On pointwise estimates involving sparse operators

Andrei Lerner

Bar-Ilan University

10th International Conference on Harmonic Analysis and PDEs
El Escorial (Spain)
June 12-17, 2016
Sparse families and operators

- Given $0 < \eta < 1$, we say that a family $S$ of cubes from $\mathbb{R}^n$ is $\eta$-sparse if for any $Q \in S$ there is a subset $E_Q \subset Q$ such that
  1. $|E_Q| \geq \eta |Q|$;
  2. the sets $\{E_Q\}_{Q \in S}$ are pairwise disjoint.
Sparse families and operators

- Given $0 < \eta < 1$, we say that a family $S$ of cubes from $\mathbb{R}^n$ is $\eta$-sparse if for any $Q \in S$ there is a subset $E_Q \subset Q$ such that
  1. $|E_Q| \geq \eta|Q|$;
  2. the sets $\{E_Q\}_{Q \in S}$ are pairwise disjoint.

- Denote $f_Q = \frac{1}{|Q|} \int_Q f$, and define the dyadic maximal operator:

$$M^D f(x) = \sup_{Q \ni x, Q \in \mathcal{D}} |f|_Q,$$

where $\mathcal{D} = \{2^{-k}([0,1]^n + j), k \in \mathbb{Z}, j \in \mathbb{Z}^n\}$. 
Sparse families and operators

- Given $0 < \eta < 1$, we say that a family $S$ of cubes from $\mathbb{R}^n$ is $\eta$-sparse if for any $Q \in S$ there is a subset $E_Q \subset Q$ such that
  1. $|E_Q| \geq \eta|Q|$;
  2. the sets $\{E_Q\}_{Q \in S}$ are pairwise disjoint.

- Denote $f_Q = \frac{1}{|Q|} \int_Q f$, and define the dyadic maximal operator:

\[
M^D f(x) = \sup_{Q \ni x, Q \in \mathcal{D}} |f|_Q,
\]

where $\mathcal{D} = \{2^{-k}([0, 1]^n + j), k \in \mathbb{Z}, j \in \mathbb{Z}^n\}$.

- The standard claim (80's): for every $f \in L^1(\mathbb{R}^n)$, there is a $\frac{1}{2}$-sparse family $S \subset \mathcal{D}$ such that

\[
M^D f(x) \leq 2^{n+1} \sum_{Q \in S} |f|_Q \chi_{E_Q}(x).
\]
Sparse families and operators

- Given $0 < \eta < 1$, we say that a family $S$ of cubes from $\mathbb{R}^n$ is $\eta$-sparse if for any $Q \in S$ there is a subset $E_Q \subset Q$ such that
  1. $|E_Q| \geq \eta|Q|$;
  2. the sets $\{E_Q\}_{Q \in S}$ are pairwise disjoint.

- Denote $f_Q = \frac{1}{|Q|} \int_Q f$, and define the dyadic maximal operator:

  $$M^D f(x) = \sup_{Q \ni x, Q \in \mathcal{D}} |f|_Q,$$

  where $\mathcal{D} = \{2^{-k}([0,1]^n + j), k \in \mathbb{Z}, j \in \mathbb{Z}^n\}$.

- The standard claim (80’s): for every $f \in L^1(\mathbb{R}^n)$, there is a $\frac{1}{2}$-sparse family $S \subset \mathcal{D}$ such that

  $$M^D f(x) \leq 2^{n+1} \sum_{Q \in S} |f|_Q \chi_{E_Q}(x).$$

- Proof: write $\Omega_k = \{x : M^D f(x) > 2^{(n+1)k}\} = \bigcup_j Q^k_j$ and set $E^k_j = Q^k_j \setminus \Omega_{k+1}$. Then the claim holds with $S = \{Q^k_j\}$.
Sparse families and operators

- **The standard claim** (80’s): for every $f \in L^1(\mathbb{R}^n)$, there is a $\frac{1}{2}$-sparse family $S \subset \mathcal{D}$ such that

  $$M^\mathcal{D} f(x) \leq 2^{n+1} \sum_{Q \in S} |f|_Q \chi_{E_Q}(x).$$

- **A one-third trick**: there are $3^n$ dyadic lattices $\mathcal{D}(j)$ such that for every cube $Q \subset \mathbb{R}^n$, there is a cube $P \in \mathcal{D}(j)$ for some $j$, containing $Q$ and such that $|P| \leq 6^n |Q|$. 
Sparse families and operators

- **The standard claim** (80’s): for every $f \in L^1(\mathbb{R}^n)$, there is a $\frac{1}{2}$-sparse family $\mathcal{S} \subset \mathcal{D}$ such that

  $$M^\mathcal{D} f(x) \leq 2^{n+1} \sum_{Q \in \mathcal{S}} |f|_Q \chi_{E_Q}(x).$$

- **A one-third trick**: there are $3^n$ dyadic lattices $\mathcal{D}^{(j)}$ such that for every cube $Q \subset \mathbb{R}^n$, there is a cube $P \in \mathcal{D}^{(j)}$ for some $j$, containing $Q$ and such that $|P| \leq 6^n |Q|$.

- **Hence**, the usual maximal operator $M$ is bounded as follows:

  $$M f(x) \leq 6^n \sum_{j=1}^{3^n} M^{\mathcal{D}^{(j)}} f(x).$$
Sparse families and operators

- **The standard claim (80’s):** for every $f \in L^1(\mathbb{R}^n)$, there is a $\frac{1}{2}$-sparse family $S \subset D$ such that
  \[ M^D f(x) \leq 2^{n+1} \sum_{Q \in S} |f|_{Q} \chi_{E_{Q}}(x). \]

- **A one-third trick:** there are $3^n$ dyadic lattices $D^{(j)}$ such that for every cube $Q \subset \mathbb{R}^n$, there is a cube $P \in D^{(j)}$ for some $j$, containing $Q$ and such that $|P| \leq 6^n |Q|$. Hence, the usual maximal operator $M$ is bounded as follows:
  \[ M f(x) \leq 6^n \sum_{j=1}^{3^n} M^{D^{(j)}} f(x). \]

- By the standard claim, for every $f \in L^1(\mathbb{R}^n)$, there are $\frac{1}{2}$-sparse families $S_j \subset D^{(j)}$, $j = 1, \ldots, 3^n$, such that
  \[ M f(x) \leq 2 \cdot 12^n \sum_{j=1}^{3^n} \sum_{Q \in S_j} |f|_{Q} \chi_{E_{Q}}(x). \]
By the standard claim, for every $f \in L^1(\mathbb{R}^n)$, there are $\frac{1}{2}$-sparse families $S_j \subset D^{(j)}$, $j = 1, \ldots, 3^n$, such that

$$Mf(x) \leq 2 \cdot 12^n \sum_{j=1}^{3^n} \sum_{Q \in S_j} |f|_{Q} \chi_{E_Q}(x).$$

We will show that almost the same pointwise domination holds for Calderón-Zygmund operators $T$:

$$|Tf(x)| \leq C(n, T) \sum_{j=1}^{3^n} \sum_{Q \in S_j} |f|_{Q} \chi_{Q}(x).$$
Sparse families and operators

- We will show that almost the same pointwise domination holds for Calderón-Zygmund operators $T$:

$$|Tf(x)| \leq C(n, T) \sum_{j=1}^{3^n} \sum_{Q \in S_j} |f|_Q \chi_Q(x).$$

- We say that $T$ is an $\omega$-Calderón-Zygmund operator if
  1. $T$ is $L^2$ bounded;
  2. $T$ is represented as

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy \quad \text{for all } x \notin \text{supp } f;$$

  3. $K$ satisfies the size condition $|K(x, y)| \leq \frac{C_K}{|x-y|^n}$, $x \neq y$;
  4. $K$ satisfies the regularity condition

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \omega \left( \frac{|x-x'|}{|x-y|} \right) \frac{1}{|x-y|^n}$$

  for $|x - y| > 2|x - x'|$, where $\omega : [0, 1] \to [0, \infty)$ is continuous, increasing, subadditive and $\omega(0) = 0$. 

If $S$ is a sparse family, then the operator $A_S f(x) = \sum_{Q \in S} f_Q \chi_Q(x)$ is called the sparse operator.
We will show that almost the same pointwise domination holds for Calderón-Zygmund operators $T$:

$$|Tf(x)| \leq C(n, T) \sum_{j=1}^{3^n} \sum_{Q \in S_j} |f_Q \chi_Q(x)|.$$

If $S$ is a sparse family, then the operator

$$A_S f(x) = \sum_{Q \in S} f_Q \chi_Q(x)$$

is called the \textit{sparse operator}.
A very brief history

- The standard methods of 70’s-80’s (good-\(\lambda\) inequalities, rearrangement inequalities, sharp-function estimates) provide an indirect relation between \(T\) and the maximal operator \(M\).
A very brief history

- The standard methods of 70’s-80’s (good-λ inequalities, rearrangement inequalities, sharp-function estimates) provide an indirect relation between $T$ and the maximal operator $M$.

- **The $A_2$ conjecture** (90’s):
  \[
  \|T\|_{L^2(w)} \leq c(n, T)[w]_{A_2},
  \]

  where $[w]_{A_2} = \sup_Q \left( \frac{1}{|Q|} \int_Q w \right) \left( \frac{1}{|Q|} \int_Q w^{-1} \right)$. 


D. Cruz-Uribe, J. Martell, C. Pérez (2010): the linear $A_2$ bound for $A_S f = \sum_{Q \in S} f_Q \chi_Q$.

A.L. (2012): Let $T$ be an $\omega$-Calderón-Zygmund operator, and assume that $\omega$ satisfies the log-Dini condition

$$\int_0^1 \frac{\omega(t)}{t} \, dt < \infty.$$
A very brief history

- The standard methods of 70’s-80’s (good-$\lambda$ inequalities, rearrangement inequalities, sharp-function estimates) provide an indirect relation between $T$ and the maximal operator $M$.

- The $A_2$ conjecture (90’s):

\[
\|T\|_{L^2(w)} \leq c(n, T)[w]_{A_2},
\]

where $[w]_{A_2} = \sup_Q \left( \frac{1}{|Q|} \int_Q w \right) \left( \frac{1}{|Q|} \int_Q w^{-1} \right)$.

A very brief history

- The standard methods of 70’s-80’s (good-\(\lambda\) inequalities, rearrangement inequalities, sharp-function estimates) provide an indirect relation between \(T\) and the maximal operator \(M\).

- **The \(A_2\) conjecture** (90’s):

  \[
  \|T\|_{L^2(w)} \leq c(n, T)[w]_{A_2},
  \]

  where \([w]_{A_2} = \sup_Q \left( \frac{1}{|Q|} \int_Q w \right) \left( \frac{1}{|Q|} \int_Q w^{-1} \right)\).


- D. Cruz-Uribe, J. Martell, C. Pérez (2010): the linear \(A_2\) bound for

  \[
  A_S f = \sum_{Q \in S} f_Q \chi_Q.
  \]
A very brief history

- **The $A_2$ conjecture (90’s):**

  \[ \|T\|_{L^2(w)} \leq c(n,T)[w]_{A_2}, \]

  where \( [w]_{A_2} = \sup_Q \left( \frac{1}{|Q|} \int_Q w \right) \left( \frac{1}{|Q|} \int_Q w^{-1} \right) \).


- D. Cruz-Uribe, J. Martell, C. Pérez (2010): the linear $A_2$ bound for

  \[ A_S f = \sum_{Q \in S} f_Q \chi_Q. \]

- A.L. (2012): Let \( T \) be an $\omega$-Calderón-Zygmund operator, and assume that $\omega$ satisfies the \textit{log-Dini} condition \( \int_0^1 \omega(t) \log \frac{1}{t} \frac{dt}{t} < \infty \). Then for every Banach function space $X$ over $\mathbb{R}^n$,

  \[ \|Tf\|_X \leq c(n,T) \sup_{\mathcal{D},\mathcal{S}} \|A_S f\|_X. \]
A very brief history

- A.L. (2012): Let $T$ be an $\omega$-Calderón-Zygmund operator, and assume that $\omega$ satisfies the log-Dini condition $\int_0^1 \omega(t) \log \frac{1}{t} \frac{dt}{t} < \infty$. Then for every Banach function space $X$ over $\mathbb{R}^n$, 
  \[ \|Tf\|_X \leq c(n, T) \sup_{\mathcal{D}, \mathcal{S}} \|A_{\mathcal{S}}|f|\|_X. \]

- J. Conde-Alonso and G. Rey, A.L. and F. Nazarov (2014): if $\int_0^1 \omega(t) \log \frac{1}{t} \frac{dt}{t} < \infty$, then for every $f \in L^1$, there are $\eta_n$-sparse families $\mathcal{S}_j \subset \mathcal{D}(j), j = 1, \ldots, 3^n$, such that for a.e. $x$, 
  \[ |Tf(x)| \leq c(n, T) \sum_{j=1}^{3^n} A_{\mathcal{S}_j} |f|(x). \]
A very brief history

- A.L. (2012): Let $T$ be an $\omega$-Calderón-Zygmund operator, and assume that $\omega$ satisfies the log-Dini condition $\int_0^1 \omega(t) \log \frac{1}{t} \frac{dt}{t} < \infty$. Then for every Banach function space $X$ over $\mathbb{R}^n$,

$$\|Tf\|_X \leq c(n, T) \sup_{\mathcal{D}, \mathcal{S}} \|A_S |f||_X.$$  

- J. Conde-Alonso and G. Rey, A.L. and F. Nazarov (2014): if $\int_0^1 \omega(t) \log \frac{1}{t} \frac{dt}{t} < \infty$, then for every $f \in L^1$, there are $\eta_n$-sparse families $S_j \subset \mathcal{D}(j), j = 1, \ldots, 3^n$, such that for a.e. $x$,

$$|Tf(x)| \leq c(n, T) \sum_{j=1}^{3^n} A_{S_j} |f|(x).$$

- M. Lacey (2015): the same estimate (for compactly supported $f$) holds under the usual Dini condition $[\omega]_{\text{Dini}} = \int_0^1 \omega(t) \frac{dt}{t} < \infty$. 


J. Conde-Alonso and G. Rey, A.L. and F. Nazarov (2014): if \( \int_0^1 \omega(t) \log \frac{1}{t} \frac{dt}{t} < \infty \), then for every \( f \in L^1 \), there are \( \eta_n \)-sparse families \( S_j \subset D(j), j = 1, \ldots, 3^n \), such that for a.e. \( x \),

\[
|Tf(x)| \leq c(n, T) \sum_{j=1}^{3^n} A_{S_j} |f|(x).
\]

M. Lacey (2015): the same estimate (for compactly supported \( f \)) holds under the usual Dini condition \([\omega]_{\text{Dini}} = \int_0^1 \omega(t) \frac{dt}{t} < \infty\).

A quantitative form: (T. Hytönen, L. Roncal and O. Tapiola (2015)) denote \( C_T = \|T\|_{L^2 \to L^2} + C_K + [\omega]_{\text{Dini}} \). Then

\[
|Tf(x)| \leq c_n C_T \sum_{j=1}^{3^n} A_{S_j} |f|(x).
\]
A very brief history

  if \( \int_0^1 \omega(t) \log \frac{1}{t} \frac{dt}{t} < \infty \), then for every \( f \in L^1 \), there are \( \eta_n \)-sparse families \( S_j \subset \mathcal{D}(j), j = 1, \ldots, 3^n \), such that for a.e. \( x \),
  \[
  |Tf(x)| \leq c(n, T) \sum_{j=1}^{3^n} A S_j |f|(x).
  \]

- M. Lacey (2015): the same estimate (for compactly supported \( f \)) holds under the usual Dini condition \( \omega_{\text{Dini}} = \int_0^1 \omega(t) \frac{dt}{t} < \infty \).

- A quantitative form: (T. Hytönen, L. Roncal and O. Tapiola (2015))
  denote \( C_T = \|T\|_{L^2 \rightarrow L^2} + C_K + [\omega]_{\text{Dini}} \). Then
  \[
  |Tf(x)| \leq c_n C_T \sum_{j=1}^{3^n} A S_j |f|(x).
  \]

A very brief history

- M. Lacey (2015): the same estimate (for compactly supported $f$) holds under the **usual Dini** condition $[\omega]_{\text{Dini}} = \int_0^1 \omega(t) \frac{dt}{t} < \infty$.

- **A quantitative form**: (T. Hytönen, L. Roncal and O. Tapiola (2015)) denote $C_T = \|T\|_{L^2 \rightarrow L^2} + C_K + [\omega]_{\text{Dini}}$. Then

  $$|Tf(x)| \leq c_n C_T \sum_{j=1}^{3^n} A_{S_j} |f|(x).$$


- The key idea behind of all approaches is an iteration:
  - iteration of the distribution function (rearrangement) of $f$ with a substitution $f \rightarrow Tf$ (70's-80's);
  - iteration of $f$ (**pointwise, “a median decomposition”**) with a substitution $f \rightarrow Tf$;
  - iteration of $Tf$ (M. Lacey).
The key recursive claim: there exist pairwise disjoint cubes \( P_j \in \mathcal{D}(Q_0) \) such that \( \sum_j |P_j| \leq \frac{1}{2} |Q_0| \) and for a.e. on \( Q_0 \),

\[
|T(f \chi_{3Q_0})(x)|_{\chi_{Q_0}} \leq c_n C_T |f|_{3Q_0} + \sum_j |T(f \chi_{3P_j})|_{\chi_{P_j}}.
\]
Main steps of the proof

- **The key recursive claim:** there exist pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and for a.e. on $Q_0$,

\[
|T(f \chi_{3Q_0})(x)|\chi_{Q_0} \leq c_n C_T |f|_{3Q_0} + \sum_j |T(f \chi_{3P_j})|\chi_{P_j}.
\]

- After iteration we obtain that there exists a $\frac{1}{2}$-sparse family $\mathcal{F} \subset \mathcal{D}(Q_0)$ such that

\[
|T(f \chi_{3Q_0})(x)|\chi_{Q_0} \leq c_n C_T |f|_{3Q} \sum_{Q \in \mathcal{F}} |f|_{3Q} \chi_Q(x).
\]
Main steps of the proof

- **The key recursive claim:** there exist pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that $\sum_j |P_j| \leq \frac{1}{2} |Q_0|$ and for a.e. on $Q_0$,
  \[
  |T(f\chi_{3Q_0})(x)|\chi_{Q_0} \leq c_n C_T |f|_{3Q_0} + \sum_j |T(f\chi_{3P_j})|\chi_{P_j}.
  \]

- For arbitrary pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$,
  \[
  |T(f\chi_{3Q_0})|\chi_{Q_0} \leq |T(f\chi_{3Q_0})|\chi_{Q_0 \cup_j P_j} + \sum_j |T(f\chi_{3Q_0\setminus 3P_j})|\chi_{P_j} + \sum_j |T(f\chi_{3P_j})|\chi_{P_j}.
  \]
Main steps of the proof

- **The key recursive claim:** there exist pairwise disjoint cubes \( P_j \in \mathcal{D}(Q_0) \) such that \( \sum_j |P_j| \leq \frac{1}{2}|Q_0| \) and for a.e. on \( Q_0 \),
  \[
  |T(f \chi_{3Q_0})(x)| \chi_{Q_0} \leq c_n C_T |f|_{3Q_0} + \sum_j |T(f \chi_{3P_j})| \chi_{P_j}.
  \]

- For arbitrary pairwise disjoint cubes \( P_j \in \mathcal{D}(Q_0) \),
  \[
  |T(f \chi_{3Q_0})| \chi_{Q_0} \leq |T(f \chi_{3Q_0})| \chi_{Q_0 \setminus \bigcup_j P_j} + \sum_j |T(f \chi_{3Q_0 \setminus 3P_j})| \chi_{P_j}
  \]
  \[
  + \sum_j |T(f \chi_{3P_j})| \chi_{P_j}.
  \]

- Hence, it suffices to find a set \( E \subset Q_0 \) and a covering of \( E \) by disjoint cubes \( P_j \in \mathcal{D}(Q_0) \) such that
  1. \( \sum_j |P_j| \leq \frac{1}{2}|Q_0| \);
  2. \( |T(f \chi_{3Q_0})(x)| \leq c_n C_T |f|_{3Q_0} \) for a.e. \( x \in Q_0 \setminus E \);
  3. \( |T(f \chi_{3Q_0 \setminus 3P_j})(x)| \leq c_n C_T |f|_{3Q_0} \) for a.e. \( x \in P_j \).
Main steps of the proof

- Hence, it suffices to find a set $E \subset Q_0$ and a covering of $E$ by disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that
  
  1. $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$;
  
  2. $|T(f\chi_{3Q_0})(x)| \leq c_n C_T |f|_{3Q_0}$ for a.e. $x \in Q_0 \setminus E$;
  
  3. $|T(f\chi_{3Q_0 \setminus 3P_j})(x)| \leq c_n C_T |f|_{3Q_0}$ for a.e. $x \in P_j$.

- The following local “grand maximal truncated” operator

$$\mathcal{M}_{T,Q_0} f(x) = \sup_{P \ni x, P \subset Q_0} \text{ess sup}_{\xi \in P} |T(f\chi_{3Q_0 \setminus 3P})(\xi)|$$

controls condition 3.
Main steps of the proof

- Hence, it suffices to find a set \( E \subset Q_0 \) and a covering of \( E \) by disjoint cubes \( P_j \in \mathcal{D}(Q_0) \) such that
  1. \( \sum_j |P_j| \leq \frac{1}{2} |Q_0| \);
  2. \( |T(f \chi_{3Q_0})(x)| \leq c_n C_T |f|_{3Q_0} \) for a.e. \( x \in Q_0 \setminus E \);
  3. \( |T(f \chi_{3Q_0 \setminus 3P_j})(x)| \leq c_n C_T |f|_{3Q_0} \) for a.e. \( x \in P_j \).

- The following local “grand maximal truncated” operator
  \[
  M_{T,Q_0} f(x) = \sup_{P \ni x, P \subset Q_0} \text{ess sup}_{\xi \in P} |T(f \chi_{3Q_0 \setminus 3P})(\xi)|
  \]
  controls condition 3.

- We have \( \|M_{T,Q_0}\|_{L^1 \to L^{1,\infty}} \leq \alpha_n C_T \) and
  \[
  |T(f \chi_{3Q_0})(x)| \leq \alpha_n \|T\|_{L^1 \to L^{1,\infty}} |f(x)| + M_{T,Q_0} f(x).
  \]
Main steps of the proof

- Hence, it suffices to find a set \( E \subset Q_0 \) and a covering of \( E \) by disjoint cubes \( P_j \in \mathcal{D}(Q_0) \) such that
  1. \( \sum_j |P_j| \leq \frac{1}{2} |Q_0| \);
  2. \(|T(f \chi_{3Q_0})(x)| \leq c_n C_T |f|_{3Q_0} \) for a.e. \( x \in Q_0 \setminus E \);
  3. \(|T(f \chi_{3Q_0 \setminus 3P_j})(x)| \leq c_n C_T |f|_{3Q_0} \) for a.e. \( x \in P_j \).

- The following local “grand maximal truncated” operator

\[
\mathcal{M}_{T,Q_0} f(x) = \sup_{P \ni x, P \subset Q_0} \text{ess sup} |T(f \chi_{3Q_0 \setminus 3P})(\xi)|
\]

controls condition 3.

- We have \( \|\mathcal{M}_{T,Q_0}\|_{L^1 \rightarrow L^{1,\infty}} \leq \alpha_n C_T \) and

\[
|T(f \chi_{3Q_0})(x)| \leq \alpha_n \|T\|_{L^1 \rightarrow L^{1,\infty}} |f(x)| + \mathcal{M}_{T,Q_0} f(x).
\]

- Set

\[
E = \{ x \in Q_0 : \mathcal{M}_{T,Q_0} f(x) > c_n C_T |f|_{3Q_0} \lor |f(x)| > c_n |f|_{3Q_0} \},
\]

where \( c_n \) is such that \( |E| \leq \frac{1}{2^{n+2}} |Q_0| \).
Main steps of the proof

- Hence, it suffices to find a set $E \subset Q_0$ and a covering of $E$ by disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that
  1. $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$;
  2. $|T(f \chi_{3Q_0})(x)| \leq c_n C_T |f|_{3Q_0}$ for a.e. $x \in Q_0 \setminus E$;
  3. $|T(f \chi_{3Q_0 \setminus 3P_j})(x)| \leq c_n C_T |f|_{3Q_0}$ for a.e. $x \in P_j$.

- The following local “grand maximal truncated” operator
  $$\mathcal{M}_{T,Q_0} f(x) = \sup_{P \ni x, P \subset Q_0} \text{ess sup}_{\xi \in P} |T(f \chi_{3Q_0 \setminus 3P})(\xi)|$$
  controls condition 3.

- Set
  $$E = \{ x \in Q_0 : \mathcal{M}_{T,Q_0} f(x) > c_n C_T |f|_{3Q_0} \lor |f(x)| > c_n |f|_{3Q_0} \},$$
  where $c_n$ is such that $|E| \leq \frac{1}{2^{n+2}}|Q_0|$.

- Apply the Calderón-Zygmund decomposition to $\chi_E$ with $\lambda = \frac{1}{2^{n+1}}$. 
Main steps of the proof

- Hence, it suffices to find a set $E \subset Q_0$ and a covering of $E$ by disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that
  1. $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$;
  2. $|T(f \chi_{3Q_0})(x)| \leq c_n C_T |f|_{3Q_0}$ for a.e. $x \in Q_0 \setminus E$;
  3. $|T(f \chi_{3Q_0 \setminus 3P_j})(x)| \leq c_n C_T |f|_{3Q_0}$ for a.e. $x \in P_j$.

- The following local "grand maximal truncated" operator
  $$
  M_{T,Q_0} f(x) = \sup_{P \ni x, P \subset Q_0} \text{ess sup} |T(f \chi_{3Q_0 \setminus 3P})(\xi)|
  $$
  controls condition 3.

- Set
  $$
  E = \{ x \in Q_0 : M_{T,Q_0} f(x) > c_n C_T |f|_{3Q_0} \lor |f(x)| > c_n |f|_{3Q_0} \},
  $$
  where $c_n$ is such that $|E| \leq \frac{1}{2^{n+2}} |Q_0|$.

- Apply the Calderón-Zygmund decomposition to $\chi_E$ with $\lambda = \frac{1}{2^{n+1}}$.
  We obtain disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that
  $$
  \frac{1}{2^{n+1}} < \frac{|P_j \cap E|}{|P_j|} \leq \frac{1}{2},
  $$
  which easily implies 1, 2 and 3.
The proof shows that if $T$ is a sublinear operator of weak type $(1, 1)$ and

$$\mathcal{M}_T f(x) = \sup_{Q \ni x} \text{ess sup}_{\xi \in Q} |T(f \chi_{\mathbb{R}^n \setminus 3Q})(\xi)|$$

is of weak type $(1, 1)$, then $\|T\|_{L^2(w)} \leq c(n, T)[w]_{A_2}$.
The proof shows that if $T$ is a sublinear operator of weak type $(1, 1)$ and
\[
\mathcal{M}_T f(x) = \sup_{Q \ni x} \text{ess sup}_{\xi \in Q} |T(f \chi_{\mathbb{R}^n \setminus 3Q})(\xi)|
\]
is of weak type $(1, 1)$, then $\|T\|_{L^2(w)} \leq c(n, T)[w]_{A_2}$.

T. Hytönen, L. Roncal and O. Tapiola (2015): for a class of rough homogeneous singular integrals $T_\Omega$,
\[
\|T_\Omega\|_{L^2(w)} \leq c(n, T)[w]_{A_2}^2.
\]
The proof shows that if $T$ is a sublinear operator of weak type $(1, 1)$ and

$$
\mathcal{M}_T f(x) = \sup_{Q \ni x} \text{ess sup} |T(f \chi_{\mathbb{R}^n \setminus 3Q})(\xi)|
$$

is of weak type $(1, 1)$, then $\|T\|_{L^2(w)} \leq c(n, T)[w]_{A_2}$.

T. Hytönen, L. Roncal and O. Tapiola (2015): for a class of rough homogeneous singular integrals $T_\Omega$,

$$
\|T_\Omega\|_{L^2(w)} \leq c(n, T)[w]_{A_2}^2.
$$

A. Seeger (1996): $T_\Omega$ is of weak type $(1, 1)$. 
The proof shows that if $T$ is a sublinear operator of weak type $(1, 1)$ and

$$
\mathcal{M}_T f(x) = \sup_{Q \ni x} \text{ess sup}_{\xi \in Q} |T(f \chi_{\mathbb{R}^n \setminus 3Q})(\xi)|
$$

is of weak type $(1, 1)$, then $\|T\|_{L^2(w)} \leq c(n, T)[w]_{A_2}$.

T. Hytönen, L. Roncal and O. Tapiola (2015): for a class of rough homogeneous singular integrals $T_\Omega$,

$$
\|T_\Omega\|_{L^2(w)} \leq c(n, T)[w]^2_{A_2}.
$$

A. Seeger (1996): $T_\Omega$ is of weak type $(1, 1)$.

It is natural to ask whether $\mathcal{M}_{T_\Omega}$ is of weak type $(1, 1)$, too.
The proof shows that if $T$ is a sublinear operator of weak type $(1, 1)$ and
\[
\mathcal{M}_T f(x) = \sup_{Q \ni x} \text{ess sup}_{\xi \in Q} |T(f \chi_{\mathbb{R}^n \setminus 3Q})(\xi)|
\]
is of weak type $(1, 1)$, then $\|T\|_{L^2(w)} \leq c(n, T)[w]_{A_2}$.

T. Hytönen, L. Roncal and O. Tapiola (2015): for a class of rough homogeneous singular integrals $T_\Omega$,
\[
\|T_\Omega\|_{L^2(w)} \leq c(n, T)[w]_{A_2}^2.
\]

A. Seeger (1996): $T_\Omega$ is of weak type $(1, 1)$.

It is natural to ask whether $\mathcal{M}_{T_\Omega}$ is of weak type $(1, 1)$, too.

Observe that the question whether the maximal singular integral operator $T^*_\Omega$ is of weak type $(1, 1)$ is still open.
Some words about the commutators

- Let \([b, T]\) denote the commutator of a Calderón-Zygmund operator \(T\) with a locally integrable function \(b\):

\[
[b, T]f(x) = bTf(x) - T(bf)(x).
\]
Some words about the commutators

- Let $[b, T]$ denote the **commutator** of a Calderón-Zygmund operator $T$ with a locally integrable function $b$:
  \[
  [b, T]f(x) = bTf(x) - T(bf)(x).
  \]

- Introduce the sparse operator $\mathcal{T}_{S,b}$ defined by
  \[
  \mathcal{T}_{S,b}f(x) = \sum_{Q \in S} |b(x) - b_Q|f_Q\chi_Q(x).
  \]

Let $\mathcal{T}_{S,b}^*$ be the adjoint operator to $\mathcal{T}_{S,b}$:

\[
\mathcal{T}_{S,b}^*f(x) = \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q |b - b_Q|f \right) \chi_Q(x).
\]
Some words about the commutators

- Let \([b, T]\) denote the **commutator** of a Calderón-Zygmund operator \(T\) with a locally integrable function \(b\):
  \[
  [b, T]f(x) = bT f(x) - T(bf)(x).
  \]

- Introduce the sparse operator \(T_{S,b}\) defined by
  \[
  T_{S,b} f(x) = \sum_{Q \in S} |b(x) - b_Q| f_Q \chi_Q(x).
  \]

Let \(T_{S,b}^*\) be the adjoint operator to \(T_{S,b}\):
  \[
  T_{S,b}^* f(x) = \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q |b - b_Q| f \right) \chi_Q(x).
  \]

- A.L., S. Ombrosi, I. Rivera-Ríos (2016): for every compactly supported \(f \in L^\infty(\mathbb{R}^n)\), there are \(\frac{1}{2.9^n}\)-sparse families \(S_j \subset \mathcal{D}(j), j = 1, \ldots, 3^n\), such that for a.e. \(x \in \mathbb{R}^n\),
  \[
  |[b, T]f(x)| \leq c_n C_T \sum_{j=1}^{3^n} (T_{S_j,b} |f|(x) + T_{S_j,b}^* |f|(x)).
  \]
Some words about the commutators

- A.L., S. Ombrosi, I. Rivera-Ríos (2016): for every compactly supported $f \in L^\infty(\mathbb{R}^n)$, there are $\frac{1}{2.9^n}$-sparse families $S_j \subset \mathcal{D}(j)$, $j = 1, \ldots, 3^n$, such that for a.e. $x \in \mathbb{R}^n$,

$$|[b, T]f(x)| \leq c_n C_T \sum_{j=1}^{3^n} (T_{S_j,b} |f|(x) + T_{S_j,b}^* |f|(x)).$$

- In particular, we obtain the following result: if $\mu, \lambda \in A_p$, $1 < p < \infty$, $\nu = (\mu/\lambda)^{1/p}$ and

$$\|b\|_{BMO(\nu)} := \sup_Q \frac{1}{\nu(Q)} \int_Q |b(x) - b_Q| \, dx < \infty,$$

then

$$\|[b, T]f\|_{L^p(\lambda)} \leq c_{n,p} C_T ([\mu]_{A_p} [\lambda]_{A_p})^{\max(1, \frac{1}{p-1})} \|b\|_{BMO(\nu)} \|f\|_{L^p(\mu)}.$$
Some words about the commutators

- A.L., S. Ombrosi, I. Rivera-Ríos (2016): for every compactly supported \( f \in L^\infty(\mathbb{R}^n) \), there are \( \frac{1}{2.9^n} \)-sparse families \( S_j \subset \mathcal{D}(j), j = 1, \ldots, 3^n \), such that for a.e. \( x \in \mathbb{R}^n \),

\[
[b, T]f(x) \leq c_n C_T \sum_{j=1}^{3^n} (T_{S_j, b}|f|(x) + T_{S_j, b}^* |f|(x)).
\]

- In particular, we obtain the following result: if \( \mu, \lambda \in A_p, 1 < p < \infty \), \( \nu = (\mu/\lambda)^{1/p} \) and

\[
\|b\|_{BMO(\nu)} := \sup_Q \frac{1}{\nu(Q)} \int_Q |b(x) - b_Q| dx < \infty,
\]

then

\[
\|[b, T]f\|_{L^p(\lambda)} \leq c_{n,p} C_T ([\mu]_{A_p} [\lambda]_{A_p})^{\max(1, \frac{1}{p-1})} \|b\|_{BMO_\nu} \|f\|_{L^p(\mu)}.
\]

- This provides a quantitative form of the two-weighted bound due to S. Bloom (1985) and I. Holmes, M. Lacey and B. Wick (2015).
Some related "sparse domination" works

Some related “sparse domination” works


Thank you for your attention!