Unexpected solutions of the compressible (and incompressible) Euler equations

Camillo De Lellis

Universität Zürich - Institut für Mathematik.
The incompressible Euler equations

They describe the motion of an incompressible fluid under the assumption that friction is absent.

The unknowns are the pressure (a scalar field) and the velocity:

- The velocity $v(x, t)$ is a vector function of the time variable $t$ and of the space $x$: it is the velocity of the fluid particle which occupies the position $x$ at time $t$;
- The pressure $p(x, t)$ is a scalar: its gradient is a force; it can be thought as a “reaction” of the fluid due to its incompressibility.
\[ \partial_t \mathbf{v} + \text{div} (\mathbf{v} \otimes \mathbf{v}) + \nabla p = 0 \]
\[ \text{div} \mathbf{v} = 0 \]

The \( i \)-th component of the advective term \( \text{div} (\mathbf{v} \otimes \mathbf{v}) \) is given by

\[ \sum_j \partial_{x_j} (v_j v_i) . \]
In this talk we will consider solutions which are defined on the entire 3-dimensional (resp. 2-dimensional) space and over some time interval $I$.

$I$ might be

- a bounded interval or a half line; in this case the left endpoint will be 0 and the equations will be complemented with an initial condition (**Cauchy problem**):

$$v(\cdot, 0) = v_0$$

- the entire real line (**ancient solutions**)
The Euler equations were derived more than 250 years ago (really by Euler!)

Nonetheless several fundamental and outstanding open questions are still open: the most famous one is the blow-up problem for 3-dimensional solutions of the Cauchy problem.

This talk WILL NOT touch that issue.
The conservation of energy

If \((v, p)\) is a \(C^1\) solution, we can scalar multiply the first equation by \(v\):

\[
\frac{\partial}{\partial t} \frac{|v|^2}{2} + \text{div} \left( \left( \frac{|v|^2}{2} + p \right) v \right) = 0
\]

Integrate in space (and by parts!) to derive the dissipation law for the kinetic energy:

\[
\frac{d}{dt} \int |v|^2(x, t) \, dx = 0 \tag{1}
\]
Three possible definitions of generalized solutions:

1: use the theory of distributions to define derivatives. Assume square summability of \( \mathbf{v} (\mathbf{v} \in L^2) \) to safely define \( \mathbf{v} \otimes \mathbf{v} \).

2: use Fourier series (periodic setting) in space and reduce the PDE to an (infinite-dimensional) system of ODEs for the Fourier coefficients. The minimal assumption to give a meaning: \( \mathbf{v}(\cdot, t) \in L^2 \) (with some uniformity in \( t \)).

3: take the "point of view of continuum physics" and use conservation laws on any "fluid element \( \Omega \)":
Balance of mass: the flux of fluid leaving/entering \( \Omega \) through \( \partial \Omega \) is 0.

Conclusion: \[ \int_{\partial \Omega} \mathbf{v} \cdot \mathbf{n} = 0 \]

Balancing the momentum: \[ \frac{d}{dt} \int_{\Omega} \mathbf{v} = \int_{\partial \Omega} \mathbf{v} (\mathbf{v} \cdot \mathbf{n}) + \int_{\partial \Omega} p \mathbf{n} \]

These integral identities make sense if, for instance \((\mathbf{v}, p) \in C\).
All these notions are equivalent and from now on: weak = generalized.

**Theorem (Scheffer 1993)**

There are compactly supported nonzero weak solutions in $L^2(\mathbb{R}^2 \times \mathbb{R})$.

A different proof in the periodic setting given by Shnirelman in 1998.

Obviously these solutions do not preserve the total kinetic energy.

**Theorem (Shnirelman 2000)**

There are weak solutions in 3-space dimension with total kinetic energy which is strictly decreasing.
Bounded "bad" weak solutions

Theorem (D-Székelyhidi 2008)

There are compactly supported nontrivial bounded weak solutions in any space dimension.

Several developments (see later and next talks).

Similar conclusions hold for other equations of fluid dynamics, where analogous methods can be used: Cordoba-Faraco-Gancedo, Shvidkoy, Wiedemann, Chiodaroli, Bardos-Titi-Wiedemann, Chiodaroli-Kreml, ...
Differential inclusions

Our 2008 paper plunged Scheffer’s nonuniqueness Theorem in a long tradition of counterintuitive examples in differential inclusions and in differential geometry.

In the theory of differential inclusions you are looking at problems of the following type.

Problem

Given a set $K$ of $k \times n$ matrices study maps $u : \mathbb{R}^n (\text{or } \Omega \subset \mathbb{R}^n) \to \mathbb{R}^k$ such that

$$\nabla u(x) \in K \quad \text{for all } x \in \Omega.$$  \hspace{1cm} (2)

It happens in several situations that $C^1$ solutions are not so interesting because they are forced to be affine. In these cases we can look at Lipschitz solutions (which are differentiable a.e.) and we turn (2) into

$$\nabla v(x) \in K \quad \text{for almost all } x \in \Omega.$$  \hspace{1cm} (3)
Let us look at two “cousins” of the D-S Theorem.

**Exercise**

Consider two $2 \times 2$ matrices $A$ and $B$: is there a Lipschitz planar map $u : \mathbb{R}^2 \to \mathbb{R}^2$ with $\nabla u = A$ “on the left” and $\nabla u = B$ “on the right”? 

\[
\begin{align*}
\nabla u &= A \\
\nabla u &= B
\end{align*}
\]
Solution: It exists if and only if the direction of \( \ell \) is in the kernel of \( A - B \). However...

**Theorem (Kirchheim 2003)**

There are \( 2 \times 2 \) matrices \( A_1, A_2, \ldots, A_5 \) and a Lipschitz map \( u : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) such that

- \( \text{rank}(A_i - A_j) = 2; \)
- \( \nabla u \in \{A_1, \ldots, A_5\} \) almost everywhere;
- \( u \) is not affine.

NB: Not possible with 2 (Ball-James), 3 (Šverak) or 4 (Chlebik-Kirchheim).
Another cousin: [Müller - Šverak 1999] there are Lipschitz solutions of nonlinear strongly elliptic systems of PDEs which are nowhere $C^1$.

Connection between differential inclusions and elliptic systems?? I.e.: the Cauchy-Riemann equations are a differential inclusion:

$$u : \mathbb{R}^2 \to \mathbb{R}^2$$ is holomorphic if $Du \in \bigcup_{\lambda} \lambda SO(2)$!

The techniques used by Kirchheim and Müller-Šverak have a long tradition: Cellina, Bressan, Bressan-Flores, Dacorogna-Marcellini, Sychev, Székelyhidi (Tartar, DiPerna).

[D-S 2008] This framework can be adapted to the Scheffer-Shnirelman nonuniqueness theorem.
A toy example

Ω ⊂ \(\mathbb{R}^2\) smooth bounded open set. We look for planar (Lipschitz!) real-valued maps \(\alpha : \Omega \rightarrow \mathbb{R}\) such that

\[|\nabla \alpha| = 1\]  \hspace{1cm} (4)

(+ maybe some boundary conditions...).

**PLAN:** Start from some smooth map \(\varphi_0\) with |\nabla \varphi_0| < 1.

Set up an iteration scheme producing \(\varphi_0 \rightarrow \varphi_1 \rightarrow \varphi_2 \rightarrow \ldots\)

such that

\[|\nabla \varphi_k| < 1;\]

\[\int_\Omega (1 - |\nabla \varphi_k|) \leq (1 - \beta) \int_\Omega (1 - |\nabla \varphi_{k-1}|)\]

Prove convergence for \(\varphi_k\) (strong convergence not at all obvious: see later!).
The iteration: from $\varphi_k = \varphi$ to $\varphi_{k+1} = \psi$

A region $R$ where $\nabla \varphi$ is almost constant
We make the slope of $\varphi$ “steeper in average” in $R$ by adding a periodic function which oscillates rapidly (in the direction of $\nabla \varphi$): we see below a cross section of $\varphi$ and of the perturbed function $x \mapsto \varphi(x) + \frac{1}{\lambda} \rho(\lambda x)$.

Next, cut off the perturbation to make it compactly supported in the region $R$: 

$$\psi(x) = \varphi(x) + \frac{1}{\lambda} \rho(\lambda x) c(x)$$

(the cut-off $c$ is compactly supported in $R$ but mostly 1 in there).
We are now ready for the key computation:

\[ \nabla \psi(x) = \nabla \varphi(x) + c(x) \nabla p(\lambda x) + \frac{1}{\lambda} p(\lambda x) \nabla c(x) \]

The Improvement “pushes” the slope towards 1 (at least in most places!).

The error can be made as small as we wish if \( \lambda \) is very large: this is not destroying what we gained with the Improvement.

Take care, do not get immediately to slope 1 (or above!) with the Improvement: for \( \lambda \) large we will keep the inequality \( |\nabla \psi| < 1 \).
Repeat now this in many many small balls which cover a substantial portion of the region where $|\nabla \varphi|$ is “far” from 1.
The upshot is: in all these “crazy” constructions the final (more or less counterintuitive) map is achieved through the addition of very fine oscillations to some underlying “subsolution”: the oscillations “pile up” and we reach the desired map only after infinitely many steps.

The Müller-Šverak paper is a landmark result also because the authors realized that similar ideas had already been used in geometry.

In particular Müller and Šverak introduced a suitable variant of Gromov’s convex integration, a tool to prove $h$-principle results.
An older tradition of counterintuitive construction indeed exists in differential geometry (Nash-Kuiper, Smale’s paradox, Gromov, Eliashberg, ...).

Rather than trying to introduce the $h$-principle let me give an example, (maybe the “mother” of all these constructions?). Consider a (smooth) Riemannian manifold $(M, g)$: an isometry $u : M \to \mathbb{R}^N$ is a map preserving the length of curves.

In what follows we deal with $C^1$ maps which are also embeddings: isometric embeddings.

**Corollary**

*Consider the standard sphere $(S^2, \sigma)$ or the flat square $([0, 1]^2, f)$. For any given $\varepsilon > 0$ there are $C^1$ isometric embeddings of these manifolds in a euclidean three-dimensional ball of radius $\varepsilon$, $B_\varepsilon(0) \subset \mathbb{R}^3$!*
The theorem of Nash-Kuiper is quite striking, since in fact with $C^2$ replacing $C^1$ is false.

A practical demonstration follows
Look at the speaker
Indeed the Theorem of Nash-Kuiper is much more general and much more precise: every short embedding (i.e. which shrinks the length of curves) of a compact Riemannian manifold can be uniformly approximated by $C^1$ isometric embeddings.

In the framework introduced by Gromov this can be translated into a “$C^0$-dense $h$-principle” (combining Nash-Kuiper with the Hirsch-Smale $h$-principle).

[D-S 2008] “Ultimately” there exists a similar dense $h$-principle statement for weak solutions of the Euler equations.

Something like that holds for all the results mentioned in the theory of differential inclusions... With a big caveat:

- in differential geometry people work in a $C^0$-type space;
- in analysis people work in a $L^\infty$-type space.
Onsager’s conjecture

Conjecture (Onsager 1949)

(A) Assume $v$ is a (periodic) weak solution of the Euler equations satisfying an Hölder condition with exponent $\alpha > \frac{1}{3}$:

$$|v(x, t) - v(y, t)| \leq C|x - y|^{\alpha}$$

Then the total kinetic energy of $v$ is conserved:

$$E(t) = \int |v|^2(x, t) \, dx \equiv \text{const.}$$

(B) Let $\alpha < \frac{1}{3}$. Then there are weak solutions satisfying the Hölder condition with exponent $\alpha$ such that

the total kinetic energy is not constant
Who cares about weak solutions?

The solutions considered by Onsager are not differentiable... A physicist (with a Nobel prize in chemistry) considers indeed weak solutions in our modern sense in 1949.

And he gave a very rigorous definition, following “road 2” from some slides ago, i.e. via a Fourier series expansion.

Part (A) of the Conjecture has been proved by Eyink and Constantin-E-Titi in 1993. (see also Cheskidov-Constantin-Friedlander-Shvidkoy 2008)

Concerning Part (B), Scheffer’s example is the first rigorous instance (although the velocity is not even bounded!).
Continuous and Hölder dissipative solutions

Theorem (D-Székelyhidi 2012)

Let \( I \) be a compact interval and \( e : I \to \mathbb{R} \) any given smooth positive function. Then there is a continuous solution \((v, p)\) of the Euler equations in \( \mathbb{T}^3 \times I \) such that

\[
\int |v|^2(x, t) \, dx = e(t) \quad \forall t \in I.
\]

Hölder solutions:

- \( \alpha < \frac{1}{10} \) [D-S 2013];
- \( \alpha < \frac{1}{5} \) [Isett 2013, Buckmaster-D-S 2014];
- \( L^1(I, C^{1/3-}) \) [Buckmaster-D-S 2015].
Q: Is there a way to rule out “bad solutions”? 
For instance: Scheffer’s solution violates energy conservation. In fact his solution has infinite energy at some times!

Possible requirements:

(a) $\int |v|^2(x, t) \, dx$ must be finite;

(b) $\int |v|^2(x, t) \, dx$ is noncreasing (resp. constant);

(c) $\partial_t \frac{|u|^2}{2} + \text{div} \left((\frac{|u|^2}{2} + p)v\right) \leq 0$ (resp. $= 0$);

(d) The solution must dissipate as much as possible.
Theorem

*If there is a smooth solution any weak dissipative solution must coincide with it.*

Conclusion: bad solutions must start from irregular initial data.

Theorem (D-S 2010)

*There are bounded $L^2$ initial data which have infinitely many dissipative (and conservative!) bounded weak solutions.*

The initial data of the theorem are *VERY* irregular.
Consider, in 2 space dimensions, the “shear flow”

\[ v_0(x_1, x_2) := \begin{cases} 
(1, 0) & \text{if } x_2 > 0 \\
(-1, 0) & \text{if } x_2 < 0
\end{cases} \]

Note: \( v(x, t) = v_0(x) \) is a stationary solution of Euler

**Theorem (Székelyhidi 2012)**

*There are infinitely many dissipative weak solutions with \( v(\cdot, 0) = v_0 \). Some of them dissipate the kinetic energy!*
Q: Are these two worlds completely apart?

In incompressible Euler maybe.
In compressible Euler no.
The equations of isentropic gas dynamics

\[
\begin{align*}
\partial_t \rho + \text{div}_x (\rho \mathbf{v}) &= 0 \\
\partial_t (\rho \mathbf{v}) + \text{div}_x (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_x [p(\rho)] &= 0
\end{align*}
\]

(5)

Full Euler: a third equation for the conservation of energy.

The pressure \( p \) is a function of \( \rho \) determined from thermodynamics (NB: \( p' > 0 \)).

A common choice is the polytropic pressure law \( p(\rho) = \kappa \rho^\gamma \) with constants \( \kappa > 0 \) and \( \gamma > 1 \).
Riemann problem

Riemann data:

\[
(\rho^0(x), \nu^0(x)) := \begin{cases} 
(\rho_-, \nu_-) & \text{if } x_2 < 0 \\
(\rho_+, \nu_+) & \text{if } x_2 > 0,
\end{cases}
\]  

Riemann problem: determine self-similar solutions with Riemann data.

Theorem (Riemann!)

There is a unique self-similar solution if we impose the energy inequality:

\[
\partial_t \left( \rho \varepsilon(\rho) + \rho \frac{|\nu|^2}{2} \right) + \text{div}_x \left[ \left( \rho \varepsilon(\rho) + \rho \frac{|\nu|^2}{2} + p(\rho) \right) \nu \right] \leq 0.
\]

NB: \( p(r) = r^2 \varepsilon'(r) \).
Non self-similar solutions

Theorem (Chiodaroli, PhD thesis 2013)

There are non self-similar weak solutions which satisfy the entropy inequality (unexpected!) for some pressure laws $p$.

Well known, compression waves: some Riemann data arise as finite-time blow-up of classical (locally Lipschitz) solutions.

Q: Can we produce a bad initial data from a compression wave?

Theorem (Chiodaroli-D-Kreml 2015)

Yes! And for a “classical” pressure law: $p(\rho) = \rho^2$. 
Corollary (Chiodaroli-D-Kreml 2015)

\[ p(\rho) = \rho^2. \] There are Lipschitz initial data for which the (unique!) Lipschitz solution develops a singularity in finite time, after which there are infinitely many ways to continue the solution fulfilling the energy condition.

Theorem (Chiodaroli-Kreml 2015)

Some of these solutions dissipate the energy faster than the “classical” solution.
Plan for the next two lectures

- The incompressible Euler equations as a differential inclusion.
- Tartar’s wave cone analysis, subsolutions.
- Baire category argument, existence of Scheffer-Shnirelman solutions.
- Pressureless solutions and entropy conditions.
- Matching initial data.
- Székelyhidi-Shnirelmann trick.
- Some computations for compressible Euler, if time allows.
Thank you
for your attention!