A DIRECT PROOF OF THE TONELLI’S PARTIAL REGULARITY RESULT

ALESSANDRO FERRIERO

UAM – ICMAT

Departamento de Matemáticas

Universidad Autónoma de Madrid

Campus de Cantoblanco

28049 Madrid, Spain

Abstract. The aim of this work is to give a simple proof of the Tonelli’s partial regularity result which states that any absolutely continuous solution to the variational problem

\[ \min \left\{ \int_a^b L(t, u(t), \dot{u}(t)) \, dt : u \in W^{1,1}_0(a, b) \right\} \]

has extended-values continuous derivative if the Lagrangian function \( L(t, u, \xi) \) is strictly convex in \( \xi \) and Lipschitz continuous in \( u \), locally uniformly in \( \xi \) (but not in \( t \)). Our assumption is weaker than the one used in [2, 4, 5, 6, 13] since we do not require the Lipschitz continuity of \( L \) in \( u \) to be locally uniform in \( t \), and it is optimal as shown by the example in [12].

1. Introduction. A solution \( \bar{u} \) in the Sobolev space \( W^{1,1}_0(a, b) \) to the variational problem

\[ \min \left\{ I(u) := \int_a^b L(t, u(t), \dot{u}(t)) \, dt : u \in W^{1,1}_0(a, b) \right\} \]

is said to be regular in the sense of Tonelli if its derivative \( \dot{\bar{u}} \) is extended-values continuous on \([a, b]\). The main advantage of this type of regularity is that the set of singularities of \( \dot{\bar{u}} \) is a closed set of \([a, b]\). In other words, the interval \([a, b]\) can be covered a.e. by the numerable union of disjoint open intervals such that \( \dot{\bar{u}} \) has continuous derivative on each of them.

The purpose of this paper is to give a simple proof of the Tonelli’s partial regularity result for Lagrangian functions \( L(t, u, \xi) \) which are strictly convex in \( \xi \) and Lipschitz continuous in \( u \), locally uniformly in \( \xi \). To be more precise, we require that, for each \( R > 0 \), there exists an integrable function \( C_R : [a, b] \to \mathbb{R}^+ \) such that

\[ |L(t, u, \xi) - L(t, v, \xi)| \leq C_R(t)|u - v|, \quad (1) \]

for a.e. \( t \) in \([a, b]\) and any vectors \( u, v, \xi \) in \( \mathbb{R}^d \) with modulus smaller than \( R \).

Up to now [2, 4, 5, 6, 13], the Lipschitz continuity of \( L \) on \( u \), locally uniform in \( t \) and \( u \), has been the weakest condition imposed for proving this kind of regularity.

2000 Mathematics Subject Classification. Primary: 49B05, 49A05, 49C05, 35J20.

Key words and phrases. Variational problems, regularity, quasi-linear elliptic differential equations.
In [12], the author gives an example of a Lagrangian which is Lipschitz continuous in \( u \), locally uniformly in \( \xi \), but with non-integrable Lipschitz constant \( C_R \), which admits solutions which do not enjoy the Tonelli’s regularity. Our result is therefore optimal in this sense.

In [10], we relax the Lipschitz continuity of \( L \) in \( u \) and we prove the Tonelli’s partial regularity assuming an invariance property on the functional \( \mathcal{I} \) as in the Noether’s theorem. In [9], we prove a weaker regularity result than the Tonelli’s partial regularity, namely the local Lipschitz continuity of any solutions \( \bar{u} \), releasing the strict convexity of \( L \) in \( \xi \) and assuming a weaker hypothesis called the bounded slope condition (which roughly speaking means that the convexified function of \( L \) in \( \xi \) does not admit unbounded affine pieces).

The proof we propose here is based on the following direct approach. Suppose that \( t_0 \) in \( (a, b) \) is a jump point for \( \bar{u} \), i.e.

\[
\dot{\bar{u}}_-(t_0) := \lim_{t \to t_0^-} \dot{\bar{u}}(t) \neq \lim_{t \to t_0^+} \dot{\bar{u}}(t) =: \dot{\bar{u}}_+(t_0).
\]

We can then show that the competitor \( \bar{w} \) of \( \bar{u} \) obtained by taking the average \( 2^{-1} (\dot{\bar{u}}_-(t_0) + \dot{\bar{u}}_+(t_0)) \) of the right and left limit values of \( \bar{u} \) at \( t_0 \) on a neighbourhood of \( t_0 \) and identically equal to \( \bar{u} \) elsewhere has, by the strict convexity of \( L \), energy strictly lower than \( \bar{u} \). Hence, \( \bar{u} \) cannot be a minimizer for \( \mathcal{I} \). That is, if the Lagrangian \( L \) is strictly convex on \( \xi \) and \( \bar{u} \) is a minimizer, the derivative of \( \bar{u} \) cannot present jumps or oscillations but it must be extend-values continuous. Strict convexity of \( L \) and discontinuities of \( \dot{\bar{u}} \) are linked by the minimality of \( \mathcal{I}(\bar{u}) \) through the convexity of the integral operation.

Our proof has the great value to be simple, direct and brief. It is based on the idea of averaging afore sketched and a local generalized weak form of the Jensen inequality.

The paper is organized as follows. In section 2, we present the generalized form of the Jensen inequality, the main result concerning the Tonelli’s partial regularity of solutions and a counterexample showing that strict convexity of \( L \) in \( \xi \) is a necessary hypothesis. We also show that the result is not true under Carathéodory condition on \( L \), i.e. \( L \) measurable in \( t \) and continuous with respect to \( u \) and \( \xi \).

2. Main Result. In the present paper we shall deal with Lagrangian functions

\[
L(t, u, \xi) : ([a, b] \setminus \Sigma_L) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R},
\]

where \( \Sigma_L \) is a closed set of zero Lebesgue measure in \( [a, b] \), which are continuous with respect to \( t, u \) and \( \xi \). In case \( \Sigma_L = \emptyset \), \( L \) is continuous on \( [a, b] \times \mathbb{R}^d \times \mathbb{R}^d \). We further assume that, for each \( R > 0 \), there exists a function \( C_R : [a, b] \to \mathbb{R}^+ \) in \( L^1(a, b) \) such that

\[
|L(t, u, \xi) - L(t, v, \xi)| \leq C_R(t)|u - v|, \tag{2}
\]

for a.e. \( t \) in \( [a, b] \) and every vector \( u, v, \xi \) in \( \mathbb{R}^d \) with modulus smaller than \( R \). Thus, \( L \) is Lipschitz continuous in \( u \), locally uniformly in \( \xi \) but not in \( t \). Notice that this is weaker than the assumption in [6] where it is required that the Lipschitz continuity of \( L \) in \( u \) is locally uniform also in \( t \).

The variational problem we are interested in is

\[
\min \left\{ \mathcal{I}(u) := \int_a^b L(t, u(t), \dot{u}(t))dt : u \in W^{1,1}_0(a, b) \right\}, \tag{3}
\]
where $W^{1,1}_0(a,b)$ denotes as usual the Sobolev space of functions $u$ from $[a,b]$ to $\mathbb{R}^d$, $d \geq 1$, with zero boundary conditions.

The general Dirichlet boundary data $u(a) = A$, $u(b) = B$, that is taking $r + W^{1,1}_0(a,b)$ as space of admissible functions, where $r(t) := (t-a)(B-A)/(b-a)+A$, can be deduced by the zero boundary case (preserving the assumptions on $L$) by considering the modified Lagrangian $\tilde{L}(t,u,\xi) := L(t,u-r(t),\xi - \dot{r}(t))$. For any solution $\bar{u}$ in $W^{1,1}_0(a,b)$ to $\mathcal{I}$ corresponds a solution $\bar{u} = r + \bar{u}$ in $r + W^{1,1}_0(a,b)$ to $\mathcal{I}$, and conversely.

In what follows,

- $\mathbb{R}$ denotes the two points compactification $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ of $\mathbb{R}$,
- $\mathbb{R}^d$ denotes the one point compactification $\mathbb{R}^d \cup \{\infty\}$ of $\mathbb{R}^d$, for $d \geq 2$.

$C([a,b] \setminus \Sigma_L; \mathbb{R}^d)$ denotes the space of continuous functions from $[a,b] \setminus \Sigma_L$ to $\mathbb{R}^d$. We will also refer to these functions as extended-values continuous functions. Notice that, for any $\xi$ in $C([a,b] \setminus \Sigma_L; \mathbb{R}^d)$, the preimage $\xi^{-1}(\mathbb{R}^d)$ is an open set of $[a,b] \setminus \Sigma_L$.

**Definition 1.** We say that a solution $\tilde{u}$ in $W^{1,1}_0(a,b)$ to (3) is regular in the sense of Tonelli if its derivative $\dot{\tilde{u}}$ belongs to $C([a,b] \setminus \Sigma_L; \mathbb{R}^d)$.

For a function $\tilde{u}$ that enjoys the properties of the definition, recalling that $\Sigma_L$ is a closed set of zero measure, there exist numerable many disjoint open intervals $I_n$ of $[a,b]$ such that $\tilde{u}|_{I_n}$ belongs to $C^1(I_n)$, for any $n$, and also $|\bigcup_n I_n| = b-a$. In other words, the set $S_{\tilde{u}}$ of singular points of the derivative of $\tilde{u}$, i.e.

$$S_{\tilde{u}} := \{ t \in [a,b] \setminus \Sigma_L : \tilde{u} \text{ is not continuous at } t \},$$

is a closed set of $[a,b] \setminus \Sigma_L$ of zero measure.

We present the proof of the Tonelli’s partial regularity result in Theorem 3 and 4; in Theorem 3 we deal with dimensions $d$ greater than 1, meanwhile in Theorem 4 we deal with dimension $d = 1$. Notice that our result for $d = 1$ is finer than the one for $d \geq 2$ since $C([a,b] \setminus \Sigma_L; \mathbb{R})$ is included in $C([a,b] \setminus \Sigma_L; \mathbb{R} \cup \{\infty\})$.

Throughout all the paper, $\{o_n(1)\}$ denotes any sequence which converges to 0, as $n$ goes to $\infty$, and $B(u_0;R) \subset \mathbb{R}^d$ is the closed ball with center $u_0$ and radius $R$.

We start showing a Lemma that will be important in the proofs of Theorem 3 and 4. It is a local weak version of the Jensen inequality adapted to our problem.

**Lemma 2.** Let $\delta > 0$, $t_0 \in (a,b) \setminus \Sigma_L$ and $\{E_n\}$ be a sequence of measurable sets in the interval $I_\delta := [t_0 - \delta, t_0 + \delta] \subset [a,b] \setminus \Sigma_L$ such that the diameter of $E_n \cup \{t_0\}$ converges to 0, as $n \to \infty$.

If $L(w,\cdot)$ is convex and $L(w,\xi) \geq |\xi|$, for every $w$ in $I_\delta \times B(0;R)$ and any $\xi$ in $\mathbb{R}^d$, then, for any continuous function $w : I_\delta \to I_\delta \times B(0;R)$ and any $\xi$ in $L^1(I_\delta)$ such that $|E_n|^{-1} \int_{E_n} \xi(t) dt \to \xi \in \mathbb{R}^d$, there exists a subsequence $\{k_n\}$ for which

$$\frac{1}{|E_{k_n}|} \int_{E_{k_n}} L(w(t),\xi(t)) dt \geq L(w(t_0),\tilde{\xi}) + o_n(1).$$

**Proof.** If there exists a subsequence $\{k_n\}$ such that $|E_{k_n}|^{-1} \int_{E_{k_n}} |\xi(t)| dt \to \infty$, then

$$\frac{1}{|E_{k_n}|} \int_{E_{k_n}} L(w(t),\xi(t)) dt \geq \frac{1}{|E_{k_n}|} \int_{E_{k_n}} |\xi(t)| dt \geq L(w(t_0),\tilde{\xi}) + o_n(1).$$

Otherwise, assume that $\{|E_n|^{-1} \int_{E_n} |\xi(t)| dt\}$ is bounded.

Since convex functions are a.e. differentiable and since the subgradients of convex functions are bounded on bounded sets, there exists a sequence $\{\xi_n\} \subset \mathbb{R}^d$, which
converges to \( \bar{\xi} \), such that, for every \( n \), the subgradient of \( L(w(t_0), \cdot) \) at \( \xi_n \) is a singleton, i.e. \( \partial \xi L(w(t_0), \xi_n) = \{ p_n(t_0) \} \), and \( p_n(t_0) \to p(t_0) \), \( p(t_0) \) in \( \partial \xi L(w(t_0), \xi) \).

By Proposition 2 in [3], \( \lim_{n \to \infty} p_n(t_0) = p_n(t_0) \), where \( p_n(t) \in \partial \xi L(w(t), \xi_n) \), and theref ore there exists a subsequence \( \{ k_n \} \) for which \( \| p_n - p_n(t_0) \|_{L^{\infty}(E_{k_n})} \to 0 \).

Then, \( \| p_n - p(t_0) \|_{L^{\infty}(E_{k_n})} \to 0 \).

By the convexity of \( L \) in \( \xi \), we have
\[
L(w(t), \xi(t)) \geq L(w(t), \xi_n) + \langle p_n(t), \xi(t) - \xi_n \rangle,
\]
for every \( t \) in \( E_{k_n} \) and any \( n \). By taking the integral over \( E_{k_n} \) of the inequality above, we obtain
\[
\int_{E_{k_n}} L(w(t), \xi(t)) dt \geq |E_{k_n}| L(w(t_0), \bar{\xi}) + \int_{E_{k_n}} [L(w(t), \xi_n) - L(w(t), \bar{\xi})] dt
+ \int_{E_{k_n}} \| p_n(t) - p(t_0) \|_{L^{\infty}(\xi_n)} dt.
\]
By the boundedness of \( \{ \xi_n \} \) and of the sequence of the averages of \( |\xi| \) on \( E_n \), the two integrals at the second member of the inequality above are equal to \( |E_{k_n}| n(1) \).
That concludes the proof.

**Theorem 3.** Let \( d \geq 2 \). If \( L(t, u, \cdot) \) is strictly convex, for any \( t \) in \( [a, b] \setminus \Sigma_L \) and any \( u \) in \( \mathbb{R}^d \), then any solution \( \bar{u} \) in \( W^{1,1}_0(a, b) \) to the variational problem (3) is regular in the sense of Tonelli.

**Proof.** Let \( \bar{u} \) in \( W^{1,1}_0(a, b) \) be a solution to (3).

**Part 1.** Let \( t_0 \) be any point in \( [a, b] \setminus \Sigma_L \), \( I(\epsilon) \) be the closed interval \( [t_0, t_0 + \epsilon] \), \( \epsilon > 0 \), and fix \( C \) to be a compact set of \( [a, b] \setminus \Sigma_L \) big enough to contain \( t_0 \) as interior point. We claim that
\[
\exists \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{I(\epsilon)} \bar{u}(t) dt =: \bar{\xi} \text{ in } \mathbb{R}^d \cup \{ \infty \}.
\]
Observe that the claim is trivially true at a.e. point \( t_0 \) in \( [a, b] \setminus \Sigma_L \), that is at the Lebesgue points of \( \bar{u} \). What we claim is that this is true for all points of \( [a, b] \setminus \Sigma_L \).

Here, for \( \bar{\xi} = \infty \), we mean that
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{I(\epsilon)} \bar{u}(t) dt = \infty.
\]
Suppose, on the contrary, that there exist two sequences of positive numbers \( \{ \epsilon_n^1 \}, \{ \epsilon_n^2 \}, \epsilon_n^1 + 1 < \epsilon_n^1 < \epsilon_n^2 \), for every \( n \), which converge to 0, such that
\[
\lim_{n \to \infty} \frac{1}{\epsilon_n^1} \int_{I(\epsilon_n^1)} \bar{u}(t) dt =: \bar{\xi}_1 \text{ in } \mathbb{R}^d,
\]
and
\[
\lim_{n \to \infty} \frac{1}{\epsilon_n^2} \int_{I(\epsilon_n^2)} \bar{u}(t) dt =: \bar{\xi}_2 \text{ in } \mathbb{R}^d,
\]
and
\[
\bar{\xi}_1 \neq \bar{\xi}_2.
\]
Since \( \bar{\xi}_1 \) and \( \bar{\xi}_2 \) cannot be both \( \infty \) by (4), one of the two, say \( \bar{\xi}_1 \), belongs to \( \mathbb{R}^d \).
Furthermore, by the continuity of the integral function
\[
M(I(\epsilon)) := \frac{1}{|I(\epsilon)|} \int_{I(\epsilon)} \bar{u}(t) dt, \quad \text{for } \epsilon \text{ in } [\epsilon_n^1, \epsilon_n^2],
\]
we can assume that $\hat{\xi}_2$ belongs to $\mathbb{R}^d$, too, by moving $\epsilon_n^2$ closer to $\epsilon_n^1$.

We can also suppose, by replacing $L(t, u, \xi)$ with $\alpha L(t, u, \xi) + \langle \overline{p}_1, \xi - \xi_1 \rangle + \beta$, for suitable $\alpha, \beta \geq 0$ and $\overline{p}_1$ in $\mathbb{R}^d$, that

$$L(t, u, \xi) \geq |\xi|, \quad \text{for any } (t, u, \xi) \in C \times B(0; \|\overline{u}\|_{L^\infty(a,b)} + 1) \times \mathbb{R}^d,$$

(5)

where recall that $C$ is the compact set fixed at the beginning of Part I (see [4, 6]).

We claim that the sequence of functions $\{\hat{u}(t_0 + \epsilon_n^2 t)\}_n$ converges in measure, for $t$ in $(0, 1)$, to the constant function $\hat{\xi}_2$.

Indeed, on the contrary, there exists $\delta > 0$ for which the set $E_n(\delta) := \{t \in (0, 1) : |\hat{u}(t_0 + \epsilon_n^2 t) - \hat{\xi}_2| \geq \delta\}$ is such that $|E_n(\delta)| \geq \lambda > 0$, for every $n$. Define

$$E_n^2(\delta) := t_0 + \epsilon_n^2 E_n(\delta) = \{t \in I(\epsilon_n^2) : |\hat{u}(t) - \hat{\xi}_2| \geq \delta\}.$$ 

Analogously as in the proof of Lemma 2, there exists a sequence $\{\xi_n\} \subset \mathbb{R}^d$, which converges to $\hat{\xi}_2$, such that, for every $n$, $\partial_k L(t_0, \overline{u}(t_0), \xi_n) = \{p_n(t_0)\}$, and $p_n(t_0) \rightarrow p(t_0)$, $p(t_0)$ in $\partial_k L(t_0, \overline{u}(t_0), \hat{\xi}_2)$. Therefore, for $p_n(t) \in \partial_k L(t, \overline{u}(t), \xi_n)$, there exists a subsequence $\{k_n\}$ for which $\|p_n - p_0(t_0)\|_{L^\infty(I(\epsilon_n^2))} \rightarrow 0$. Then,

$$\|p_n - p(t_0)\|_{L^\infty(I(\epsilon_n^2))} \rightarrow 0.$$ 

Reindex $\{I(\epsilon_n^2)\}$ and $\{E_n^2(\delta)\}$ with $\{k_n\}$.

By the strict convexity of $L$ on $\xi$, the functions $L(t, \overline{u}(t), \xi) - \langle p_n(t), \xi - \xi_n \rangle$ and $L(t_0, \overline{u}(t_0), \xi) - \langle p(t_0), \xi - \hat{\xi}_2 \rangle$ have their unique minima over $\overline{\xi}$ in $\mathbb{R}^d$ respectively at $\xi = \xi_n$ and $\xi = \hat{\xi}_2$. Since the sequence of the averages of $|\hat{u}|$ over $I(\epsilon_n^2)$ is bounded\(^1\) and $|E_n^2(\delta)|/\epsilon_n^2 \geq \lambda$, there exist $h > |\hat{\xi}_2| + \delta$, $\lambda > 0$ for which $\{|t \in E_n^2(\delta) : |\hat{u}(t)| \leq h\}/\epsilon_n^2 \geq \lambda$, for every $n$. Let $c > 0$ be such that

$$L(t, \overline{u}(t), \xi) - \langle p_n(t), \xi - \xi_n \rangle = L(t_0, \overline{u}(t_0), \xi) - \langle p(t_0), \xi - \hat{\xi}_2 \rangle + c + o_n(1) \geq L(t_0, \overline{u}(t_0), \hat{\xi}_2) + c + o_n(1),$$ 

for every $|\xi - \hat{\xi}_2| \geq \delta$, with $|\xi| \leq h$, and any $t$ in $I(\epsilon_n^2)$. Therefore,

$$\begin{aligned}
\int_{I(\epsilon_n^2)} [L(t, \overline{u}(t), \hat{u}(t)) - \langle p_n(t), \hat{u}(t) - \xi_n \rangle]dt \\
\geq \left(\{t \in E_n^2(\delta) : |\hat{u}(t)| \leq h\} \right) [L(t_0, \overline{u}(t_0), \hat{\xi}_2) + c + o_n(1)] \\
+ \left[I(\epsilon_n) \setminus \{t \in E_n^2(\delta) : |\hat{u}(t)| \leq h\} \right) [L(t_0, \overline{u}(t_0), \hat{\xi}_2) + o_n(1)] \\
\geq \epsilon_n^2 L(t_0, \overline{u}(t_0), \hat{\xi}_2) + \hat{\lambda} c + o_n(1).
\end{aligned}$$

(6)

On the other hand, by the minimality of $\overline{u}$,

$$\begin{aligned}
\int_{I(\epsilon_n^2)} L(t, \overline{u}(t), \hat{u}(t))dt \\
\leq \int_{I(\epsilon_n^2)} L(t, \overline{u}(t_0) + M(I(\epsilon_n^2))(t - t_0), M(I(\epsilon_n^2))dt \\
= \epsilon_n^2 L(t_0, \overline{u}(t_0), \hat{\xi}_2) + o_n(1)].
\end{aligned}$$

(7)

Comparing (7) and (6), we reach a contradiction since the integral of $\langle p_n, \hat{u} - \xi_n \rangle$ over $I(\epsilon_n^2)$ is equal to $\epsilon_n^2 o_n(1)$ and $\hat{\lambda} c > 0$. This proves the claim.

---

\(^1\)Observe that in general, by the minimality of $\overline{u}$, (5) and (7) imply that the sequence of averages of $|\overline{u}|$ over intervals $I_n \subset [a, b] \setminus \Sigma_L$, $|I_n| \rightarrow 0$, is bounded as long as $\{M(I_n)\}$, i.e. the sequence of averages of $\overline{u}$ over $I_n$, is bounded.
From the convergence in measure of \( \{ \hat{u}(t_0 + \epsilon_n) \} \), by Egorov’s theorem and a diagonal argument, there exists a positive sequence \( \{ \delta_n \} \) such that, setting \( Z_n := E_n^2(\delta_n) \), \( |Z_n^2|/\epsilon_n^2 \to 0 \) and

\[
\delta_n = \| \hat{u} - \bar{\xi}_2 \|_{L^\infty(I(\epsilon_n^2) \setminus Z_n^2)} \to 0,
\]

that is, \( \hat{u}|_{I(\epsilon_n^2) \setminus Z_n^2} \) converges uniformly to \( \bar{\xi}_2 \). This also implies that

\[
\lim_{n \to \infty} \epsilon_n^1/\epsilon_n^2 = 0.
\]

We proceed with the proof separating three cases: the first case in which \( \hat{u} \) is bounded close to \( t_0 \), i.e. in \( I(\epsilon_n^2) \), the second case in which \( \hat{u} \) is unbounded but the averages of the high values of \( \hat{u} \), i.e. \( M(E_n^2(\delta)) \) with \( \delta \) big, have a limit different from \( \xi_2 \) and the third case in which the averages of the high values of \( \hat{u} \) converge always to \( \xi_2 \).

**Case 1.** If there exists \( \delta > 0 \) such that \( |E_n^2(\delta)| = 0 \), for every \( n \), then we choose a measurable set \( B_n^2 \) in \( I(\epsilon_n^2) \setminus I(\epsilon_n^1) \cup Z_n^2 \) of the same measure as \( I(\epsilon_n^1) \), and we define the competitor \( w_n \) in \( W_0^{1,1}(a, b) \) by \( w_n(t) := \int_a^t \hat{w}_n(\tau)d\tau \) where

\[
\hat{w}_n(\tau) := \begin{cases} 
[M(I(\epsilon_n^1)) + M(B_n^2)]/2, & \tau \in I(\epsilon_n^1) \cup B_n^2, \\
\hat{u}(\tau), & \text{otherwise.}
\end{cases}
\]

Observe that \( w_n \) coincides with \( \hat{u} \) on \( [a, b] \setminus I(\epsilon_n^2) \).

We claim that

\[
\gamma_n := \| w_n - \bar{u} \|_{L^\infty(I(\epsilon_n^2))} = \epsilon_n^1 O_n(1),
\]

where \( O_n(1) \) is a bounded sequence. Indeed, by (5), the minimality of \( \bar{u} \) and the Lipschitz continuity of \( L \) in \( u \), we have that

\[
\begin{align*}
\int_{I(\epsilon_n^1) \cup B_n^2} |\hat{u}(t)|dt & \leq \int_{I(\epsilon_n^1) \cup B_n^2} L(t, \bar{u}(t), \hat{u}(t))dt \\
& \leq \int_{I(\epsilon_n^2)} L(t, w_n(t), \hat{w}_n(t))dt - \int_{I(\epsilon_n^2) \setminus [I(\epsilon_n^1) \cup B_n^2]} L(t, \bar{u}(t), \hat{u}(t))dt \\
& \leq 2\epsilon_n^1 [L(t_0, \bar{u}(t_0), (\bar{\xi}_1 + \bar{\xi}_2)/2) + o_n(1)] + (L(t, \bar{u}(t), \hat{u}(t)))dt \\
& \leq 2\epsilon_n^1 [L(t_0, \bar{u}(t_0), (\bar{\xi}_1 + \bar{\xi}_2)/2) + o_n(1)] + \gamma_n \int_{I(\epsilon_n^2) \setminus [I(\epsilon_n^1) \cup B_n^2]} C_{|\xi_1 + \hat{\delta}|} dt.
\end{align*}
\]

Observing that

\[
\gamma_n \leq \int_{I(\epsilon_n^1) \cup B_n^2} |\hat{u}(t)|dt + \epsilon_n^1 [\bar{\xi}_1 + \bar{\xi}_2 + o_n(1)]
\]

and, by the integrability of \( C_{|\xi_1 + \hat{\delta}|} \), that the integral over \( I(\epsilon_n^2) \setminus [I(\epsilon_n^1) \cup B_n^2] \) of \( C_{|\xi_1 + \hat{\delta}|} \) converges to 0, as \( n \) goes to \( \infty \), we infer that

\[
\gamma_n[1 - o_n(1)] \leq 2\epsilon_n^1 [L(t_0, \bar{u}(t_0), (\bar{\xi}_1 + \bar{\xi}_2)/2) + |\bar{\xi}_1 + \bar{\xi}_2|/2 + o_n(1)],
\]

which implies the claim.
By Lemma 2 on $I(c_n^1)$ and on $B_n^2$, we have
\[
\int_{I(c_n^2)} L(t, \bar{u}(t), \dot{\bar{u}}(t)) dt \geq c_n^1[L(t_0, \bar{u}(t_0), \bar{\xi}) + L(t_0, \bar{u}(t_0), \bar{\xi}_2) + o_n(1)] \\
+ \int_{I(c_n^2) \setminus [I(c_n^1) \cup B_n^2]} L(t, \bar{u}(t), \dot{\bar{u}}(t)) dt.
\]
By the strict convexity of $L$ in $\xi$, there exists a positive constant $c > 0$ such that $L(t_0, \bar{u}(t_0), \bar{\xi}) + L(t_0, \bar{u}(t_0), \bar{\xi}_2) \geq 2L(t_0, \bar{u}(t_0), (\bar{\xi} + \bar{\xi}_2)/2) + c$. Hence, by the Lipschitz continuity of $L$ in $u$, and since $\gamma_n = c_n^1O_n(1)$,
\[
\int_{I(c_n^2)} L(t, \bar{u}(t), \dot{\bar{u}}(t)) dt \geq c_n^1[2L(t_0, \bar{u}(t_0), (\bar{\xi} + \bar{\xi}_2)/2) + c + o_n(1)] \\
+ c_n^1o_n(1) + \int_{I(c_n^2) \setminus [I(c_n^1) \cup B_n^2]} L(t, w_n(t), \dot{w}(t)) dt \\
= c_n^1[c + o_n(1)] + \int_{I(c_n^2)} L(t, w_n(t), \dot{w}(t)) dt.
\]
Thus, if $\bar{n}$ is such that $o_n(1) < c$, $I(\bar{u}) > I(w_{\bar{n}})$, in contradiction with the minimality of $\bar{u}$.

**Case 2.** If, for every $\delta > 0$, $|E_n^2(\delta)| > 0$, for every $n$, and there exists $\bar{\delta} > 0$ such that $M(E_n^2(\bar{\delta}))$ converges to $\bar{\xi} \neq \bar{\xi}_2$ in $\mathbb{R}^d$, possibly $\bar{\xi} = \infty$, then, by the regularity of the Lebesgue measure and the continuity of the integral, there exists a sequence of measurable sets $A_n^2 \supset E_n^2(\bar{\delta})$ in $I(c_n^2) \setminus I(c_n^1)$ such that $M(A_n^2)$ converges to $\bar{\xi}$ in $\mathbb{R}^d$ finite and $\bar{\xi} \neq \bar{\xi}_2$. Since $\{\bar{u}(t_0 + c_n^2)\}_n$ converges in measure to $\bar{\xi}_2$, then $\lim |A_n^2|/c_n^2 = 0$.

Letting $B_n^2$ be a measurable subset of $I(c_n^2) \setminus [I(c_n^1) \cup Z_n^2 \cup A_n^2]$ such that $|B_n^2| = |A_n^2| =: \alpha_n^2$, we define the competitor $w_n$ in $W^{1,1}_0(a, b)$ by $w_n(t) := \int_a^t w_n(\tau)d\tau$ where
\[
w_n(\tau) := \begin{cases} 
[M(A_n^2) + M(B_n^2)]/2, & \text{if } \tau \in A_n^2 \cup B_n^2, \\
\bar{u}(\tau), & \text{otherwise}.
\end{cases}
\]
Observe that $w_n$ coincides with $\bar{u}$ on $[a, b] \setminus [I(c_n^2) \setminus I(c_n^1)]$.

Analogously as in Case 1, one can prove that
\[\gamma_n := \|w_n - \bar{u}\|_{L^\infty(I(c_n^2))} = \alpha_n^2O_n(1).\]

Indeed, this claim depends only on (5), the minimality of $\bar{u}$, the Lipschitz continuity of $L$ in $u$ and the boundedness of $|\dot{\bar{u}}|$ over $I(c_n^2) \setminus [I(c_n^1) \cup A_n^2 \cup B_n^2]$.

Again, as in Case 1, one can prove that, by Lemma 2 on $A_n^2$ and on $B_n^2$, the strict convexity of $L$ in $\xi$ and the Lipschitz continuity of $L$ in $u$,
\[
\int_{I(c_n^2) \setminus I(c_n^1)} L(t, \bar{u}(t), \dot{\bar{u}}(t)) dt \geq \alpha_n^2[L(t_0, \bar{u}(t_0), \bar{\xi}) + L(t_0, \bar{u}(t_0), \bar{\xi}_2) + o_n(1)] \\
+ \int_{I(c_n^2) \setminus I(c_n^1) \cup A_n^2 \cup B_n^2} L(t, w(t), \dot{w}(t)) dt \\
\geq \alpha_n^2[c + o_n(1)] \\
+ \int_{I(c_n^2) \setminus I(c_n^1)} L(t, w_n(t), \dot{w}_n(t)) dt.
\]
Thus, if $\bar{n}$ is such that $o_n(1) < c$, $I(\bar{u}) > I(w_{\bar{n}})$, in contradiction with the minimality of $\bar{u}$. 

A DIRECT PROOF OF THE TONELLI’S PARTIAL REGULARITY RESULT 7
Case 3. If, for every \( \delta > 0 \), \( |E_n^2(\delta)| > 0 \), for every \( n \), and \( M(E_n^2(\delta)) \rightarrow \xi_2 \), we choose \( \delta := L(t_0, \bar{u}(t_0), \xi_2) + 1 + |\xi_2|, \ E_n^2 := E_n^2(\delta) \setminus I(\epsilon_n^2) \).

Define the competitor \( \bar{w}_n \) in \( W_{1,2}^1(a, b) \) by \( \bar{w}_n(t) := \int_a^t \bar{w}_n(\tau) d\tau \) where

\[
\bar{w}_n(\tau) := \begin{cases} 
M(E_n^2), & \tau \text{ in } E_n^2, \\
\hat{u}(\tau), & \text{otherwise}.
\end{cases}
\]

Observe that \( \bar{w}_n \) coincides with \( \bar{u} \) on \([a, b] \setminus [I(\epsilon_n^2) \setminus I(\bar{\epsilon}_n^2)]\).

Analogously as in Case 1 and Case 2, since \( |\hat{u}| \) is bounded by \( |\xi_2| + \delta \) over \( I(\epsilon_n^2) \setminus [I(\epsilon_n^1) \cup E_n^2] \), one can prove that

\[
\gamma_n := \|w_n - \bar{u}\|_{L^\infty(I(\epsilon_n^2))} = |E_n^2|O_n(1).
\]

By (5) and the Lipschitz continuity of \( L \) in \( u \),

\[
\int_{I(\epsilon_n^2) \setminus I(\epsilon_n^1)} L(t, \bar{u}(t), \dot{\bar{u}}(t)) dt \geq |E_n^2|[L(t_0, \bar{u}(t_0), \xi_2) + 1 + o_n(1)]
\]

\[
+ \int_{I(\epsilon_n^2) \setminus [I(\epsilon_n^1) \cup E_n^2]} L(t, w(t), \dot{\bar{u}}(t)) dt \geq |E_n^2|[1 + o_n(1)]
\]

\[
+ \int_{I(\epsilon_n^2) \setminus I(\epsilon_n^1)} L(t, w_n(t), \bar{w}_n(t)) dt.
\]

Thus, if \( \bar{u} \) is such that \( o_n(1) < c, \mathcal{I}(\bar{u}) > \mathcal{I}(w_n) \), in contradiction with the minimality of \( \bar{u} \). This concludes the three cases and the proof of the claim at the beginning of the proof, that is, for every \( t_0 \) in \([a, b] \setminus \Sigma_L \),

\[
\exists \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{I(\epsilon)} \dot{\bar{u}}(t) dt =: \bar{\xi}_2 =: \bar{\xi} \text{ in } \mathbb{R}^d \cup \{ \infty \},
\]

In the same way one can show that, if \( I(\epsilon) \) is the closed left interval \([t_0 - \epsilon, t_0]\) then

\[
\exists \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{I(\epsilon)} \dot{\bar{u}}(t) dt =: \bar{\xi}_2' =: \bar{\xi}' \text{ in } \mathbb{R}^d,
\]

and also, slightly modifying the proof above, that the limit of the right and the left averages of \( \dot{\bar{u}} \) at \( t_0 \) i.e. \( \bar{\xi} \) and \( \bar{\xi}' \), must be equals (replacing \( I(\epsilon_n^1) \) with \([t_0 - \epsilon_n^1, t_0]\) and \( I(\epsilon_n^2) \) with \([t_0 - \epsilon_n^2, t_0 + \epsilon_n^2]\))^2.

Part 2. Let \( \{t_n\} \subset [a, b] \setminus \Sigma_L \) be any sequence of Lebesgue points of \( \dot{\bar{u}} \) which converges to \( t_0 \).

We claim that

\[
\exists \lim_{n \to \infty} \dot{\bar{u}}(t_n) =: \bar{\xi},
\]

where \( \bar{\xi} \) is defined in Part 1 as the limit of the averages of \( \dot{\bar{u}} \) at \( t_0 \). Indeed, suppose on the opposite that (by passing to a subsequence)

\[
\lim_{n \to \infty} \dot{\bar{u}}(t_n) =: \bar{\xi}_1 \text{ in } \mathbb{R}^d, \quad \bar{\xi}_1 \neq \bar{\xi}.
\]

Assume also for simplicity that \( t_n < t_0 \), for every \( n \) (the other case \( t_n > t_0 \), for every \( n \), can be proved analogously). Being \( t_n \) a Lebesgue point for \( \dot{\bar{u}} \), there exists

\footnote{The contradiction argument of Part 1 is based on the existence of two sequences of sets of positive measure, which converge to a point \( t_0 \) and which union is an interval, such that the limit of the average of \( \dot{\bar{u}} \) on those sets are different.}
$\epsilon_n^1$ in $(0, \epsilon_n^2)$ such that
\[ \frac{1}{\epsilon_n^1} \int_{t_n}^{t_n+\epsilon_n^1} \dot{u}(t) dt = \dot{u}(t_n) + o_n(1) = \bar{\xi}_1 + o_n(1). \] (9)

Proceeding similarly as in Part 1 (replacing $I(\epsilon_n^1)$ with $[t_n, t_n + \epsilon_n^1]$ and $I(\epsilon_n^2)$ with $[t_n, t_0]$), one can prove the claim.

We have therefore obtained by Part 1 and Part 2 that, for every point $t_0$ in $[a, b] \setminus \Sigma_L$ and for any sequence of Lebesgue points $\{n\}$ of $\dot{u}$ which converges to $t_0$,
\[ \lim_{n \to \infty} \dot{u}(t_n) = \bar{\xi} \text{ in } \mathbb{R}^d, \]
where $\bar{\xi}$ depends only on $t_0$, i.e. $\bar{\xi} = \bar{\xi}(t_0)$. Indeed, by Part 1, denoting by $E$ the set of Lebesgue points of $\dot{u}$, the functions $\dot{u}$ restricted to $E$ coincides with $\bar{\xi}$. Therefore, by Part 2, the function $\bar{\xi}$ can be extended by continuity to $[a, b] \setminus \Sigma_L$. Hence, $\dot{u}$ is a.e. equal on $[a, b] \setminus \Sigma_L$ to an extended-values continuous functions on $[a, b] \setminus \Sigma_L$. \qed

From Theorem 3, the set $[a, b] \setminus (\Sigma_L \cup S_g)$ is a measure union of disjoint open intervals $I_n$ of $[a, b]$ such that $\dot{u}|_{I_n}$ belongs to $C^1(I_n)$, for any $n$, and $|\bigcup_n I_n| = b - a$.

Notice that the proof of Theorem 3 is valid also for $d = 1$ and it shows that $\dot{u}$ belongs to $C([a, b] \setminus \Sigma_L; \mathbb{R} \cup \{\infty\})$. Nevertheless, in Theorem 4, we propose a finer Tonelli’s partial regularity result for the scalar case. Before, we present two examples that show the optimality of our result.

First, strict convexity is a necessary condition for Theorem 3. Indeed, consider the Lagrangian
\[ L(t, u, \xi) := |u - f(t)|(\xi + 1)^2 \]
where $f$ is a function in $W_0^{1,1}(0, 1)$ with $S_f$ a non-closed set of $[0, 1]$. Notice that the function $L$ is continuous on $t$ and $\xi$, smooth with respect to $u$, with $\Sigma_L = \emptyset$, and convex with respect to $\xi$ but not strictly convex as $u = f(t)$. The variational problem
\[ \min \int_0^1 |u(t) - f(t)|[\dot{u}(t) + 1]^2 dt, \]
for $u$ in $W_0^{1,1}(0, 1)$, admits the unique solution $\ddot{u} = f$ which is not regular in the sense of Tonelli.

Second, the continuity of $L$ with respect to $t$ in $[a, b] \setminus \Sigma_L$, with $\Sigma_L$ closed, is an optimal condition and the continuity assumption on $L$ cannot be replaced by a Carathéodory condition. In fact, consider the Lagrangian
\[ L(t, u, \xi) := |\xi - f(t)|^2 \]
where $f$ is a function in $C([0, 1] \setminus \Sigma_L)$, $f_0^1 f = 0$, $\Sigma_L$ is a non-closed set in $[a, b]$, $|\Sigma_L| = 0$. The function $L$ is strictly convex with respect to $\xi$ and continuous on $[a, b] \setminus \Sigma_L$ with respect to $t$. The variational problem
\[ \min \int_0^1 |\dot{u}(t) - f(t)|^2 dt, \]
for $u$ in $W_0^{1,1}(0, 1)$, admits the unique solution $\ddot{u}(t) = f$ which is not regular in the sense of Tonelli.

Third, the integrability of the Lipschitz continuity constant $C_R$ of $L$ in $u$ is an optimal condition as the example in [12] shows.
Theorem 4. Let \( d = 1 \). If \( L(t,u,\cdot) \) is strictly convex, for any \( t \) in \([a,b] \setminus \Sigma_L\) and any \( u \) in \( \mathbb{R} \), then any solution \( \bar{u} \) in \( W^{1,1}_{0}(a,b) \) to the variational problem (3) is regular in the sense of Tonelli.

Proof. The key feature of the scalar case is that any continuous path in \( \mathbb{R} \) that goes from \( -\infty \) to \( +\infty \) is forced to pass by a finite point of \( \mathbb{R} \), which is not the case in the multidimensional setting. The proof is very similar to the one of Theorem 3. We present here the main differences.

Part 1. Let \( t_0 \) be a point in \([a,b] \setminus \Sigma_L\) and let \( I(\epsilon) \) be the closed right interval \([t_0, t_0 + \epsilon]\), \( \epsilon > 0 \).

We claim that

\[
\exists \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{t_0}^{t_0 + \epsilon} \dot{u}(t) \, dt =: \bar{\xi} \text{ in } \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}.
\]

Indeed, suppose, on the contrary, that there exist two sequences of positive numbers \( \{\epsilon_n^1\}, \{\epsilon_n^2\} \), \( \epsilon_n^2 < \epsilon_n^1 < \epsilon_n^{1+1} \), which converge to 0, such that

\[
\lim_{n \to \infty} \frac{1}{\epsilon_n^1} \int_{t_0}^{t_0 + \epsilon_n^1} \dot{u}(t) \, dt =: \bar{\xi}_1 \text{ in } \mathbb{R},
\]

\[
\lim_{n \to \infty} \frac{1}{\epsilon_n^2} \int_{t_0}^{t_0 + \epsilon_n^2} \dot{u}(t) \, dt =: \bar{\xi}_2 \text{ in } \mathbb{R},
\]

and

\[
\bar{\xi}_1 \neq \bar{\xi}_2.
\]

Suppose, without loss of generality, that \( \bar{\xi}_1 = +\infty \) and \( \bar{\xi}_2 = -\infty \). Then, by the continuity of the integral function \( M(I(\epsilon)) \), for \( \epsilon \) in \([\epsilon_n^1, \epsilon_n^2]\) (see the proof of Theorem 3), we can choose \( \epsilon_n^2 \) such that

\[
\lim_{n \to \infty} \frac{1}{\epsilon_n^2} \int_{I(\epsilon_n^2)} \dot{u}(t) \, dt = 1.
\]

The proof therefore follows as in Part 1 of Theorem 3.

Part 2. Let \( \{t_n\} \subset [a,b] \setminus \Sigma_L \) be a sequence of Lebesgue points of \( \dot{u} \) which converges to \( t_0 \).

We claim that

\[
\exists \lim_{n \to \infty} \dot{u}(t_n) = \bar{\xi},
\]

where \( \bar{\xi} \) is defined in Part 1 as the limit of the averages of \( \dot{u} \) at \( t_0 \). Indeed, suppose on the opposite that (by passing to a subsequence)

\[
\lim_{n \to \infty} \dot{u}(t_n) =: \bar{\xi}_1 \text{ in } \mathbb{R}, \quad \bar{\xi}_1 \neq \bar{\xi}.
\]

Assume also for simplicity that \( t_n < t_0 \), for every \( n \) (the other case \( t_n > t_0 \), for every \( n \), can be proved analogously). Being \( t_n \) a Lebesgue point for \( \dot{u} \), there exists \( \epsilon_n^1 \) in \((0,\epsilon_n^2)\) such that

\[
\frac{1}{\epsilon_n^1} \int_{t_n}^{t_n + \epsilon_n^1} \dot{u}(t) \, dt = \dot{u}(t_n) + o_n(1) = \bar{\xi}_1 + o_n(1).
\]

Suppose, without loss of generality, that \( \bar{\xi} = +\infty \) and \( \bar{\xi}_1 = -\infty \). Then, by the continuity of \( M(I(\epsilon)) \), for \( \epsilon \) in \([\epsilon_n^1, \epsilon_n^2]\), we can choose \( \epsilon_n^1 \) such that

\[
\lim_{n \to \infty} \frac{1}{\epsilon_n^1} \int_{I(\epsilon_n^1)} \dot{u}(t) \, dt = 1.
\]
Proceeding similarly as in Part 1 of Theorem 3, one can prove the claim.

We then obtain by Part 1 and Part 2 that \( \dot{\bar{u}} \) is a.e. equal to a continuous function from \([a, b] \setminus \Sigma_L\) to \( \mathbb{R} \).

As a Corollary of Theorem 3 and 4, we obtain that any Lipschitz solution to the variational problem (3) is actually \( C^1 \).

**Corollary 5.** Let \( d \geq 1 \). If \( L(t, u, \cdot) \) is strictly convex, for any \( t \) in \([a, b] \setminus \Sigma_L\) and any \( u \) in \( \mathbb{R}^d \), then any solution \( \bar{u} \) in \( W^{1, \infty}_{0}(a, b) \) to the variational problem (3) is \( C^1([a, b] \setminus \Sigma_L) \).

**Proof.** It follows directly from Theorem 3 and 4.

**Acknowledgement.** The author wishes to thank the Spanish Ministry of Science and Innovation (MICINN) for supporting the research contained in this work.

**REFERENCES**


E-mail address: alessandro.ferriero@gmail.com