) & \leftarrow \text{(III)} \end{cases}$$

For each of the three symmetries it is possible to construct energy formulas (similar to the mass formulas of particle physics) that give the energies of the states in terms of the quantum numbers characterizing the representations of the algebras

$$g \supset g' \supset g'' \supset \ldots$$

A. Arima and F. Iachello, Ann. Phys. (N.Y.) 99, 253 (1976).
A. Arima and F. Iachello, Ann. Phys. (N.Y.) 111, 201 (1978).
A. Arima and F. Iachello, Ann. Phys. (N.Y.) 123, 468 (1979).

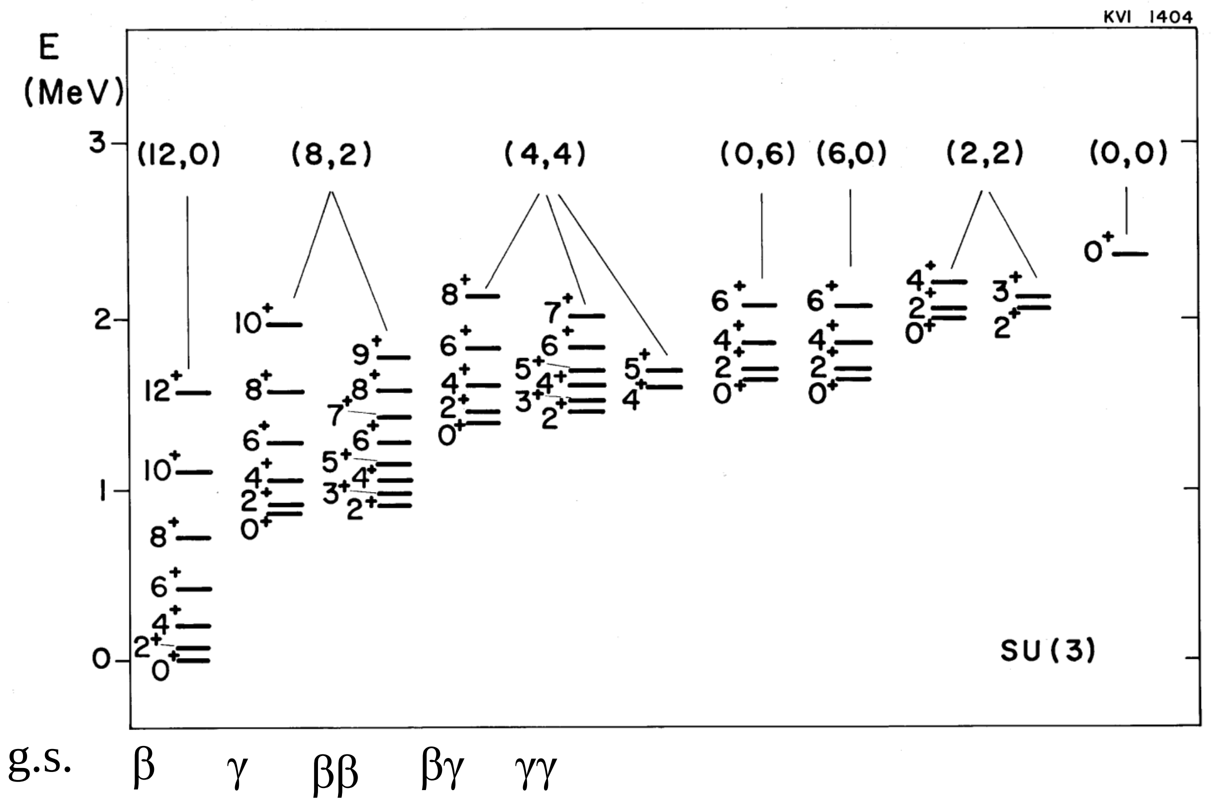
ENERGY FORMULAS

$$E^{(I)}(N,n_d,v,n_\Delta,L,M_L)=E_0+\alpha n_d(n_d+4)+\beta v(v+3)+\gamma L(L+1)$$

$$E^{(II)}(N,\lambda,\mu,K,L,M_L)=E_0+\kappa(\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu)+\kappa' L(L+1)$$

$$E^{(III)}(N,\sigma,\tau,v_\Delta,L,M_L)=E_0+A\sigma(\sigma+4)+B\tau(\tau+3)+CL(L+1)$$

(II):



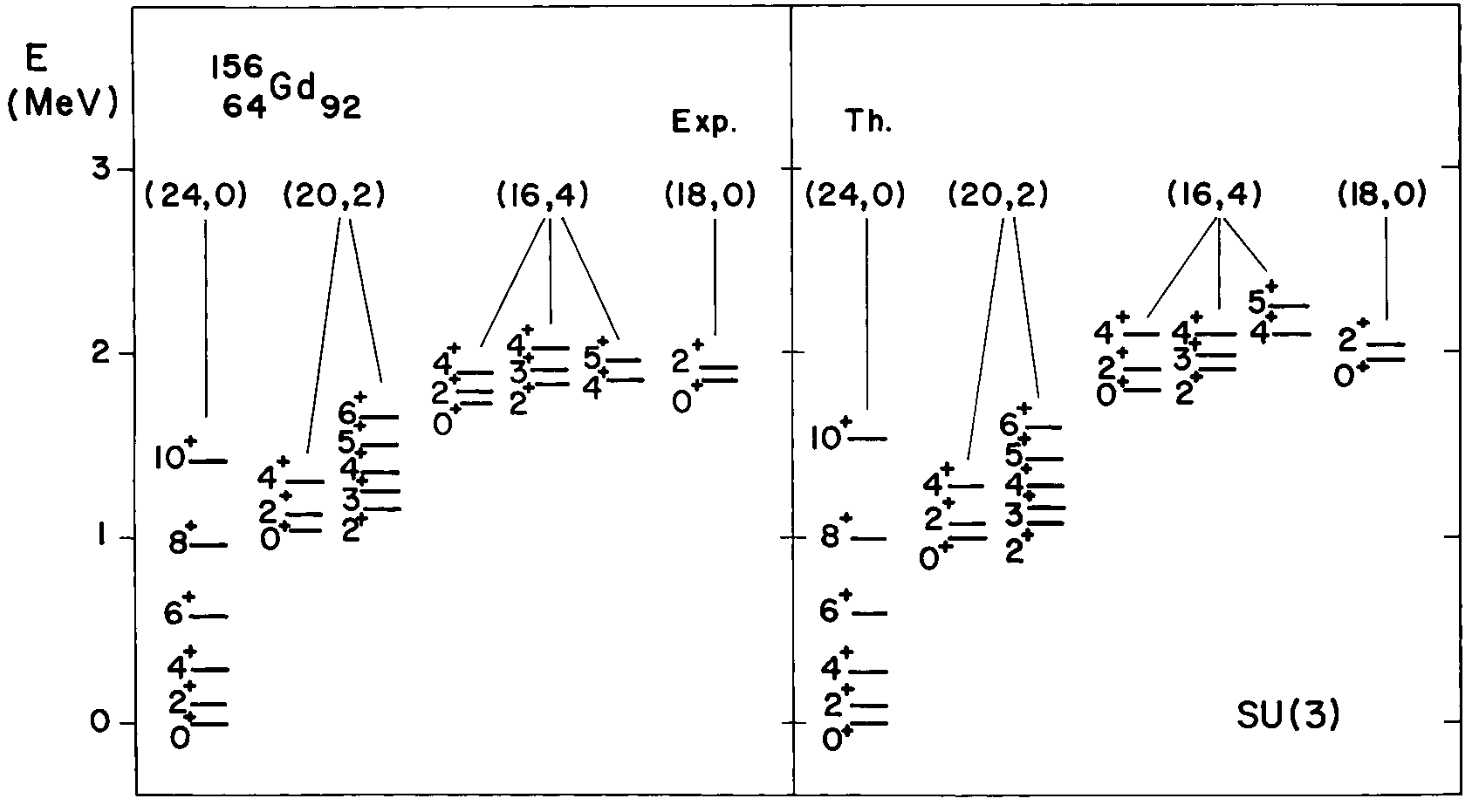
Rotational bands with L=0,2,4,…, max{λ,μ} for K=0 and L=K, K+1,…, K+max{λ,μ} for K>0, where K=integer=min{λ,μ}, min{λ,μ},…,1 or 0.

Many examples of the three symmetries of the Interacting Boson Model have been found in nuclei (1974-…)

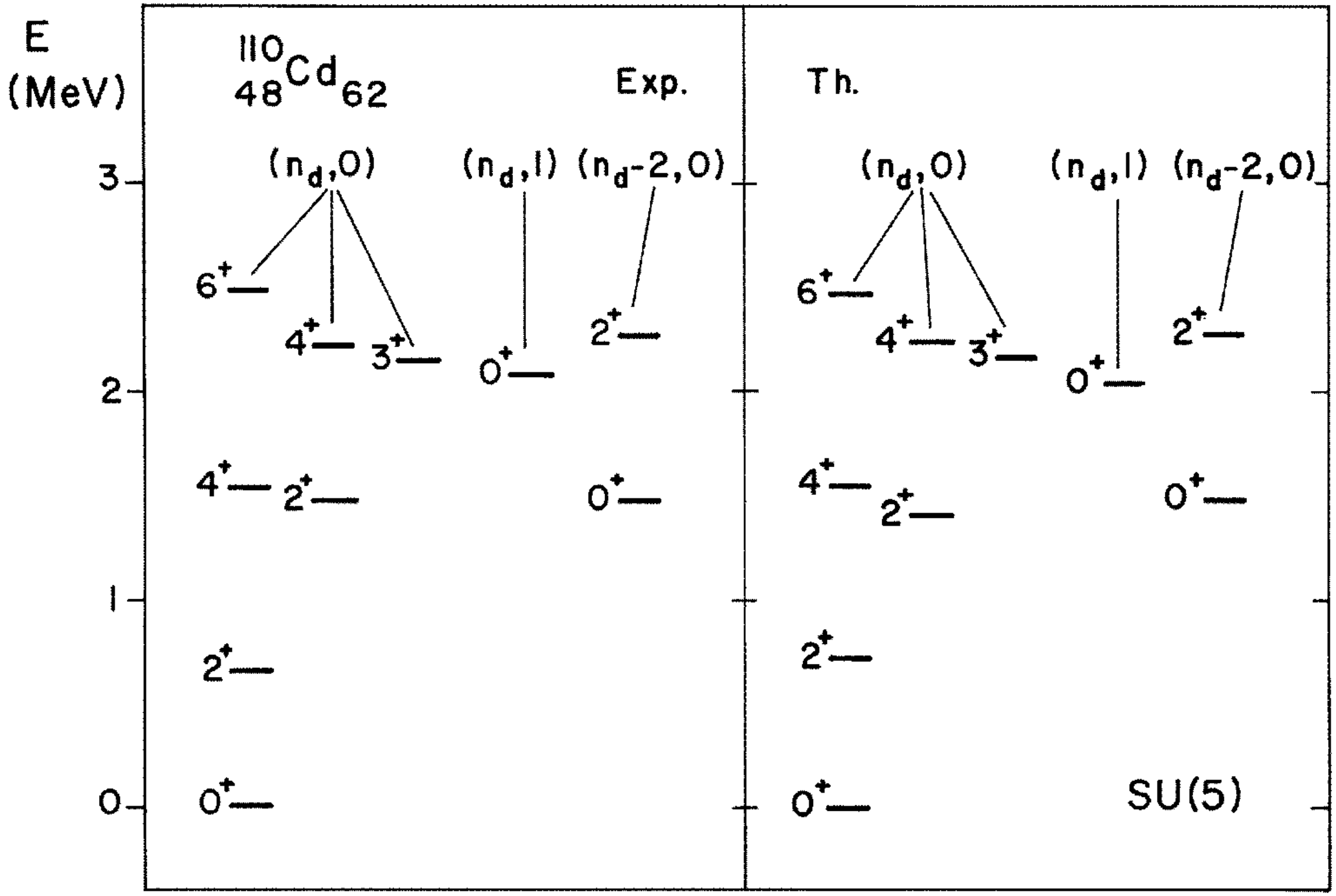
It has been found that symmetry extends to higher energies than originally thought!

Among the best examples of dynamic symmetry in physics!

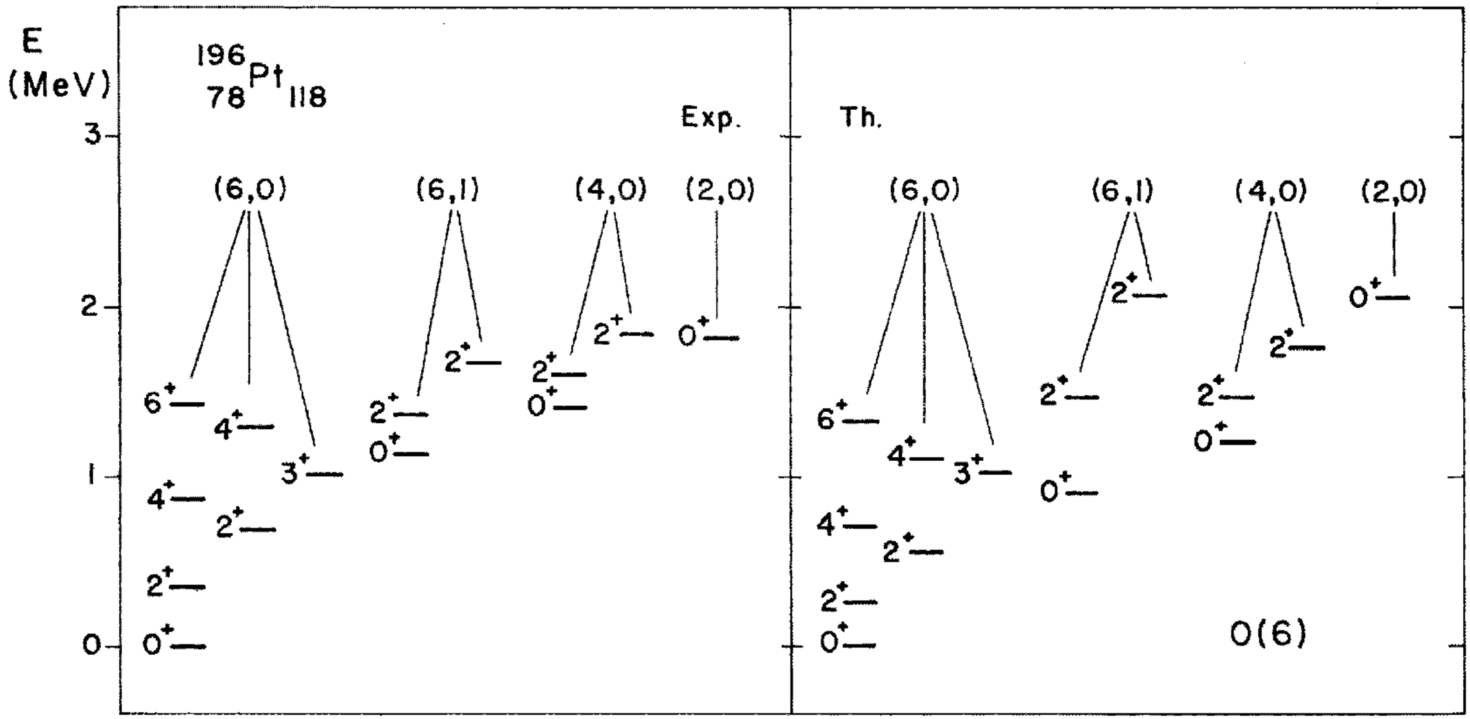
EVIDENCE FOR IBM-SU(3) SYMMETRY IN NUCLEI



EVIDENCE FOR IBM-U(5) SYMMETRY IN NUCLEI



EVIDENCE FOR IBM-SO(6) SYMMETRY IN NUCLEI



Another example: Molecules (Iachello, 1981; Iachello and Levine, 1982)

THE VIBRON MODEL

Spectra of molecules can be constructed from elementary excitations, called vibrons,
Vibron model [¶] with algebraic structure u(4).
Dynamic symmetries in this model are obtained by breaking u(4) into its subalgebras

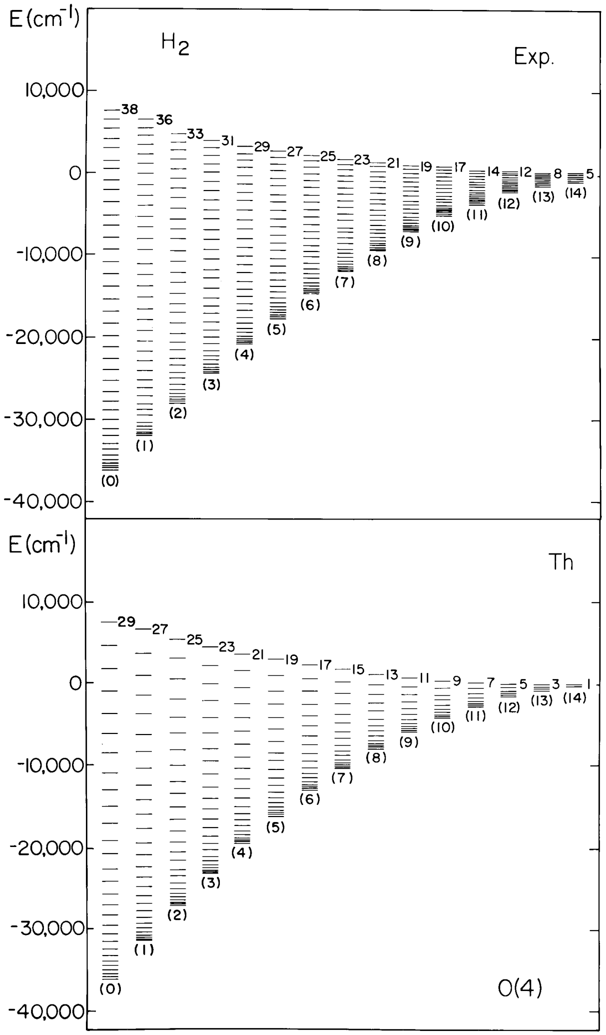
$$u(4) \supset u(3) \supset so(3) \supset so(2) \qquad \text{(I)}$$

$$u(4) \supset so(4) \supset so(3) \supset so(2) \qquad \text{(II)}$$

Several examples of dynamic symmetries in molecules have been found

[¶] For a review see F. Iachello and R.D. Levine, *Algebraic Theory of Molecules*, Oxford University Press, Oxford, 1995.

EVIDENCE FOR VM-SO(4) SYMMETRY IN MOLECULES



The spectrum of the H_2 molecule is shown as an example of dynamic symmetry in molecules

SUPERSYMMETRY IN PHYSICS

In the 1970's, in an attempt to further unify the laws of physics, a new concept was introduced: supersymmetry (Miyazawa, 1966; Ramond, 1971; Neveu and Schwartz, 1971; Volkov and Akulov, 1973; Wess and Zumino, 1974).

Permutation symmetry: bosons and fermions
Discussed previously: systems of bosons or systems of fermions. Symmetry operations change bosons into bosons or fermions into fermions.
Supersymmetry: symmetry operations change also bosons into fermions and viceversa (appropriate for mixed systems of bosons and fermions). A very strange type of symmetry!

(From M.C. Escher, *Fish*, circa 1942)

Also in the 1970's the mathematical language needed to describe supersymmetry, Graded Lie algebras and groups, was developed and a classification of these algebraic structures was given (Kac, 1975).

Supersymmetry and its language, Graded Lie algebras and groups, also called superalgebras, is used today in a variety of ways.

One of them is the use of superalgebras as a tool to solve problems in quantum mechanics, called Supersymmetric Quantum Mechanics (Witten, 1970's).

Most importantly, in the last 30 years, supersymmetry has become a guiding principle in constructing theories of Nature.

SOME OF THE WAYS OF SUPERSYMMETRY

1. Space-time (fundamental) supersymmetry

A generalization of Lorentz-Poincare' symmetry

Space-time coordinates x,t (bosonic)
Super space-time coordinates θ (fermionic) (Grassmann variables)

Transformations mix x,t and θ

Mathematical framework: SuperPoincare' group

2. Gauge supersymmetry

Fixes the form of the equations satisfied by the fields.
Fixes the form of the interaction between particles and external fields.

An example is the Wess-Zumino Lagrangian (1974).

$$L = L_B + L_F + L_{BF}$$

$$L_B = -\frac{1}{2}\left(\partial_\mu A(x)\right)^2 - \frac{1}{2}\left(\partial_\mu B(x)\right)^2 - \frac{1}{2}m^2 A^2(x) - \frac{1}{2}m^2 B^2(x)$$

$$-g\, mA(x)\left[A^2(x) + B^2(x)\right] - \frac{1}{2}g^2\left[A^2(x) + B^2(x)\right]$$

$$L_F = -\frac{1}{2}i\overline{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - \frac{1}{2}im\overline{\psi}(x)\psi(x)$$

$$L_{BF} = -ig\overline{\psi}(x)\left[A(x) - \gamma_5 B(x)\right]\psi(x)$$

Fundamental space-time and gauge supersymmetries are one of the most active areas of research in particle physics at the present time.

Consequence of supersymmetry:
To each particle there corresponds a super-particle (quarks-squarks, gluon-gluino, …)

The Large Hadron Collider (LHC) at CERN has been built in part to search for supersymmetric partners of the known particles.

3. Dynamic supersymmetry

Fixes the boson-boson, fermion-fermion and boson-fermion interactions in a mixed system of bosons and fermions

Determines spectral properties of mixed systems of bosons and fermions

An example of dynamic supersymmetry has been discovered in atomic nuclei (Iachello, 1980; Balantekin, Bars and Iachello, 1980) and will now be briefly discussed.

DYNAMIC SUPERSYMMETRY

The concept of dynamic supersymmetry, introduced implicitly by Miyazawa in 1966, was formalized in 1980 ¶

Definition: Dynamic supersymmetry is a situation in which the Hamiltonian and other operators of a mixed system of bosons and fermions can be written in terms of elements of a Graded Lie Algebra, also called a superalgebra, g*, and furthermore the Hamiltonian contains only invariant (Casimir) operators of the algebra g* and its graded (or not) subalgebras

$$g^* \supset g' \supset g'' \supset \ldots$$

Consequences of a dynamic supersymmetry:
All properties for mixed systems of bosons and fermions can be calculated in explicit analytic form!

The mathematical framework needed to describe supersymmetries is Graded Lie Algebras.

¶ F. Iachello, Phys. Rev. Lett. 44, 772 (1980).

GRADED LIE ALGEBRAS (also called superalgebras)

A set of operators, X (bosonic) and Y (fermionic), satisfying the commutation relations

$$\left[X_\alpha, X_\beta\right] = \sum_\gamma c_{\alpha\beta}^{\gamma} X_\gamma$$

$$\left[X_\alpha, Y_\beta\right] = \sum_\gamma d_{\alpha\beta}^{\gamma} Y_\gamma$$

$$\left\{Y_\alpha, Y_\beta\right\} = \sum_\gamma f_{\alpha\beta}^{\gamma} X_\gamma$$

together with the super-Jacobi identity is said to form a graded Lie algebra g*.

Graded Lie algebras U(n/m) can be simply constructed as bilinear products of boson and fermion creation and annihilation operators.

$$G_{\alpha\beta}^{B} = b_\alpha^\dagger b_\beta$$

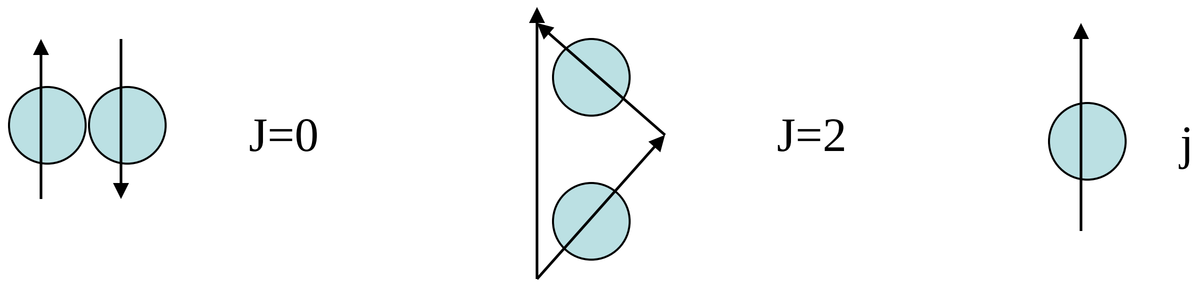
$$G_{ik}^{F} = a_i^\dagger a_k$$

$$F_{\alpha i}^\dagger = b_i^\dagger a_i$$

$$F_{i\alpha} = a_i^\dagger b_\alpha$$

SUPERSYMMETRY IN NUCLEI

In odd-even nuclei at least one particle is unpaired, and at higher energies pairs may break. A more accurate description of nuclei is then in terms of correlated pairs, treated as bosons, plus unpaired fermions: Interacting Boson-Fermion Model ¶



Hamiltonian $$H = H_B + H_F + V_{BF}$$

$$H_B = E_0 + \sum_{\alpha\beta} \varepsilon_{\alpha\beta} b_\alpha^\dagger b_\beta + \sum_{\alpha\alpha'\beta\beta'} v_{\alpha\alpha'\beta\beta'} b_\alpha^\dagger b_{\alpha'}^\dagger b_\beta b_{\beta'}$$

$$H_F = E'_0 + \sum_{ik} \eta_{ik} a_i^\dagger a_k + \sum_{ii'kk'} u_{ii'kk'} a_i^\dagger a_{i'}^\dagger a_k a_{k'}$$

$$V_{BF} = \sum_{\alpha\beta ik} w_{\alpha\beta ik} b_\alpha^\dagger b_\beta a_i^\dagger a_k$$

A supersymmetry occurs if the couplings in H are related by a supersymmetry transformation.

¶ For a review see, F. Iachello and P. van Isacker, *The Interacting Boson Fermion Model*, Cambridge University Press, Cambridge, 1991.

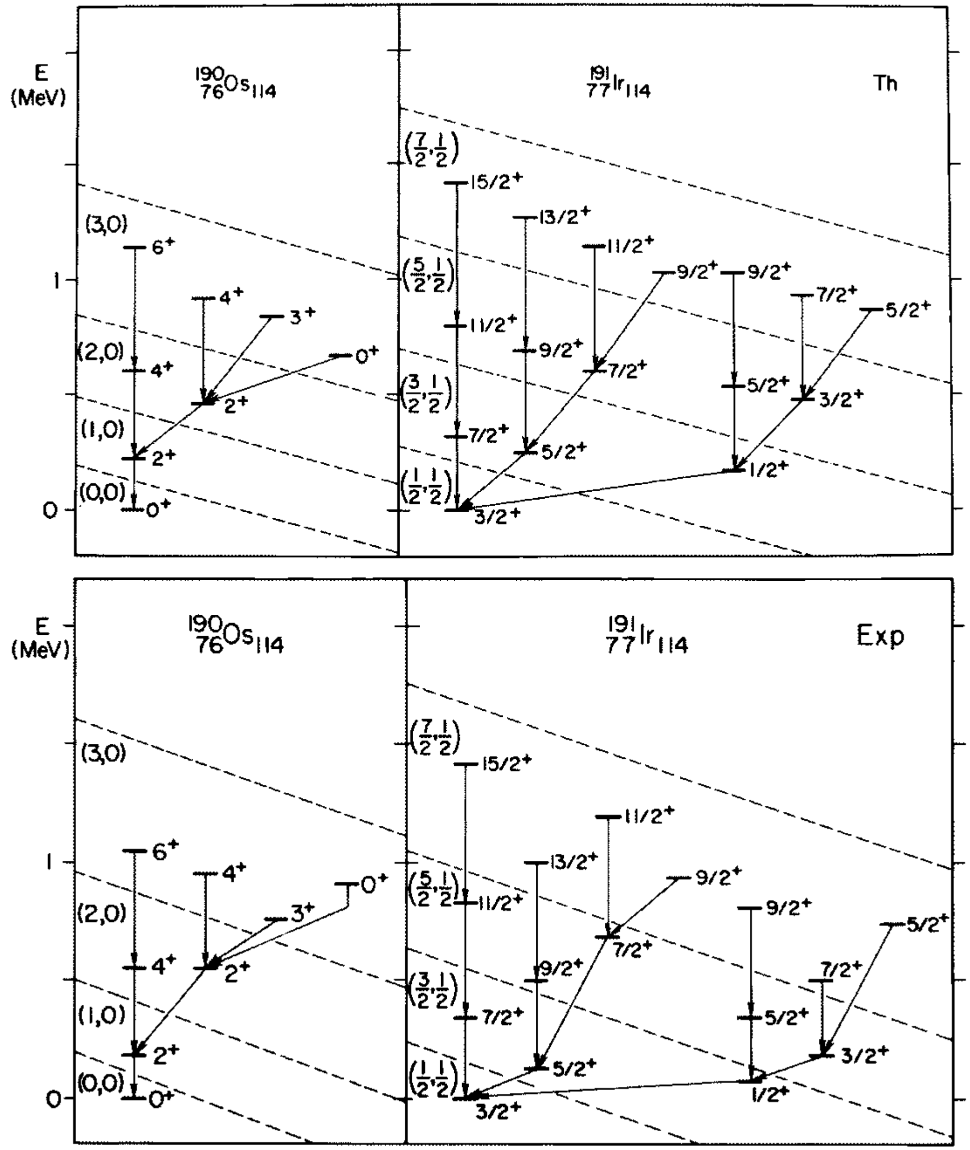
Consequences of supersymmetry:
If bosonic states are known, one can predict fermionic states! Both are given by the same formula! A most intricate type of symmetry!
[In particle physics, for example, if one knows properties of particles, one can predict those of their supersymmetric partners. Quarks-Squarks. Gluons-Gluinos.]
Here, if one knows the spectra of even-even nuclei (bosonic), one can predict those of odd-even nuclei (odd-proton or odd-neutron) (fermionic).

In the 1980's several cases of spectra with supersymmetric properties were found.

Supersymmetry in nuclei discovered!

The only experimental example of supersymmetry so far!

AN EXAMPLE OF U(6/4) SUPERSYMMETRY IN NUCLEI: ^{190}Os-^{191}Ir



SUPERSYMMETRY IN NUCLEI CONFIRMED

An improvement in the description of nuclei is obtained by the explicit introduction of proton and neutron degrees of freedom, in particular here by the introduction of proton pairs and neutron pairs. The corresponding model is known as Interacting Boson Model-2, with algebraic structure

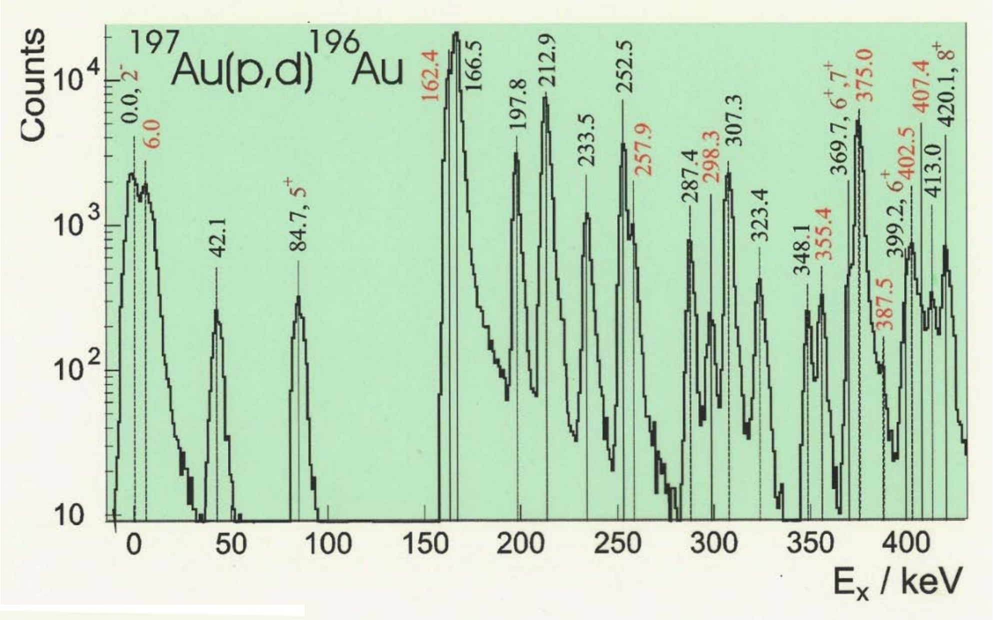
$$U_\pi(6) \otimes U_\nu(6)$$

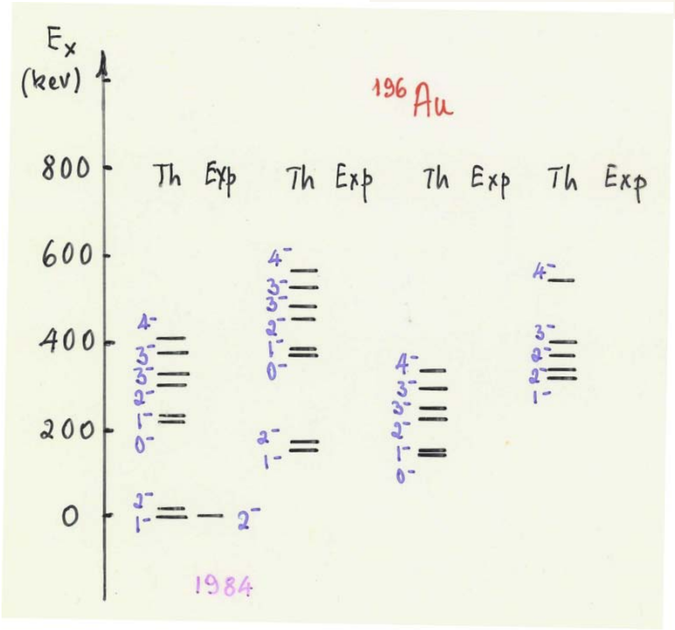
Consequently, when going to nuclei with unpaired particles, one has a model with two types of bosons (protons and neutrons), called Interacting Boson-Fermion Model-2. If supersymmetry occurs for this very complex system, one expects to have supersymmetric partners composed of a quartet of nuclei, even-even, even-odd, odd-even and odd-odd.

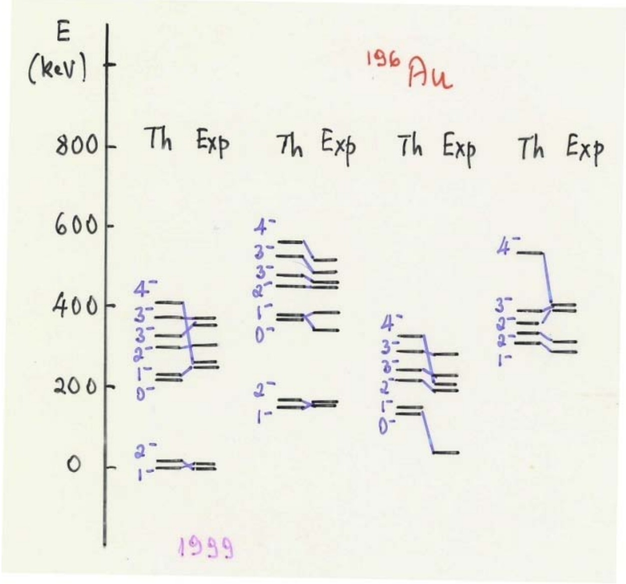
^{195}Au	^{196}Au
^{194}Pt	^{195}Pt

Spectra of even-even and even-odd nuclei have been known for some time. However, spectra of odd-odd nuclei are very difficult to measure, since the density of states in these nuclei is very high and the resolution of most detectors is not sufficiently good.

In a major effort that has involved several laboratories for several years, it has been possible to measure spectra of odd-odd nuclei. In particular, the magnetic spectrometer at the Ludwig-Maximilians Universität in Münich, Germany can separate levels only a few keV apart. It has thus been possible to measure the spectrum of 196Au, the missing supersymmetric partner of 194Pt, 195Pt and 195Au. (Metz et al, 1999; Metz et al, 2000; Gröger et al, 2000).

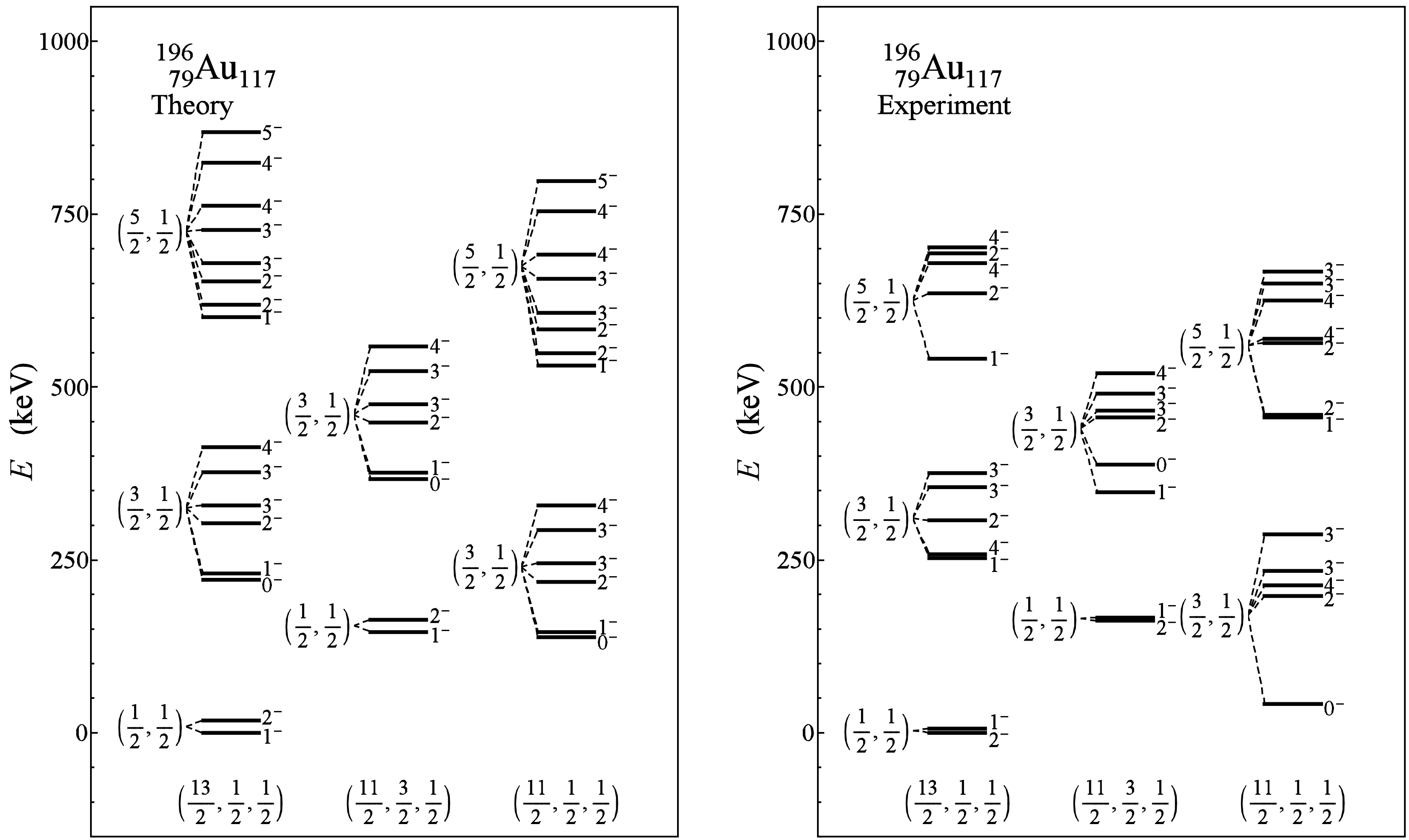






From A. Metz *et al.*, Phys. Rev. Lett. 83, 1542 (1999).

AN EXAMPLE OF SUPERSYMMETRY IN NUCLEI: 196Au



Implications of the discovery of supersymmetry in nuclei to other fields

(a) Particle Physics

Supersymmetry has been sought in Particle Physics for decades. The occurrence of supersymmetry in nuclei indicates that this type of symmetry can occur in Nature, and thus gives hope that, although badly broken, supersymmetry may play a role in Particle Physics. However, supersymmetry in nuclei is between composite bosons (pairs) and fundamental fermions.

Composite supersymmetry in Particle Physics?

(Catto and Gürsey, 1985)

(b) Condensed matter physics

Supersymmetry has been used here mostly as a mathematical tool (Parisi and Sourlas, 1979). Nambu (1985) has suggested that supersymmetry may occur in Type II superconductors with supersymmetric partners Cooper pairs-Impurities, identical to what encountered in nuclei.
The possibility of supersymmetry occurring in high-Tc superconductors has also been mentioned (Müller, 2002; Iachello, 2002).

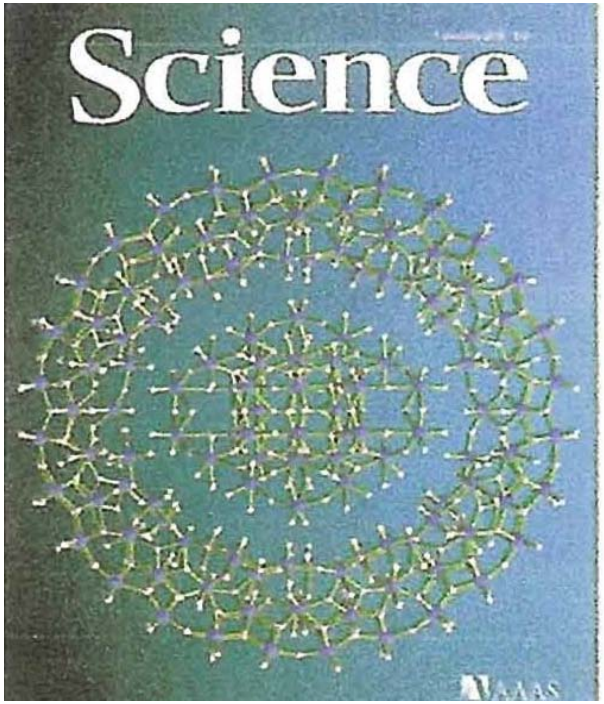
THE FUTURE OF SYMMETRY IN PHYSICS

1. Fundamental symmetries: Experimental verification of supersymmetry in particle physics

Supersymmetry has been observed in nuclei. Is this an accidental fact?
Does supersymmetry, even if badly broken, exists in particle physics?
Does supersymmetry play a fundamental role in Nature?

2. Geometric symmetries: Unraveling further the role of symmetry in complex materials

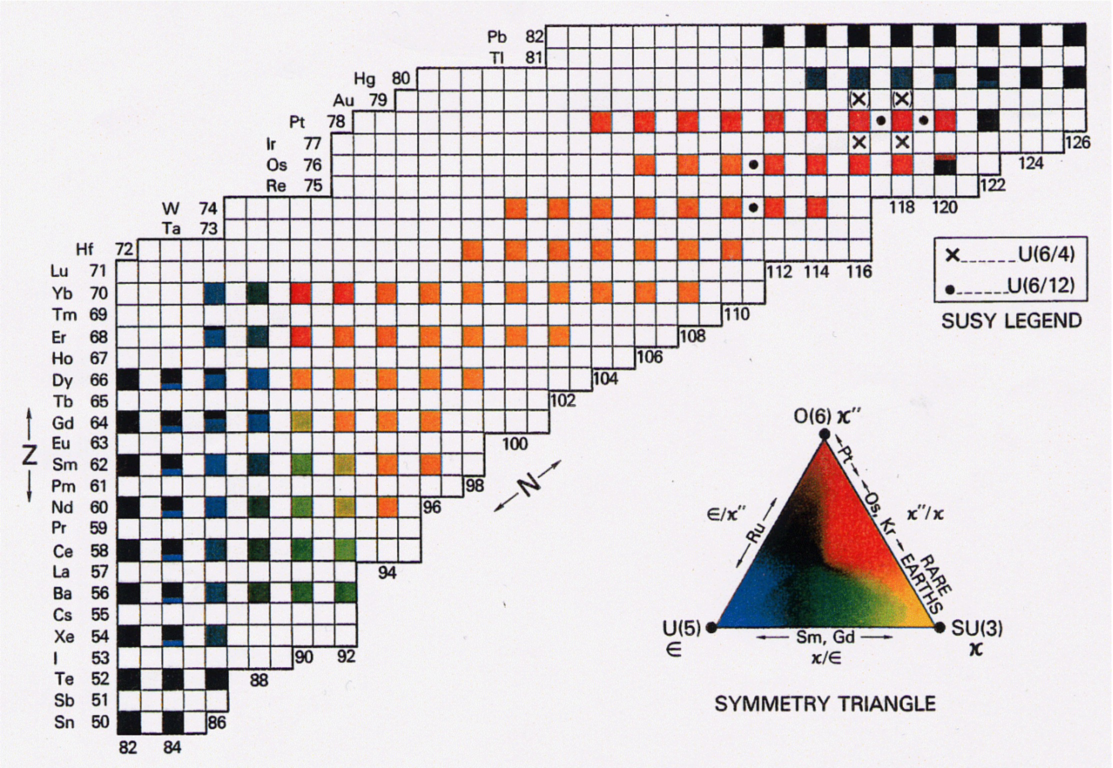
As new materials are discovered, more and more the role of geometric symmetries becomes apparent.



Crystal structure of a molybdenum oxide nanowheel (Miras *et al.*, 2010)
(From the cover of Science, 1 Jan 2010)

3. Dynamic symmetry: Application to complex systems. Simplicity in complexity program.

Most applications so far of dynamic symmetry have been in molecules, atoms, nuclei and hadrons

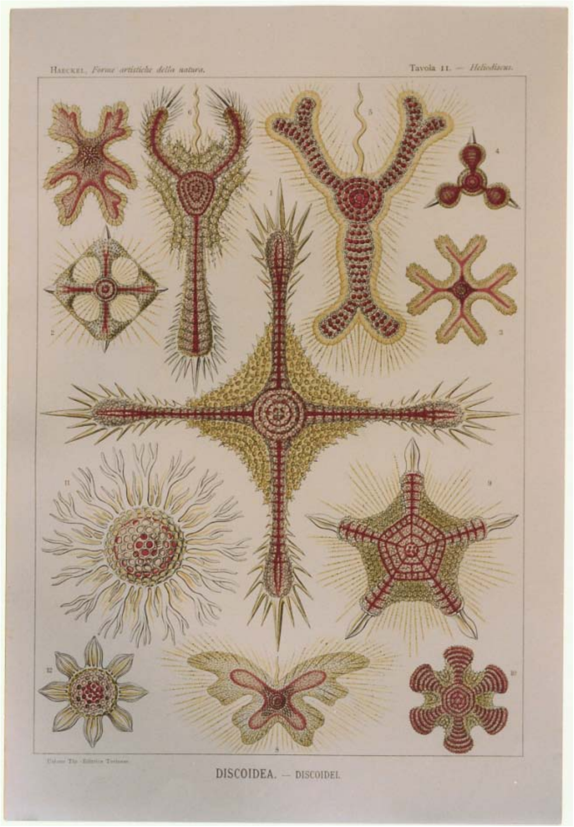


Classification of spectra of nuclei according to dynamic symmetry groups (Figure from Casten and Feng, 1984)

OTHER APPLICATIONS?

Macromolecules, polymers, atomic (Bose and Bose-Fermi) condensates, …
Biological molecules

At the macroscopic level, many forms of Nature, even the most complex, are often ordered



(From E. Haeckel, *Kunstformen der Natur*, Leipzig, 1899)

Order in the bio-world at the microscopic level?

CONCLUSIONS

Symmetry in its various forms has become a guiding principle in the description of Nature.

The 20th Century has seen the development of space-time and gauge symmetries as a tool in determining the fundamental laws of physics.

It has also seen the emergence of dynamic symmetry (and supersymmetry) as a way to classify the structure of physical systems.

The 20^{th} Century has also seen the development of new mathematical tools needed to describe symmetries (and supersymmetries).

[Galileo: The book of Nature is written in the language of Mathematics].

Nature and physics appear to display order at all levels:
The fundamental laws of Nature are dictated by symmetry principles (space-time and gauge symmetries).

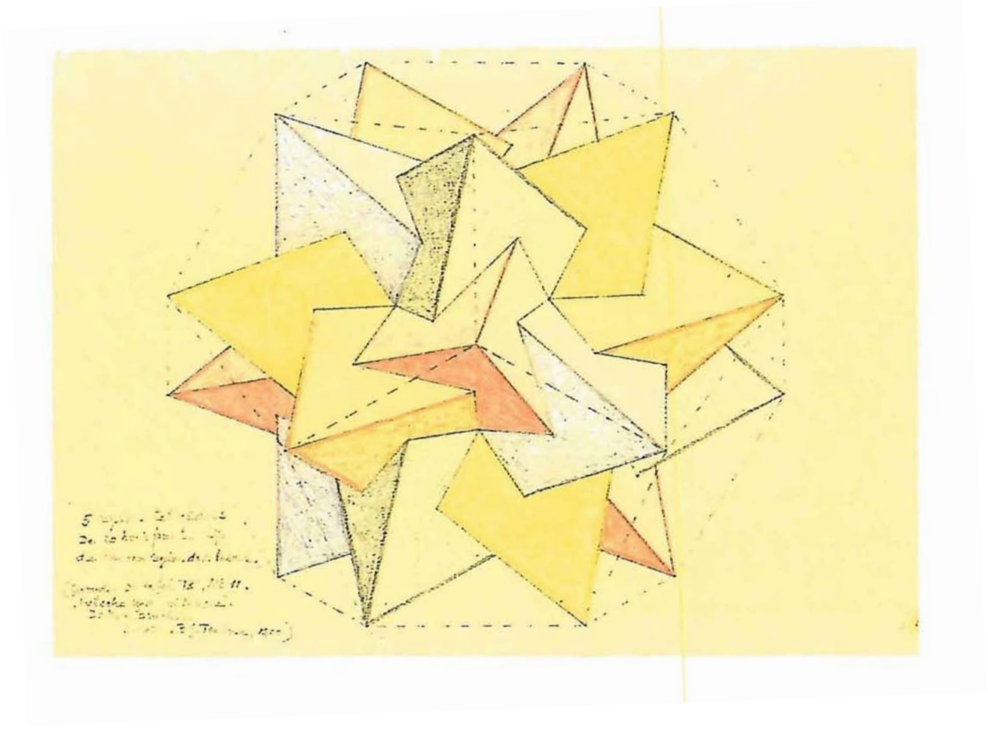
Spectra of quantum systems are often ordered as seen in molecules, atoms, nuclei and hadrons (dynamic symmetries).

Constituent particles often aggregate in ordered structures (geometric symmetries).

As Herman Weyl wrote: Nature loves symmetry!

In the 21st Century, as the complexity of the phenomena that we are studying increases, symmetry may play an equally important role.

In fact, one of the lessons we have learned is that the more complex the structure, the more useful is the concept of symmetry.



5 regular tetrahedra whose 20 vertices are those of a regular dodecahedron (From M.C. Escher, 1950)

I am therefore looking forward to many more applications of symmetry concepts to Physics!