MINICOURSES

The Mathematical Foundations of Compressed Sensing

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Abstract

Compressed Sensing is a new area of signal and image processing which has as its goal to minimize the number of measurements (samples) that we need to take of a function/signal/image for accurate reconstruction. For example, it replaces the Shannon model of bandlimited signals by a model of signals as having a sparse representation in some dictionary of wave forms. The ideas for compressed sensing have their origin in certain constructions in approximation and finite dimensional geometry which showed the optimality of random sampling. This course will develop compressed sensing from its mathematical origins to its current implementation in signal and image processing.

The course will focus on the discrete sensing problem where we are given a vector in $x \in \mathbb{R}^N$ with N large and we wish to capture it through a small number n of measurements given by inner products with fixed vectors. Such a measurement system can be represented by an $n \times N$ matrix Φ. The vector $y = \Phi x$ is the vector of n measurements we make of x. The information that y holds about x is extracted through a decoder Δ. So $\Delta(\Phi x)$ should be designed to be a faithful approximation to x.

The lectures will introduce different ways to evaluate a sensing system (Φ, Δ) and then describe which such systems are best. The best systems will depend on the criteria to measure performance and in particular if we ask for accuracy with certainty or only with high probability.

Local Tb Theorems and applications in PDE

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Abstract

A Tb Theorem is a boundedness criterion for singular integrals, which allows the L^2 boundedness of a singular integral operator T to be deduced from sufficiently good behavior of T on some suitable non-degenerate test function b. However, in some PDE applications, including, for example, the solution of the Kato problem for square roots of divergence form elliptic operators, it may be easier to test the operator T locally (say on any given dyadic cube Q), on a test function b_Q that depends upon Q, rather than on a single, globally defined b. Or to be more precise, in the applications, it may be easier to find a family of b_Q's for which Tb_Q is locally well behaved, than it is to find a single b for which Tb is nice globally. In these lectures, we'll discuss some versions of local Tb theorems, as well as some applications to PDE.

Some recent developments in nonlinear dispersive equations

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Abstract

In these lectures I will describe the concentration-compactness/rigidity method for critical evolution problems, which I have been developing with Frank Merle. The concrete examples that I will focus on are the energy critical, focusing nonlinear Schrodinger and non linear wave equations, but other examples may also be discussed. The issues adressed are global existence and scattering.

Complex Interpolation between Banach, Hilbert and Operator spaces

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Abstract

The talks will describe various results proved using the complex interpolation method either in Banach Space Theory or in Operator Space Theory. Whenever possible special emphasis will be placed on illustrations in Harmonic Analysis. We will also include a characterization of the so-called θ-Hilbertian spaces: those Banach spaces B that can be written as $(B_0, B_1)_\theta$ where B_1 is Hilbert and B_0 an arbitrary Banach space.

In particular, we will present the results of a recent paper (available on Arxiv) where we describe the spaces

$$(B(\ell_{p_0}^n), B(\ell_{p_1}^n))_\theta \ , \ (B(\ell_{p_0}), B(\ell_{p_1}))^\theta \text{ or } (B(L_{p_0}), B(L_{p_1}))^\theta$$

for any pair $1 \leq p_0, p_1 \leq \infty$ and $0 < \theta < 1$. In the same vein, given a locally compact Abelian group G, let $M(G)$ (resp. $PM(G)$) be the space of complex measures (resp. pseudo-measures) on G equipped with the usual norm $\|\mu\|_{M(G)} = |\mu|(G)$ (resp.

$$\|\mu\|_{PM(G)} = \sup\{|\hat{\mu}(\gamma)| \ \Big| \ \gamma \in \widehat{G}\}).$$

We describe similarly the interpolation space $(M(G), PM(G))^\theta$. Various extensions and variants of this result will be given, e.g. to Schur multipliers on $B(\ell_2)$ and to operator spaces.

INVITED TALKS

On a square function estimate related to the Kato problem and solvability of elliptic PDE's

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Abstract

We consider the following theorem: If B is a bounded and strictly accretive multiplication operator on $L^2(R^n, C^N)$ and D is a self-adjoint first order differential operator with constant coefficients on $L^2(R^n, C^N)$ such that $\|Df\|_2 \sim \|f\|_{\dot{W}^{1,2}}$ for f in the range and the domain of D, then DB has quadratic estimates.

If D is one-one then this goes back to Coifman-McIntosh-Meyer's work and can be proved by standard $T(b)$ arguments. If D is not one-one, this is more difficult and was first proved by Axelsson, Keith and McIntosh as a consequence of their results on perturbed Dirac operators and elaboration of the local $T(b)$ for square functions as in proof of the Kato square root problem. It turns out that this estimate implies the Kato square root problem. It also implies solvability of boundary value problems as explained in Alan McIntosh's lecture. Here we sketch a direct proof and give some further consequences. This is joint with Andreas Axelsson and Alan McIntosh.

Heat-flow monotonicity related to some norm inequalities in analysis

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Abstract

We will discuss the manner in which certain classical norms arising in euclidean harmonic analysis are monotonic as their inputs evolve under certain "nonlinear" heat-flows. Among the examples discussed will be some of the familiar Strichartz norms associated with the free Schrödinger equation.

Affine inequalities in harmonic analysis

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Abstract

We discuss two affine inequalities and their role in two specific problems in harmonic analysis. The aim is to clarify the role that affine inequalities may play in harmonic analysis rather than to present any definitive results.

Noncommutative Riesz transforms for Fourier multipliers

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Abstract

In this talk we study Riesz transforms for discrete groups and Fourier or Herz-Schur multipliers acting on the von Neumann algebra given by left regular representation. For a commutative G this von Neumann algebra is simply $L_\infty(\hat{G})$. Given a semigroup (T_t) of positive maps with negative generator $-A$ we can follow Meyer's approach and define the gradient form

$$2\Gamma(f_1, f_2) = A(f_1)^* f_2 + f_1^* A(f_2) - 2A(f_1^* f_2)\,.$$

This gradient generalizes the usual scalar product of the gradient given by the Laplace or Laplace Beltrami operator. We given an overview on results concerning Meyer's general Riesz transforms estimates

$$\|\Gamma(f,f)^{1/2}\|_p \sim \|A^{1/2}f\|_p$$

in the context of Fourier multipliers.

A new approach to solvability of some elliptic pde's with square integrable boundary data

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Abstract

I shall survey some recent results of Auscher, Axelsson and myself, concerning second order elliptic divergence form equations with complex measurable coefficients A that are independent of the transversal coordinate. In particular, we prove that the set of $A \in L^\infty(R^n; C^{n+1})$ for which boundary value problems with L^2 Dirichlet or Neumann data are well posed, is an open set. This work is based on results of Axelsson, Keith and myself concerning perturbed Dirac operators, which in turn builds on the solution to the Kato square root problem by Auscher, Hofmann, Lacey, Tchamitchian and myself. Recent papers on this topic, also using techniques from harmonic analysis which were developed to solve the Kato square root problem, are by Alfonseca, Auscher, Axelsson, Hofmann and Kim, and by Auscher, Axelsson and Hofmann.

Flag Paraproducts

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Abstract

We shall speak about a new class of multi-linear operators which we named "flag paraproducts" and their connection to the theory of differential equations

On the univalent Bloch-Landau constant

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Abstract

Landau in the 30's estimated the univalent Bloch-Landau constant $\mathcal{U}$, i.e., the biggest radius R such that $f(D(0,1))$ always contains a disk of radius R for any univalent f normalized with $|f'(0)| = 1$. Although the exact value of $\mathcal{U}$ is not known, many authors have provided upper and lower bounds. In a joint work with T. Carroll we have studied fine properties of the extremal functions and shown the connection with other well studied question, the Polya-Cebotarev problem. This relationship has been exploited to improve (very slightly) the upper bound for the constant and to clarify the nature of the extremal domains.

Iterated Riesz commutators and a characterization of product BMO

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Abstract

This is joint work with Lacey, Pipher and Wick. We give a characterization of product BMO in the several real variable multi-parameter setting. Iterated commutators of multiplication by a symbol b with all combinations of Riesz transforms are bounded if and only if the symbol b belongs to product BMO.

Local smoothing inequalities and radial Fourier multipliers

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Abstract

The so called local smoothing problem for the wave equation (as formulated by Sogge) is the question whether the inequality

$$\left(\int_0^1 \|e^{it\sqrt{-\Delta}} f\|^q_{L^q(\mathbb{R}^d)}\, dt \right)^{1/q} \lesssim \|f\|_{L^q_\beta(\mathbb{R}^d)}$$

holds for $\beta = (d-1)(1/2 - 1/q) - 1/q$, for suitable $q \gg 2$. Here L^q_β is the usual Sobolev space.

We shall report progress on this and related questions, obtained in work with Gustavo Garrigos and Wilhelm Schlag, and in work with Fedya Nazarov.

The paper with F. Nazarov also gives, for fixed $p \neq 2$, a characterization of all radial multipliers of $\mathcal{F}L^p(\mathbb{R}^d)$ provided that the dimension d is large enough.

On the unreasonable effectiveness of Gutzmer's formula

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Abstract

In this expository talk we plan to demonstrate the usefulness of "Gutzmer's formula" in the study of various problems related to the Segal-Bargmann transform. Gutzmer's fomula has been known in several contexts: compact Lie groups, compact and noncompact Riemannian symmetric spaces, Heisenberg groups and Hermite expansions. We apply Gutzmer's formula to study holomorphic Sobolev spaces, local Peter-Weyl theorems, Paley-Wiener theorems and Poisson semigroups.

Bilinear virial identities and applications

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Abstract

I'll present some recent work with F. Planchon on bilinear virial identities for the nonlinear Schrodinger equation, which are extensions of the Morawetz interaction inequalities firstly obtained by Colliander, Keel, Staffilani, Takaoka and Tao. We recover and extend known bilinear improvements to Strichartz inequalities and provide applications to various nonlinear problems, most notably on domains with boundaries.

Principal Invariant Subspaces for Unitary Representations of LCA Groups

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Abstract

We will present the basic properties of shift invariant subspaces on the real line from a very simple point of view that reduces them to the properties of weighted L^2 spaces on the interval [0,1). The methods extend to general locally compact Abelian Groups and unitary representations of the groups acting on appropriate Hilbert spaces. Even in the classical case there are results that are new. This is work that is being conducted in a collaboration with Eugenio Hernandez, Hrvoje Sikic and Ed Wilson.

Vector-valued and noncommutative aspects of Littlewood-Paley theory

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Abstract

We discuss in this talk two aspects of the classical Littlewood-Paley theory. Let f be a function on $\mathbb{R}$. The Littlewood-Paley g-function of f is defined by

$$G(f)(x) = \left(\int_0^\infty \left|\frac{\partial}{\partial t} P_t * f(x)\right|^2 \frac{dt}{t}\right)^{1/2}, \quad x \in \mathbb{R},$$

where P_t denotes the Poisson semigroup on $\mathbb{R}$. The classical Littlewood-Paley inequalities read as

$$c_p \|G(f)\|_p \leq \|f\|_p \leq C_p \|G(f)\|_p\,, \quad 1 < p < \infty.$$

Here P_t can be replaced by the Poisson semigroup subordinated to any symmetric Markovian semigroup on a measure space. This is the so-called Littlewood-Paley-Stein theory.

The first aspect discussed in this talk concerns the vector-valued case of the preceding inequalities. Now f takes values in a Banach space X and $G(f)$ is defined as above just with the norm of X in place of the absolute value. Then it is not hard to show that the previous two-sided inequality holds iff X is isomorphic to a Hilbert space. However, the validity of only one of the two inequalities is a much subtler matter and is equivalent to the existence of an equivalent 2-uniformly convex or smooth norm of X. We also discuss the extreme case $p = \infty$ where BMO and Carleson measures are involved.

The second aspect deals with the noncommutative setting, where the usual L_p-spaces are replaced by noncommutative L_p-spaces and P_t is a quantum symmetric Markovian semigroup of maps preserving a given trace (or state). The resulting theory is closely related to noncommutative martingale/ergodic inequalities recently developed in noncommutative probability. It is also related to the noncommutative aspect of McIntosh's H_∞-functional calculus.