Lecture 3

Local Tb Theorem For SIO's. (Recall Thin of u. (Christ).)

Also AY

Thm (essentially AHMTT). Let T be an SIO w/ \( \mathcal{K}(x,y) \in L^\infty \).

(qualitatively) Suppose \( a \& b \leq 3, i = 1, 2 \) s.t.

(i) \( \sum_{Q} b_i^2 \leq C_0 |Q| \), \( i = 1, 2 \)

(ii) \( \sum_{Q} (\sum_{Q} b_i^2)^2 + \sum_{Q} |T^* b_i^2| \leq C_0 |Q| \)

(iii) \( \sum_{Q} 8 \leq \sum_{Q} b_i^2 \), \( i = 1, 2 \)

Then \( T : L^2 \to L^2 \).

Remarks: Proved in AHMTT. For "perfect dyadic" SIO's:

If \( y \) supported in a dyadic \( Q \), \( \|y\| = 0 \), then \( T^* y \) supported in \( Q \). Proof extends to \( \sum_{Q} b_i^2 < \infty \) for standard SIO's. Alternatively: Prove decoupling of dyadic standard SIO into perfect dyadic + \( L^2 \) bounded, so can extend Thin to standard case.

Care of \( L^p \) central \( \mathfrak{u} \) Christ's Thin.

Extends to spaces of homogeneous type (dyadic cube structure).

Applications: Bounds for Diagonal potentials in setting where 

\( D^k \) is not available.
Application:

Let \( E \) be an ADR set of dimension \( n \), \( E \subset \mathbb{R}^{m+1} \), i.e.,

\[
\exists C_0 \geq 1 \text{ s.t. } A \times \subset E, \quad \forall r \leq C_0 r_0,
\]

\[
\frac{1}{C_0} r^{n} \leq h^n(\overline{B(r_0)} \cap E) \leq C_0 r^{n}.
\]

Suppose that \( E = \partial \Omega \) for some domain \( \Omega \subset \mathbb{R}^{m+1} \) (e.g., \( \Omega = \mathbb{R}^{m+1} \setminus E \)). In particular, \( E \) is a space of homogeneous type, and has a "dyadic cube" structure.

Suppose that, given \( x \notin E \) and \( 0 < r \leq r_0 \) (where we have

\[(*) \quad \left| B(x, r) \cap \Omega \right| \geq \frac{1}{C_1} r^{n+1} \]

(Note: this is of course true if \( \Omega = \mathbb{R}^{m+1} \setminus E \).)

(Where: we do not impose (*) in (5.2).)

Then by ADR and Pisacw-Helson, \( \exists \) "corkscrew pt" \( A_{x, r} \) s.t.

\[ \text{dist}(A_{x, r}, E) = |A_{x, r} - x| \sim r. \]

Prop. (Bourgain) For \( E, \Omega \) as above, given \( Q \) ("dyadic cube") \( \subset E \), \( \exists A_Q \) w/ dist \( (A_Q, Q) \sim \text{dist}(A_Q, E) \)

w/ diam \( Q \) s.t.

\[ w(A_Q) = s > 0 \quad (s \text{ unit} \in \mathbb{Q}) \]

\( w = \text{harmonic measure}. \)
we now impose a further condition on $\Omega$, namely that $\mathcal{I}$ an outer unit normal $\nu$ at a.e. $x \in \partial \Omega$; and that the Gauss--Green formula holds in $\Omega$
(technically, this amounts to saying that $\Omega$ has "locally finite perimeter", and that its "measure theoretic boundary" coincides with $2\partial \mathcal{I}$ a.e.)

Remark: For such $\Omega$, Wiener's criterion holds at a.e. $pt.$ on $\partial \Omega$.
(Assuming ADR). Also, ADR $\Rightarrow$ locally finite perimeter.

Proposition: Suppose $w^A$ is absolutely continuous w.r.t. $H^N|\partial \Omega$ and that

$$f^A = \frac{dw^A}{dH^N} = -\partial G(\cdot, A_0) \in L^2_{loc}$$

with

$$\int_Q |f^A|^2 \leq C \lambda Q^{-1} \quad (\text{uni.t. in } Q)$$

(Here $G =$ Green's function).

Then $\nabla G : L^2(\partial \Omega) \rightarrow C(\partial \Omega)$,

where

$$\nabla G(x) = \int_{\partial \Omega} G(x, y) \nu_y \, dH^N(y)$$

($G(x) = C_{n-1} x^{2-n}$)
Remark: We do not assume that $\Omega$ contains a convex cube $Q$. For each cube $Q = x_iR$, nor even that $\Omega$ is uniformly rectifiable.

Sketch of proof (formal -- to make rigorous seems to require truncations of the kernel, and this is messy).

We apply local Tb theorem with $b = \frac{A_{x}}{2}$.

(Observe: $K(x+y) = O(x+y) = \text{standard C}_{-2}$)

Verify hypotheses of local Tb.

(iii) $\frac{b}{2} \leq b_{x}$. This is just Bourgain's Lemma.

(i) \[ \frac{b_{x}}{2} \leq C_{0}. \]
(by hypothesis).

(iii) \[ \frac{1/2}{b_{x}} \leq C_{0}. \]

(this is the part where we really need to truncate.)

Instead, suppose $x \in Q$, consider

\[ \nabla \cdot b_{x}(x). \]

Typically

\[ 101 \left( \nabla \cdot b_{x}(x) - \nabla \cdot S b_{x}(x) \right) \]

\[ = 101 \left( \nabla b_{x}(x) - \nabla S b_{x}(x) \right) \]
Sketch of Proof of (ii).

Observe that for \( x \) near \( A \), since \( |\nabla_x f(x,y)| \leq \frac{c}{|x-y|^n} \),

\[ \| \nabla_x f(x,y) - \nabla f(x,y) \| \leq \frac{c}{(\text{diam} A)^n} \sim \frac{1}{121} \]

Also, if \( \tilde{A} \) is a “fattened” version of \( A \), then for \( x \) near \( \tilde{A} \), we have

\[ \left| \int_{\tilde{A}} \nabla_x f(x,y) \, dy \right| \leq \frac{1}{(\text{diam} A)^n} \approx \frac{1}{121} \]

Ignoring the fact that

\[ b_0 = 121 \int_{\tilde{A}} A_0 \neq 121 \int_{\tilde{A}} A_0 + \tilde{A} \]

we have (morally) reduced matters to considering

\[ \int_{\tilde{A}} \nabla_x f(x,y) \, dy \]

by (†), enough to treat even

\[ \nabla_x f(x,A_0) = \int f(x,y) \, dy \]

\[ = S(x) := \text{dist}(x, A) \] where

\[ |\nabla_x g(x,A_0)| \leq \frac{g(x,A_0)}{\text{dist}(x,A)} = \frac{\Delta x}{\text{dist}(x,A)} \leq M \Delta x \]

Applying estimates (Bourgain + max., principle)

\[ A_x = \mathcal{S} \cap B(x, RS(x)) \]

Apply now that by hypothesis, \( \int_{\tilde{A}} A_0 \leq \frac{1}{121} \).
Some ideas of proof of the local $T^*$ for SLOCCs.

We shall ignore issues of WBP - in practice, we seek to establish

\[ (*) \]

\[ \frac{\theta_T^{1 \wedge 0}}{Q} \leq C_0, \]

which implies both $T^*_{1 \wedge 0}$ and WBP (also, we note that for $T^*_{1 \wedge 0}$).

... as in local $T^*$. Thus it's for square functions, it is enough to show $T^*_{1 \wedge 0}$, $T^*_{1 \wedge 3}$ and (really $(+)$).

Let

\[ \Delta_t^{X \wedge Y} := \int_{\mathbb{R}^n} u \left( \frac{x-y}{t} \right) \, dx \]

where $u \in C^0(\mathbb{R}^n \setminus \{0\})$, $\int u = 0$, and

\[ \int |u(tz)|^2 \, dt = 1 \]

hence

\[ \int_0^\infty \Delta_t \frac{dt}{t} = \mathbf{I} \quad \text{in strong operator topology on $L^2$} \]

" Calderón reproducing formula"

To show:

\[ \int \int \frac{\theta_T^{1 \wedge 0}}{Q} \, dx \, dt \]

is a Carleman measure.

We would like to apply local $T^*$ to square functions to

\[ \theta_t = \Delta_t T \]
Problem: \( \Delta_T \) does not have a standard L-P kernel in general,
(but o.k. if \( T^*1 = 0 \)).

Solution: build \( \Delta_t \) adapted to \( b_2 \), locally,
since \( T^*b_2 \) is good locally on \( \mathbb{A} \) (maybe not \( 0 \))
but still o.k.,

easier to do in discrete setting.

Let \( D_h = \) grid of dyadic cubes \( \mathbb{A} \), \( l(\mathbb{A}) = 2^{-h} \)

\[
E_m f = \sum_{Q \in D_h} 1_Q f_Q \quad Q \in D_h
\]

\( \Delta_h = E_{m_h}^{-} \Delta_h E_{m_h}^{+} \)

Adapted \( E_{m_h}, \Delta_h \) (following C.J.S)

Given approximate \( b \), set

\[
E_{m_h}^{-} f = \frac{E_{m_h}(b f)}{E_{m_h}(b)}
\]

note \( E_{m_h}^{-} 1 = 1 \)

\( \Delta_h = E_{m_h}^{-} - E_{m_h}^{+} \)
Fact: For accuracy \( b \), there is a good "Littlewood-Paley Theory" for \( \Delta^b \), i.e.

\[
\sum_{\vec{h}} |\Delta^b_{\vec{h}} f|^2 \leq \alpha \|

In our case, in view of \( \| \Delta^b T \|^2 \) is a discrete custom measure, we try to reduce we essentially locally on \( \Omega \), to proceed considering

\[
\sum_{\vec{h}} |\Delta^b_{\vec{h}} T|^2 \leq \alpha
\]

Problem \( \Delta^b \) is not good except where \( b \) is accurate. (see \( \Theta^b_{\vec{h}}(B) = \frac{E_{\vec{h}}(b^2 f)}{E_{\vec{h}}(b^2)} \))

So we first need to extract, via stopping time, an ample sawtooth on which

\[
|E_{\vec{h}}(b^2 a)| \geq \alpha
\]

Note: working w/ \( \Delta^b_{\vec{h}} (T T) \) allows us to exploit good behavior of \( T^a \) on \( \Omega \), so that hand of

Although, we can make it less precise using

\[
\Theta = \Delta^b_{\vec{h}} T
\]

is "close" to standard \( L^1 \),

so that we can proceed here as in local \( T \) for sq. functions.
OPEN PROBLEMS

1. Prove local $T_b$. For standard $C^2$ channels with $L^8$ control, $g > 1$, i.e. with
   \[ \|b_q \|_8 \leq C_0 \|a_1\|_1. \]

   We have seen that this works for square functions. Also, argument of $AHMTT$ works for perfect dyadic $SIOs$ (but with $T_q b_q, y_q \in L^8$).

   However, the error terms arising from treating standard $SIOs$ seem intractable if $g < 2$.

2. Prove a matrix-valued version of local $T_b$. For $SIOs$ (i.e. for $B_q$ that are matrix-valued).

   We have seen that this works for square functions. Difficulty for $SIOs$: when using adapted expectation operators, one would need to consider

   \[ E_{2a} f = (E_{2a} B_q)^{-1} E_{2a} (B_q f). \]

   How do we extend a result in which $E_{2a} B_q$ is invertible (with uniform control of inverse)? (2-D von Neumann idea? Direct proof somehow?)

- mention potential applications
3) Recall that we have observed that to estimate the Poisson kernels can be used to prove boundedness of layer potentials if \( \log A \leq 100 \log \frac{1}{\delta} \), or perhaps \( \log \frac{1}{\delta} \leq 100 \log \frac{1}{\delta} \).

O.T.O.H. It is known by abstract functional analysis that, given an \( r \)-bdd. \( S \), \( T \) is pseudodifferential systems \( r \geq 1, 1 \leq \delta \) adapted to \( T \) and \( T^* \) \((x, y) \in \text{control})

Problem: can we make this explicit in this case of layer potentials and Poisson kernels;

i.e., if layer potentials are bounded on \( L^2 \),

does this imply something good about it (e.g., \( \log \frac{1}{\delta} \leq 100 \log \frac{1}{\delta} \))?

- Very significant application

Remark: work of MMU and B. To

provides indirect evidence in 2 dimensions.

4) Let \( L = -\Delta + \chi(\mathbb{R}^d) \) in \( \mathbb{R}^d \), \( A(\mathbb{R}^d) \mathbb{R}^d \)


carlson's elliptic (accidental).

Prove: layer potentials are bounded \( L^2 \).

Remarks:

\( \text{MAHK} \Rightarrow \text{set of boundedness} \text{ open} \).

Thus in particular, true in complex case of real symmetric.

Block case: \( A = \begin{bmatrix} B_{nn} & 0 \\ 0 & 1 \end{bmatrix} \) is Kato problem

Q: Application of \( \text{MMU} \) to prove this?

In block case, yes, because \( \text{MMU} \) includes Kato.