2. Complex Interpolation and Operator spaces

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Plan

1. Operator Spaces

2. Complex interpolation for Operator Spaces
Definition

An operator space is a Banach space given with an isometric embedding

\[ E \subset B(H) \]

into the space \( B(H) \) of all bounded operators on a Hilbert space \( H \).

Intermediate notion between “Banach space” and “\( C^* \)-algebra”. Sometimes called “non-commutative Banach spaces”. Reason behind this terminology:

\[ \forall \ B \text{ Banach} \quad B \subset C(\Omega) \]

where \( \Omega = \text{unit ball of } B^* \)

\[ x \rightarrow f_x(\xi) = \langle \xi, x \rangle, \quad \| x \| = \sup_{\Omega} |f_x| \]

But \( C(\Omega) \) is the prototype of a commutative \( C^* \)-algebra.
Although there are important references before that, we consider that the “Theory of Operator Spaces”, as such, started with the thesis of Z.J. Ruan who gave an abstract characterization of operator spaces. The theory was then developed intensively by EFFROS and RUAN and by BLECHER and PAULSEN, later on by JUNGE and myself and others (LE MERDY, OZAWA, XU, OIKHBERG, RICARD...). Any Banach space embeds isometrically into $B(H)$ for a suitable $H$, hence can be “viewed” as an operator space in at least one way (and actually in many !). So all Banach spaces appear among the “objects”. But the MORPHISMS are different:

“COMPLETELY BOUNDED MAPS”

(instead of bounded linear maps).
Successors of Completely positive maps (Stinespring 55, Arveson 69, Choi-Effros 74,...).
Notion of C.B. MAP developed since the 80’s (Paulsen, Haagerup, Wittstock, Effros, Smith, Christensen, Sinclair,...).
Notation. $E \subset B(H)$

$$M_n(E) = \{n \times n \text{ matrices with entries in } E\}.$$ 

Equipped with norm induced by

$$M_n(B(H)) \simeq B(H \oplus \cdots \oplus H).$$ 

More generally, if $E_1 \subset B(H_1), E_2 \subset B(H_2)$ we define

$$E_1 \otimes_{\min} E_2 = \overline{E_1 \otimes_{\text{alg}} E_2} \subset B(H_1 \otimes_2 H_2)$$
Definition of c.b. maps.

Let \( u_n : M_n(E_1) \to M_n(E_2) \) defined by \((a_{ij}) \mapsto (u(a_{ij}))\). Then \( u \) is called completely bounded (c.b. in short) if \( \sup \|u_n\| < \infty \). We denote

\[
\|u\|_{cb} = \sup_{n \geq 1} \|u_n\|
\]

\[CB(E_1, E_2) = \{ u : E_1 \to E_2 \mid \text{c.b. linear map} \} .\]

We say that \( u \) is completely isometric (resp. completely contractive) if \( u_n \) is isometric (resp. contractive) for all \( n \). We say that \( u \) is a complete isomorphism if \( u \) is an isomorphism such that \( u \) and \( u^{-1} \) are both c.b.
Examples of c.b. maps:

\[ E_1 \subset B(H_1) \quad E_2 \subset B(H_2) \quad u: E_1 \to E_2 \]

\[ u(x) = a\pi(x)b \]

where \( \pi: B(H_1) \to B(H) \) \( C^* \) representation
\( a, b \) bounded operators.

Wittstock-Haagerup-Paulsen:

\[ \|u\|_{cb} = \inf\{\|a\|\|b\|\} \]
Some examples:

\[ R = \overline{\text{span}}[e_{1j} \mid j = 1, 2, \ldots] \quad \text{“row”} \quad (1) \]
\[ C = \overline{\text{span}}[e_{i1} \mid i = 1, 2, \ldots] \quad \text{“column”} \quad (2) \]
\[ R_n = \overline{\text{span}}[e_{1j} \mid 1 \leq j \leq n] \quad (3) \]
\[ C_n = \text{span}[e_{i1} \mid 1 \leq i \leq n]. \quad (4) \]

Then \( R \simeq C \simeq \ell_2 \) as Banach but

\[ R \not\simeq C \]

as operator spaces.
Central idea:

\( E \) Banach space ↔ norm \( \| \cdot \|_E \)

\( E \subset B(H) \) operator space ↔ sequence of norms

\( (\| \cdot \|_{M_n(E)})_{n \geq 1} \).

Equivalently, we can define \( \forall x \in \bigcup_n M_n(E) \)

\[
\| x \|_\infty = \lim_{n} \| x \|_{M_n(E)} \quad (5)
\]

and

\[
\mathcal{K}(E) = \bigcup_n M_n(E) \quad (6)
\]
If $E = \text{span}[e_j]$ and $u : E \to F$

$$\|u\| = \sup \left\{ \left\| \sum \alpha_j u(e_j) \right\|_F \mid \left\| \sum \alpha_j e_j \right\|_E \leq 1 \right\}$$

BUT

$$\|u\|_{cb} = \sup \left\{ \left\| \sum a_j \otimes u(e_j) \right\|_{\mathcal{K}(F)} \mid \left\| \sum a_j \otimes e_j \right\|_{\mathcal{K}(E)} \leq 1 \right\}$$

where

$$\mathcal{K}(E) = \bigcup_n M_n(E) \|_{\| \cdot \|_\infty} = \mathcal{K} \otimes_{\min} E$$
Reason: The role of the scalars $\mathbb{C}$ is played instead by matrices (="quantized scalars") or by the elements of

$$\mathcal{K} = \mathcal{K}(\mathbb{C}) = \bigcup_{n} M_n(\mathbb{C})$$

Summary:

$E$ Banach Space $\leftrightarrow$ $\| \sum \alpha_j x_j \|_E$

for $\alpha_j \in \mathbb{C}$, $x_i \in E$

$E$ Operator Space $\leftrightarrow$ $\| \sum a_j \otimes x_j \|_{\mathcal{K}(E)}$

for $a_i \in \mathcal{K}$, $x_i \in E$
Assume $E \subset B(H)$ spanned by a system $(e_j)_{j \in I}$. The operator space structure is determined by the following norm on $\mathcal{K} \otimes E$

$$\|(a_j)_{j \in I}\|_{\mathcal{K}[E]} = \| \sum a_j \otimes e_j \| \quad (a_j \in \mathcal{K} \ \forall j \in I)$$

Now suppose given a pair of such norms

$$\|(a_j)_{j \in I}\|_{\mathcal{K}[E_0]}, \quad \|(a_j)_{j \in I}\|_{\mathcal{K}[E_1]}$$

using the same index set $I$.

This gives us a compatible pair of operator spaces

$$E_0 \subset B(H_0) \quad E_1 \subset B(H_1)$$

with, say, $E_0$ spanned by $(e_j^0)$ and $E_1$ spanned by $(e_j^1)$.
Theorem

There is an operator space $E_{\theta} \subset B(H)$ spanned by $(e^\theta_j)$ such that

$$\| (a_j)_{j \in I} \|_{(\mathcal{K}[E_0], \mathcal{K}[E_1])_\theta} = \| (a_j)_{j \in I} \|_{\mathcal{K}[E_\theta]},$$

i.e.

$$= \| \sum a_j \otimes e^\theta_j \|.$$
Fundamental example

\[ R = \overline{\text{span}}[e_{1j} | j = 1, 2, \ldots] \quad \text{“row”} \quad (7) \]
\[ C = \overline{\text{span}}[e_{i1} | i = 1, 2, \ldots] \quad \text{“column”} \quad (8) \]
\[ R_n = \overline{\text{span}}[e_{1j} | 1 \leq j \leq n] \quad (9) \]
\[ C_n = \text{span}[e_{i1} | 1 \leq i \leq n]. \quad (10) \]

Then we find

\[ (R, C)_{1/2} = OH \]

where

\[ \|(a_j)_{j \in I}\|_{X[OH]} = \| \sum a_j \otimes \overline{a_j} \|^{1/2} \]

and similarly in \( n \)-dim case \( (R_n, C_n)_{1/2} = OH_n \).
More explicitly: here we have $E_0 = R$, $E_1 = C$

$$\|(a_j)_{j \in I}\|_0 = \sup \| \sum a_j a_j^* \|^{1/2}$$

$$\|(a_j)_{j \in I}\|_1 = \sup \| \sum a_j^* a_j \|^{1/2}$$

one then finds

$$\|(a_j)_{j \in I}\|_\theta = \sup \left\{ \| \sum a_j^* x a_j \|^{1/2}_p \mid \| x \|_p \leq 1 \right\}$$

where $1/p = 1 - \theta$. Note when $\theta = 1/2$, $p = 2$ and

$$\|(a_j)_{j \in I}\|_{1/2} = \sup \left\{ \| \sum a_j^* x a_j \|^{1/2}_2 \mid \| x \|_2 \leq 1 \right\}$$

$$= \| \sum a_j \otimes \overline{a_j} \|^{{1/2}}_{B(\ell^2 \otimes \ell^2)}$$
Operator spaces admit a good duality theory parallel to the Banach space case:

\[
\|(a_j)_{j \in I}\|_{\mathcal{K}[E^*]} = \sup \left\{ \| \sum a_j \otimes b_j \| \ : \ \|(b_j)_{j \in I}\|_{\mathcal{K}[E]} \leq 1 \right\}
\]

Note: existence of such an \(E^*\) is not obvious...(cf. Ruan)

Then, if \(I = \mathbb{N}\) (resp. \(I = [1, \ldots, n]\)) \(OH\) appears as the precise analogue of \(\ell_2\) (resp. \(\ell_2^n\)) uniquely characterized by its self-duality.
Haagerup tensor product:

\[ E \subset B(H) \quad F \subset B(K) \]

we then map

\[ E \otimes_{\text{alg}} F \rightarrow B(H) \ast B(K)(= \text{“free product C* – algebra}} \]

\[ e \otimes f \mapsto e \cdot f \in B(H) \ast B(K) \]

gives us (after completion) a new operator space denoted by

\[ E \otimes_h F \]

Crucial properties: both injective and projective (actually self-dual) and associative (very nice multilinear behaviour) but highly non-commutative

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2. Complex Interpolation and Operator spaces
Theorem (Effros et al)

Then for any $T \in E \otimes F$

$$\|T\|_{E \otimes_h F} = \inf \{ \| \sum a_j a_j^* \|^{1/2} \| \sum b_j^* b_j \|^{1/2} \mid T = \sum a_j \otimes b_j \}$$

There is a similar formula computing the norm in

$$\mathcal{H}[E \otimes_h F]$$
Suppose given two compatible pairs of operator spaces \((E_0, E_1)\) and \((F_0, F_1)\). Fix \(0 < \theta < 1\). Let \(E_\theta\) and \(F_\theta\) be the complex interpolated operator spaces. Then

\[
(E_0 \otimes_h F_0, E_1 \otimes_h F_1)_{\theta} = E_\theta \otimes_h F_\theta
\]

completely isometrically

In otherwords:

the Haagerup tensor product \textit{commutes} with interpolation
Non-commutative analogue of measure space: $M$ von Neumann algebra equipped with a trace $\tau$ (faithful, normal, semi-finite)

predual $M_* \simeq L_1(\tau)$

Convention: $L_\infty(\tau) = M$

Example: $B(H)$ with usual trace, predual = TRACE CLASS $M_* = S_1$ with norm $x \mapsto \text{tr}(|x|)$

duality $\langle x, y \rangle = \text{tr}(^t xy)$

Then if $1/p = 1 - \theta$:

$$(S_1, B(\ell_2))_\theta = S_p \quad \text{Schatten } p \text{ – class}$$
Non-commutative $L_p$

\[(L_1(\tau), L_\infty(\tau))_\theta = L_p(\tau)\]

With O.S. duality non-comm $L_1$ becomes an operator space
(using $M_* \subset M^*$)

Therefore $S_1$ or $S_1^n$ become operator spaces
By interpolation, recall $(L_1(\tau), L_\infty(\tau))_\theta = L_p(\tau)$

$L_p(\tau)$ also becomes operator space
More generally, let $E$ op. space, assume first $\dim(E) < \infty$ 

note $M_n(E)$ is analogue of $\ell^n_\infty(E)$ 

we may define an $E$-valued $S_1$ by setting 

$$S_1^n[E] = M_n(E^*)^*$$ 

Then by interpolation we define 

$$S_p^n[E] = (S_1^n[E], M_n(E))_\theta$$ 

and then extend this by setting 

$$S_p[E] = \bigcup_n \bigcup_i S_p^n[E_i]$$ 

where 

$$E = \bigcup_i E_i$$ 

We may argue similarly to construct 

$$L_p(\tau; E)$$ 

at least in hyperfinite case (Junge has extended this further)
We set

\[ S^n_\infty = M_n \]

The operator space analogue of pair

\[ B_0 = B(\ell^n_\infty), \quad B_1 = B(\ell^n_1) \]

is:

\[ B_0 = CB(S^n_\infty), \quad B_1 = CB(S^n_1) \]

**Theorem**

\[ 1/p = \theta \]

\[ (CB(S^n_\infty), \ CB(S^n_1))_\theta = CB_r(S^n_p) \]

where regular norm on \( CB_r(S^n_p) \) is described below.
Let

$$B_0 = CB(S^n_\infty) \quad B_1 = CB(S^n_1)$$

Case $\theta = 1/2$ : If $T(x) = \sum a_k x b_k$ we have

$$\|T\|_{B_0} = \inf \{ \| \sum a_k a_k^* \|^ {1/2} \| \sum b_k b_k^* \|^ {1/2} \}$$

$$\|T\|_{B_1} = \inf \{ \| \sum a_k^* a_k \|^ {1/2} \| \sum b_k b_k^* \|^ {1/2} \}$$

Then using Kouba’s theorem one finds

$$\|T\|_{B_1/2} = \inf \{ \| \sum a_k \otimes \bar{a}_k \|^ {1/2} \| \sum b_k \otimes \bar{b}_k \|^ {1/2} \}$$

More generally

$$\|T\|_{CB_r(S^n_{\rho})} = \|T\|_{B_{\theta}} = \inf \{ \| (a_k) \|_{(R,C)_\theta} \| (b_k) \|_{(R,C)_{1-\theta}} \}$$

where

$$\| (a_k) \|_{(R,C)_\theta} = \sup \{ \| \sum a_k x a_k^* \|_{1/\theta}^{1/2} | \| x \|_{S_{1/\theta}} \leq 1 \}$$
Similarly

\[(CB(S_\infty), CB(S_1))^{\theta} = CB_r(S_p)\]

\[T \in CB_r(S_p) \iff T = T_1 - T_2 + i(T_3 - T_4)\]

with all \(T_j\) completely positive and bounded on \(S_p\)

\[\|T\|_{CB_r(S_p)} \simeq \inf\left\{ \sum_{1}^{4} \|T_j\| \right\}\]
Key fact

Let $S^n_{\infty}[E] = M_n(E)$

\[
1/p = \theta
\]

Recall we define

\[
S^n_p[E] = (S^n_{\infty}[E], S^1_1[E])_\theta
\]

\[
S^n_{\infty}[E] = C_n \otimes_h E \otimes_h R_n
\]

\[
S^n_1[E] = R_n \otimes_h E \otimes_h C_n
\]

and hence (by Kouba’s theorem)

\[
S^n_p[E] = (C_n, R_n)_\theta \otimes_h E \otimes_h (R_n, C_n)_\theta
\]
Some applications:
With vector valued non-commutative $L_p$-spaces one can now extend
maximal inequalities on non-commutative $L_p$
JUNGE : Doob’s maximal inequalities for martingales (Crelle 2002)
JUNGE and XU : Dunford-Schwartz maximal ergodic inequality (JAMS 2007)
Lemma

Consider \( T : S_p \to S_p \)

\[
\| T \|_{CB_r(S_p)} = \sup_X \{ \| T_X : S_p[X] \to S_p[X] \| \} = \sup_X \{ \| T_X \|_{cb} \}
\]

where the sup is over all possible (finite dim) operator spaces \( X \)

Note if \( p = 1 \) or \( p = \infty \) then \( \| T \|_{CB_r(S_p)} = \| T \|_{CB(S_p)} \)

Lemma

\( X = OH \) completely c-isomorphically IFF for any \( T : S_2 \to S_2 \)

\[
\| T_X : S_2[X] \to S_2[X] \| \leq c \| T : S_2 \to S_2 \|
\]
Operator space version of characterization of $\theta$-Hilbertian:

**Theorem**

Let $\mathcal{OH}(\theta)$ be the set of finite dimensional arcwise $\theta$-$0$-Hilbertian operator spaces. Let $0 < \theta < 1$. Consider $CB(\theta, n) = (CB_r(S^n_2), B(S^n_2))_\theta$. Then, for any $T : S^n_2 \rightarrow S^n_2$ we have

$$\|T\|_{CB(\theta,n)} = \sup_{X \in \mathcal{OH}(\theta)} \|TX\|_{B(S^n_2[X])} = \sup_{X \in \mathcal{OH}(\theta)} \|TX\|_{CB(S^n_2[X])}.$$  \hspace{1cm} (11)

Moreover, the supremum is unchanged if we restrict it to those $X = X(0)$ with $X(z) \simeq S^m_2$ $\forall z \in J_\theta$ and $X(z) = M_m$ $\forall z \notin J_\theta$, where $\simeq$ means here completely isometric.
The operator spaces $X$ that are $c$-isomorphically subquotients of $\theta$-0-Hilbertian operator spaces can be characterized in the same vein by
\[ \| Tx \| \leq c \| T \| (CB_r(S_2^n), B(S_2^n))_\theta \]
Moreover we can describe analogously the spaces
\[ (CB(S_{p_0}), CB(S_{p_1}))^\theta \]
\[ (CB(L_{p_0}(\tau)), CB(L_{p_1}(\tau)))^\theta \ldots \]
Embedding problems:

Span of i.i.d. Gaussian variables yields isometric embedding

\[ H \subset L_1 \quad \text{commutative case} \]

\[ 1 < p < 2 \quad p\text{-stable random variables yield isometric embedding} \]

\[ L_p \subset L_1 \quad \text{commutative case} \]

—Problem 1

\[ \text{• } OH \subset \text{non – com } L_1 ? \]

completely isometrically? completely isomorphically?

—Problem 2

\[ \text{• } \text{non – com } L_p \subset \text{non – com } L_1 ? \]

completely isometrically? completely isomorphically?
Problem 1

• \( OH \subset non \rightarrow com L_1 \) ?

answered by Marius Junge (2002, appeared in Inv. Math.)
Original proof - rather complicated - uses \( K \)-functional (i.e. real interpolation) and free probability inequalities

∃ Other proofs :
– P IMRN 2004 using complex interpolation and free probability
– Junge+Xu using real interpolation
– Haagerup and Musat (2007) using Clifford algebra

best constant of isomorphism \( \sqrt{2} \)
constructions all require Type III (necessary by [P 2004])
completely isometric case still open
Problem 2

- $non - com \ L_p \subset non - com \ L_1$?

Junge (GAFA 2000) proved that there is an isometric embedding (semi-finite)
Junge and Parcet 2007: completely isomorphic embedding (non semi-finite, i.e. non tracial) (again necessary by [P 2004])
...
completely isometric still open
Some ideas about the solution of Problem 1

Ingredients:
1. Interpolation between $R$ and $C$
2. Non-commutative analogue of Gaussian variables

Starting point uses interpolation:

**Theorem**

$OH \in SQ(R \oplus C)$ (completely isometrically)
Proof (sketch) Notation

\( H \) Hilbert : \( H_c = B(\mathbb{C}, H) \) “columns”, \( H_r = B(\overline{H}, \mathbb{C}) \) “rows”

\[ B_0 = M_n(H_r) \quad B_1 = M_n(H_c) \]

By definition :

\[ \| x \|_{B_1/2} = \inf_{f(1/2)=x, f \text{ ana}} \max \{ \| f|_{\partial_0} \|_{L_2(\partial_0;B_0)}, \| f|_{\partial_1} \|_{L_2(\partial_1;B_1)} \} \]

\[ \| f|_{\partial_0} \|_{L_2(\partial_0;B_0)} = \| f|_{\partial_0} \|_{L_2(\partial_0;M_n(H_r))} \geq \| f \|_{M_n(L_2(\partial_0;H)_r)} \]

\[ \| f|_{\partial_1} \|_{L_2(\partial_1;B_1)} = \| f|_{\partial_1} \|_{L_2(\partial_1;M_n(H_c))} \geq \| f|_{\partial_1} \|_{M_n(L_2(\partial_1;H)_c)} \]

but actually a nice surprise :

\[ \| x \|_{B_1/2} = \inf_{f(1/2)=x, f \text{ ana}} \max \{ \| f|_{\partial_0} \|_{M_n(L_2(\partial_0;H)_r)}, \| f|_{\partial_1} \|_{M_n(L_2(\partial_0;H)_c)} \} \]

\[ = \inf_{f(1/2)=x, f \text{ ana}} \| f|_{\partial} \|_{M_n(L_2(\partial_0;H)_r \oplus L_2(\partial_0;H)_c)} \]
Conclusion:

\[ M_n(OH) = M_n(E)/M_n(F) = M_n(E/F) \]

where

\[ E = \{ f|_\partial, f \text{ ana} \} \subset L_2(\partial_0; H)_r \oplus L_2(\partial_0; H)_c \quad F = \{ f \in E \mid f(1/2) = 0 \} \]

\[ OH \simeq E/F \]

and hence

**Theorem**

\[ OH \in QS(R \oplus C) = SQ(R \oplus C) \]
Then to embed $OH$ into non-commutative $L_1$ it suffices to embed into it any quotient of $R \oplus C$
Equivalently it suffices to show any subspace of $R \oplus C$
$(\simeq C \oplus R )$ is a quotient of non-commutative $L_\infty$
This is realized in free probability by Shlyakhtenko’s generalized free circular elements on the full Fock space
(with final isomorphism constant $2 \sqrt{2}$)
Haagerup and Musat 2007 got a very nice proof using instead the anti-symmetric Fock space
(with final isomorphism constant $\sqrt{2}$)

Note : Classical Gaussian case corresponds to the symmetric Fock space!