Objectives

The aim of this school is to focus on those aspects of Harmonic Analysis which recently have had a huge impact, in particular in image and signal processing. One characteristic feature is that several technological deadlocks have been solved through the resolution of deep theoretical problems in harmonic analysis and Geometric Measure Theory. It is our purpose to present the new interlaces between Geometric Measure Theory and Harmonic Analysis and how these new understandings can be applied to solve real life problems. The courses of this CIMPA school will be taught by leaders in these areas and will cover both: theoretical aspects and applications. Our goal is that the courses can be followed by PhD students and PostDocs in mathematics, and in signal and image processing; the challenge being to emulate new interactions between these communities.

Date and Location:

The conference will take place from July 31st through August 11th, 2017 in Buenos Aires, Argentina.

IMPORTANT: Registration Deadline: April 9th, 2017.


Scientific Committee:  Local Organizing Committee:

- Akram Aldroubi  - Elena Agora
- Carlos Cabrelli  - Jorge Antezana
- Stephane Jaffard  - Carlos Cabrelli
- Ursula Molter  - Ursula Molter
- Victoria Paternostro  - Ezequiel Rela
- Pablo Shmerkin
Balian-Low and Time Frequency shift invariance

El Escorial 2016

ursula molter
Universidad de Buenos Aires
IMAS UBA/CONICET
Def: A SIS (shift invariant space) is a space of functions that is invariant under integer translations.

They are often models for classes of signals and images.
Examples:

1) $\text{PW}(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : \text{supp}(\hat{f}) \subseteq \left[ \frac{1}{2}, \frac{3}{2} \right] \}$

Note: $f \in \text{PW}(\mathbb{R}) \Rightarrow t_x f \in \text{PW}(\mathbb{R})$

$$t_x f(y) = f(y - x)$$

$\text{PW}(\mathbb{R})$ is translation invariant!

*Thus [Wieners]: $S \subseteq L^2(\mathbb{R}^d)$ is translation invariant, if there exists a measurable set $A \subseteq \mathbb{R} : S := \{ f \in L^2 : \hat{f}(w) = 0 \ a.e. \ A^c \}$
\[ S = \overline{\text{span}}\left\{ t_k x_{[0,1]} : k \in \mathbb{Z}\right\} \]

\[ \text{if } s \in S \text{ then } tx \in S \quad \text{and} \quad x \in \mathbb{Z}. \]

\[ S = \overline{\text{span}}\left\{ t_h x_{[0,1]} , \ t_j (x_{[0,1]} - x_{[\frac{1}{2},1]} ) : \right\} \]

\[ \text{if } h \in \mathbb{Z} \]

\[ \text{if } s \in S \text{ then } tx \in S \quad \text{and} \quad x \in \frac{1}{2} \mathbb{Z}. \]

why? \[ V_0 \oplus W_0 = V_1 \quad (\text{MRA}) \]
In ACHRM we gave a characterization for $S$ to have 'extra' - invariance - in particular:

\[ M := \{ \theta \in \mathbb{R} : \theta \notin S \vee \theta \in S \} \]

is a subgroup of $\mathbb{R}$ that contains $\mathbb{Z}$.

**Proposition** Let $S$ be a SIS. Then either $S$ is translation-invariant, or there exists a maximum positive integer $n$ such that $S$ is $\frac{1}{n}\mathbb{Z}$-invariant.
\[ \lambda \in \Lambda \subseteq \mathbb{R} \times \mathbb{N}, \ \varphi \in L^2(\mathbb{R}), \ \Lambda = R \mathbb{Z}^2 \ (R \text{ invertible}) \]

Gabor space \[ G(\gamma, \Lambda) = \overline{\text{span}} \{ \pi(\lambda) \varphi \} \lambda \in \Lambda \]

**Question**: Can there exist \( \mu \in \mathbb{R} \times \mathbb{N} \setminus \Lambda \) such that \( \pi(\mu) \varphi \in G(\gamma, \Lambda) \)?

Look at the Zak transform:

\[ \mathcal{Z} \varphi(x, w) := \sum_{h \in \mathbb{Z}} \varphi(x + h) e^{-2\pi i k w}, \ (x, w) \in \mathbb{R} \times \mathbb{N} \]

and note that:

\[ \mathcal{Z} \varphi(x + m, w) = e^{2\pi i m w} \mathcal{Z} \varphi(x, w), \ \mathcal{Z} \varphi(x, w + m) = \mathcal{Z} \varphi(x, w) \]
Hence: \( Z(\Pi(h,l)\psi)(x,w) = e^{2\pi i (h x + k w)} Z \psi(x,w) \)

Note that \( \psi \in S_0 \Rightarrow Z \psi \) is continuous.

Assume \( \Lambda = \mathbb{Z} \times \mathbb{Z}, \rho \in \mathbb{N}, \mu = (m, \eta) \), and \( \Pi(m, \eta) \psi \in \mathcal{Y}(\psi, \mathbb{Z} \times \rho \mathbb{Z}) \).

We have: \( \Pi(m, \eta) \psi \in \mathcal{Y}(\psi, \mathbb{Z} \times \rho \mathbb{Z}) \iff \)

\[
Z(\Pi(m, \eta)\psi) \in \mathcal{Z}(\mathcal{Y}(\psi, \mathbb{Z} \times \rho \mathbb{Z})) = \overset{\text{below}}{=} \overset{\text{below}}{=} \overset{\text{below}}{=}
\]

\[
= \text{span} \left\{ e^{2\pi i (\rho l x + k w)} Z \psi(x,w), (h,l) \in \mathbb{Z}^2 \right\}
\]

\( S_0 := \{ f \in L^2: Vf(t,v) = \int |x| e^{-|x-t|^2} e^{2\pi i x v} dx \in L^1 \} \)
Back to SIS:

Aldroubi, Sun, Wang show:
If \( t \in L^2 \) and \( \{ t_k \} \) is a Riesz basis of \( S(\mathbb{Z}) \), if \( S(\mathbb{Z}) \) is translation invariant, then \( t \not \in L^1 \).
Abdouani, Sun, Wang show:
Let \( Y \in L^2 \) and \( \{ t_n Y : n \in \mathbb{Z} \} \) be a Riesz basis of \( S(Y) \), if \( S(Y) \) is translation invariant, then \( Y \notin L^1 \).

Moreover:
Let \( Y \in L^2 \) and \( \{ t_n Y \} \) be a Riesz basis of \( S(Y) \), if \( S(Y) \) is \( \frac{1}{m} \)-invariant for some \( m \geq 2 \) then
\[
\exists \varepsilon > 0 \quad \int_{\mathbb{R}} |Y(x)|^2 |x|^{1+\varepsilon} \, dx = \infty
\]

In particular, if \( \varepsilon = 1 \) \( \Rightarrow \int_{\mathbb{R}} |Y(x)x|^2 \, dx = \infty \)

Reminiscent of Balian-Low!!
Balian–Low: time–frequency concentration and non-redundancy are "incompatible":

\( f \in L^2(\mathbb{R}), \Lambda \subseteq \mathbb{R}^2 \) lattice \( \{ e^{2\pi i m x} f(x-n) : (m,n) \in \Lambda \} \) being basis

\[ a,b \quad \left( \int |x-a|^2 |f(x)|^2 \, dx \right) \left( \int |\omega-b|^2 |\hat{f}(\omega)|^2 \, d\omega \right) = \infty \]

Note that \( \text{T} \) or \( \text{F} \) could be finite.
Amalgam Bohan - Low : [BHW 95]

If \( \{ e^{2\pi i \alpha k} f(x-\beta j) : j, k \in \mathbb{Z} \} \) Riesz basis \( \Rightarrow \)

\[ p \in S_0 := \{ f \in L^2 : Vf(t, v) \in L' \} \]

where \( Vf(t, v) = \int f(x) e^{-(x-t)^2} e^{2\pi i \cdot xv} \, dx \)

Remark: Both, BLT and ABLT are statements about smoothness, but neither one is stronger nor weaker.
Recall

Assume \( \Lambda = \mathbb{Z} \times p\mathbb{Z} \), \( p \in \mathbb{N} \), \( \mu = (m, \eta) \): \( m, \eta \in \mathbb{Q} \)
and \( \Pi(m, \eta) \gamma \in \mathcal{G}(\gamma, \mathbb{Z} \times p\mathbb{Z}) \).
We have: \( \Pi(m, \eta) \gamma \in \mathcal{G}(\gamma, \mathbb{Z} \times p\mathbb{Z}) \) \( \iff \)

\[
Z(\Pi(m, \eta) \gamma) \in \mathcal{M}(\gamma, \mathbb{Z} \times p\mathbb{Z}) = \overline{\text{span} \left\{ e^{2\pi i (plx + kw)} \mathcal{G}(x, w), (h, k) \in \mathbb{Z}^2 \right\}}
\]
So \( \Pi (m, n) \in \mathcal{Y} (q, \mathbb{Z} \times \mathbb{P} \mathbb{Z}) \iff \exists C = (c_{nk}) \in \mathcal{L}^2 \)

\[
\mathcal{Z} (\Pi (m, n) \psi) = e^{2\pi i n x} \mathcal{Z} \psi (x-m, \omega-n) = \\
= \sum c_{nk} e^{2\pi i (p l x + k \omega)} \mathcal{Z} \psi (x, \omega) \\
= h (x, \omega) \mathcal{Z} \psi (x, \omega) \quad \text{with} \\
h (x, \omega) = \sum_{h, l} c_{hk} e^{2\pi i (p l x + k \omega)} \\
which \ is \ \frac{1}{p} - \text{periodic in} \ x \ \text{and} \ 1 - \text{periodic in} \ \omega .
\]

Obs.: we used the fact that \( \Pi (2) \psi \) is a Riesz basis.
With some 'busy work' we show that there exist integers $R$, $M_1$, and $M_2$: \( \not\exists (x, w) \in \mathbb{R} \times \hat{\mathbb{R}} \)

\[
\frac{R}{\prod_{r=1}^{\infty}} h(x + rM, w + rM_z) = e^{2\pi i (M_1 x - M_2 w)}
\]

and if $\mathcal{F}$ is continuous $h(x, w)$ is continuous as well.

But this in turn implies (technical lemma) that $\mathcal{F} = k \cdot P$ and $\mu \in \mathbb{Z}$.

**CONTRACTION** - since $\mu = (m, n) \in \Lambda = (\mathbb{Z}, \mathbb{R})$

So $h$ can not exist if $\mu \in S_0$!!
In general, we have the following:

**Theorem:** If \((Y, \Lambda)\) is a Riesz basis for \(Y(Y, \Lambda)\) with \(Y \in S_0(\mathbb{R})\), and the density of \(\Lambda\) is rational, then \(\Pi(m, n) \in Y(Y, \Lambda)\), for all \((m, n) \in \Lambda\).

Note that this theorem generalizes the ABLT, since if \((Y, \Lambda)\) is a Riesz basis of \(L^2(\mathbb{R}) \Rightarrow \Lambda = \alpha \mathbb{Z} \times \beta \mathbb{Z}\) satisfies \(\alpha \beta = 1 \in \mathbb{Q}\) and \(Y(Y, \Lambda) = L^2(\mathbb{R}) \Rightarrow \Pi(m, n) \in Y(Y, \Lambda) \Rightarrow (m, n) \in \mathbb{R} \times \hat{\mathbb{R}} \Rightarrow \) by our theorem \(Y \notin S_0(\mathbb{R})\).
Examples:

It is straightforward to construct:

a) a discontinuous function \( f \) such that

- \( T_{1/2} f \in \mathcal{V}(f, \mathbb{Z} \times 3\mathbb{Z}) \)
- \( \{ \prod (x) f \}_{x \in \mathbb{Z} \times 3\mathbb{Z}} \) is a Riesz basis for \( \mathcal{V} \), clearly \( f \) cannot be \( 50 \).
b) a smooth function $f$ such that $T_k f \in \mathcal{Y}(t, \mathbb{Z} \times 3 \mathbb{Z})$ and not, $\{\overline{\pi}(x) f\}_{x \in (2 \times 3 \mathbb{Z})}$ is not a ring basis for $\mathcal{Y}(t, \mathbb{Z} \times 3 \mathbb{Z})$. 
Characterization of extra invariance in $515$:
Proposition. Let $S$ be a SIS. Then either $S$ is translation-invariant, or there exists a maximum positive integer $n$ such that $S$ is $\frac{1}{n}\mathbb{Z}$-invariant.

Further, if $S$ is $\frac{1}{m}$ invariant and 

$$S = S(\Psi) = \text{span} \{ \tau_k \Psi : k \in \mathbb{Z} \}$$

then

$$\{ \tau_k \Psi \}_{k \in \mathbb{Z}}$$

is a frame for $S$. 

$$\{ \tau_l \Psi \}_{l = 0, \ldots, m-1}$$

is a frame for $S$. 

Proposition. Let $S$ be a SIS. Then either $S$ is translation-invariant, or there exists a maximum positive integer $n$ such that $S$ is $\frac{1}{n}\mathbb{Z}$-invariant.

Further, if $S$ is $\frac{1}{n}$ invariant and

$$S = S(\lambda) = \overline{\text{span}} \{ t_n \lambda : k \in \mathbb{Z} \}$$

$\{ t_k \lambda \}_{k \in \mathbb{Z}}$ is a frame for $S$.

$\{ t_k \lambda \}_{k \in \mathbb{Z}}$ is a frame for $S$ where $\lambda$ is a "cutoff" of $\lambda$. 
Theorem 4.4. If $S \subseteq L^2(\mathbb{R})$ is a SIS, then the following are equivalent.

(a) $S$ is $\frac{1}{n}\mathbb{Z}$-invariant.
(b) $U_k \subseteq S$ for $k = 0, \ldots, n-1$.
(c) If $f \in S$, then $f^k = P_k f \in S$ for each $k = 0, \ldots, n-1$.

Moreover, in case these hold we have that $S$ is the orthogonal direct sum

\[ S = U_0 \oplus \ldots \oplus U_{n-1}, \]

with each $U_k$ being a (possibly trivial) $\frac{1}{n}\mathbb{Z}$-invariant SIS.
One then has (after some work):

**Corollary**  Let \( \varphi \in L^2(\mathbb{R}) \) be given. If the SIS \( S(\varphi) \) is \( \frac{1}{n} \mathbb{Z} \)-invariant for some \( n > 1 \), then \( \hat{\varphi} \) must vanish on a set of infinite Lebesgue measure. Furthermore, for each interval \( I \subseteq \mathbb{R} \) of length \( n \), we have that

\[
\left| \{ \omega \in I : \hat{\varphi}(\omega) = 0 \} \right| \geq n |E_0| + (n-1) |E_1| \geq n - 1,
\]

where \( E_0 = \{ \omega \in [0,1) : G_\varphi(\omega) = 0 \} \) and \( E_1 = \{ \omega \in [0,1) : G_\varphi(\omega) \neq 0 \} \)

Here \( G_\varphi(\omega) := \sum_{h \in \mathbb{Z}} |\hat{\varphi}(\omega + h)|^2 \). In particular

If a nonzero function \( \varphi \in L^2(\mathbb{R}) \) has compact support, then \( S(\varphi) \) is not \( \frac{1}{n} \mathbb{Z} \)-invariant for any \( n > 1 \).

Moreover, if \( \varphi \in L^2(\mathbb{R}) \) and \( S(\varphi) \) is translation invariant, \( |\text{supp } \varphi| \leq 1 \).
On going work on Characterization of extra invariance in $G(+, \mathbb{Z} \times \mathbb{PZ})$. 

$f \in L^2(\mathbb{R})$, $p, p' \in \mathbb{N}$, $p'$ divides $p$.

$k = 0, \ldots, p/p' - 1$.

$B_k = B_k(p, p') = \bigcup_{j \in \mathbb{Z}} \left( \frac{j}{p'}, \frac{k}{p} + \left[ \frac{k}{p}, \frac{k+1}{p} \right] \right) \times \mathbb{R}$

$U_k := \{ f \in L^2(\mathbb{R}) : \mathbb{Z}f = \mathbb{Z}g \times \chi_{B_k} \text{ for some } g \in \mathcal{G}(1, \mathbb{R} \times R) \}$.

We then have the following:
Theorem 29 (cf. Theorems 4.4 and 4.7 in [ACH+10]). Let $\varphi \in L^2(\mathbb{R})$ and $P, P' \geq 1$ integers such that $P' \mid P$, that is, $P'$ is a divisor of $P$. Then the following are equivalent.

(a) $\mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z})$ is invariant under $\pi(0, P')$.
(b) $U_k \subseteq \mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z})$ for $k = 0, \cdots, P/P' - 1$.
(c) $Z\varphi \cdot \chi_{B_k} \in Z[\mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z})]$ for $k = 0, \cdots, P/P' - 1$.

Moreover in this case, $\mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z})$ is the orthogonal direct sum

$$\mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z}) = U_0 \oplus \cdots \oplus U_{P/P' - 1}$$

with each $U_k$ being a (possibly trivial) subspace of $\mathcal{G}(\varphi, \mathbb{Z} \times P\mathbb{Z})$ invariant under $\{\pi(k, P'\ell)\}_{k, \ell \in \mathbb{Z}}$. 
collaborators:

- akram aldroubi (vanderbilt)
- carlos cabrelli (imas/ uba-conicet)
- chris heil (georgia-tech, atlanta)
- keeri kornelson (grinell, iowa)
- dae gwan lee (phillips uni. marburg)
- götz pfander (phillips uni. marburg)
Thank you very much!!
Technical Lemma:

If \( h(x) \) is a function satisfying

\[
e^{2\pi i \frac{M}{R} x} = \prod_{y=1}^{R} h(x + y \cdot 0) = (h(x))^R
\]

- \( h(x) \neq 0 \).

If \( h(x) \) is \( \frac{1}{p} \)-periodic, then

- \( R \) divides \( M \)

\[
h(x) = h(x + \frac{1}{p}) = e^{2\pi i \frac{M}{R} (x + \frac{1}{p})} = e^{2\pi i \frac{M}{Rp}} h(x).
\]