Flow monotonicity and Strichartz inequalities

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(Heat flow not special here - for instance, any mass-preserving convolution semi-group will work.)
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E.g. $B(x, y) = (x - y) \log(\frac{x}{y})$ is convex, so the (symmetrised) “relative entropy”

$$H(f_1, f_2) := \int (f_1 - f_2) \log \left( \frac{f_1}{f_2} \right)$$

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Which other inequalities in harmonic analysis can be understood in such a way?
Generalised Hölder inequalities

Similar monotonicity phenomena hold for

• Multilinear Hölder:

\[ \int \prod_j f_j^{\frac{1}{p_j}} \leq \prod_j \left( \int f_j \right)^{\frac{1}{p_j}} ; \quad \sum_j \frac{1}{p_j} = 1. \]
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- Loomis–Whitney (forerunner to multilinear Kakeya):

\[ \int_{\mathbb{R}^{n}} \prod_{j=1}^{n} f_{j}(x_{1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{n})^{\frac{1}{n-1}} \, dx \leq \prod_{j=1}^{n} \left( \int_{\mathbb{R}^{n-1}} f_{j} \right)^{\frac{1}{n-1}}. \]
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- Young’s convolution:

\[
\int \int f_1(x)^{\frac{1}{p_1}} f_2(y)^{\frac{1}{p_2}} f_3(x - y)^{\frac{1}{p_3}} \; dx dy \lesssim \left( \int f_1 \right)^{\frac{1}{p_1}} \left( \int f_2 \right)^{\frac{1}{p_2}} \left( \int f_3 \right)^{\frac{1}{p_3}},
\]

\[
\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 2,
\]

along with certain common generalisations (Carlen–Lieb–Loss 2003, B–Carbery–Christ–Tao 2007).
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Taking $u_1, u_2$ to be solutions with data $f_1^{p_1}, f_2^{p_2}$ respectively, then if $p_1, p_2 \geq 1$,

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$$\|f_1 \ast f_2\|_p = \lim_{t \to 0} \mathcal{F}(u_1, u_2) \leq \lim_{t \to \infty} \mathcal{F}(u_1, u_2) = C_{p_1, p_2} \|f_1\|_{p_1} \|f_2\|_{p_2},$$

where $C_{p_1, p_2} = \|H_{\sigma_1}^{1/p_1} \ast H_{\sigma_2}^{1/p_2}\|_p$ is Beckner’s sharp constant ($H_t$ denotes the standard heat kernel on $\mathbb{R}^d$).
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Taking $u_1, u_2$ to be solutions with data $f_1^{p_1}, f_2^{p_2}$ respectively, then if $p_1, p_2 \geq 1$,

$$\| f_1 * f_2 \|_p = \lim_{t \to 0} \mathcal{F}(u_1, u_2) \leq \lim_{t \to \infty} \mathcal{F}(u_1, u_2) = C_{p_1, p_2} \| f_1 \|_{p_1} \| f_2 \|_{p_2},$$

where $C_{p_1, p_2} = \| H_{\sigma_1}^{1/p_1} * H_{\sigma_2}^{1/p_2} \|_p$ is Beckner’s sharp constant ($H_t$ denotes the standard heat kernel on $\mathbb{R}^d$).

If $p_1, p_2 \leq 1$ we obtain the sharp reverse Young’s inequality of Brascamp–Lieb.
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$$\partial_t u = \Delta u^m; \quad m = \frac{d}{d+2},$$

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Proof:

- Observe that $\mathcal{F}(f) = C_{HLS} \|f\|^2_{2d \over d+2} - c \int_{\mathbb{R}^d} f(-\Delta)^{-1} f$;
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- Apply a sharp (Gagliardo–Nirenberg–)Sobolev inequality.
Monotonicity and restriction/Kakeya

Monotonicity perspective also attractive from a restriction/Kakeya point of view...
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**Example 1**: Multilinear restriction/Kakeya.
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**Proposition (**“B–Carbery–Tao”**)**

$C_{Kak}(\delta) \lesssim C_{Kak}(\delta/\delta') C_{Kak}(\delta')$ and $C_{Rest}(R) \lesssim C_{Kak}(R^{-1/2}) C_{Rest}(R^{1/2})$. 

Jonathan Bennett (U. Birmingham)  
Flow monotonicity and Strichartz inequalities  
4th June 2014 9 / 29
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These are "discrete monotonicity" properties, that for Kakeya at least, underly genuine monotonicity properties (B–Carbery–Tao, 2006).
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For restriction such genuine monotonicity has proved more elusive.
The monotonicity perspective is also attractive in a linear setting...
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**Example 2 (philosophical):** The restriction conjecture for $\mathbb{S}^{n-1}$ may be reformulated as

$$\int_{\mathbb{R}^n} |\widehat{gd\sigma}|^q \lesssim \int_{\mathbb{R}^n} |\widehat{d\sigma}|^q$$

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So, certainly tempting to look for monotonicity of appropriate functionals, such as

$$\int_{\mathbb{R}^n} |\hat{g}d\sigma|^q \quad \text{or maybe} \quad c\|g\|_p^q - \int_{\mathbb{R}^n} |\hat{g}d\sigma|^q$$

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This is what we’ll take for the rest of the talk.
Define the operator $e^{is\Delta}$ acting on functions $f : \mathbb{R}^d \to \mathbb{C}$ by

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**Strichartz estimates:**

$$\|e^{is\Delta} f\|_{L^p_s L^q_x(\mathbb{R} \times \mathbb{R}^d)} \leq C_{p, q} \|f\|_{L^2(\mathbb{R}^d)}$$
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$$I(s) = \int_{\mathbb{R}} \int_{\mathbb{R}} |u(s, x)|^2 |u(s, y)|^2 |x - y| \, dx \, dy$$

then Planchon and Vega show that

$$I'(s) = 2 \int_{\mathbb{R}} \int_{\mathbb{R}} (\mathbb{S}(\overline{u} \partial_x u)(x) |u(y)|^2 - \mathbb{S}(\overline{u} \partial_y u)(y) |u(x)|^2) \, \text{sign}(x - y) \, dx \, dy$$

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In particular, by the FTC and $L^2$ conservation, we have the Sobolev–Strichartz estimate

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(Much more can be said using such “Morawetz interaction functionals” - see PV.)
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Monotonicity results are known for certain spatial norms of solutions of Schrödinger equations. E.g. if \( u \) is a solution on the line and

\[
l(s) = \int_{\mathbb{R}} \int_{\mathbb{R}} |u(s, x)|^2 |u(s, y)|^2 |x - y| \, dx \, dy
\]

then Planchon and Vega show that

\[
l'(s) = 2 \int_{\mathbb{R}} \int_{\mathbb{R}} (\mathcal{S}(\bar{u} \partial_x u)(x)|u(y)|^2 - \mathcal{S}(\bar{u} \partial_y u)(y)|u(x)|^2) \text{sign}(x - y) \, dx \, dy
\]

and

\[
l''(s) = 4 \int_{\mathbb{R}} (\partial_x(|u|^2))^2.
\]

In particular, by the FTC and \( L^2 \) conservation, we have the Sobolev–Strichartz estimate

\[
4 \int_{\mathbb{R}} \int_{\mathbb{R}} (\partial_x(|u|^2))^2 \, dx \, ds = 4 \int_{\mathbb{R}} l''(s) \, ds \lesssim \sup_s |l'(s)| \lesssim \|u_0\|_2^3 \|\partial_x u_0\|_2.
\]

(Much more can be said using such “Morawetz interaction functionals” - see PV.)

Let’s return to diffusions...
Heat-flow monotonicity of Strichartz norms
Theorem (B-Bez-Carbery-Hundertmark; 2009)

If \((p, q, d)\) is Schrödinger admissible, \(q \in 2\mathbb{N}\) and \(q \nmid p\) then

\[
\| e^{is\Delta} (e^{t\Delta} |f|^2)^{1/2} \|_{L_p^s L_x^q} \]

is nondecreasing.
Theorem (B-Bez-Carbery-Hundertmark; 2009)

If \((p, q, d)\) is Schrödinger admissible, \(q \in 2\mathbb{N}\) and \(q|p\) then

\[ t \mapsto \| e^{is\Delta} (e^{t\Delta} |f|^2)^{1/2} \|_{L^p_t L^q_x} \]

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Proof in the case \((p, q, d) = (4, 4, 2)\).
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Proof in the case \((p, q, d) = (4, 4, 2)\). Multiplying out the 4th power, noting the Fourier-invariance, gives

\[
\| e^{is\Delta} f \|_{L^4_{s,x}(\mathbb{R} \times \mathbb{R}^2)}^4 = \frac{1}{4} \langle PF, F \rangle
\]

where \(F = f \otimes f\) and \(P : L^2(\mathbb{R}^2 \times \mathbb{R}^2) \to L^2(\mathbb{R}^2 \times \mathbb{R}^2)\) is given by

\[
PG(x) = 4 \int_{(\mathbb{R}^2)^2} G(y) \delta(x_1 + x_2 - y_1 - y_2) \delta(|x_1|^2 + |x_2|^2 - |y_1|^2 - |y_2|^2) dy
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Heat-flow monotonicity of Strichartz norms

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\]

where \(O\) denotes the group of orthogonal transformations \(\rho\) of \(\mathbb{R}^4\) fixing \((1, 0, 1, 0)\) and \((0, 1, 0, 1)\).
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Suppose that $u$ solves the heat equation and write $U = u \otimes u$. 
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$$\|e^{is\Delta}u^{1/2}\|_{L^4_{s,x}(\mathbb{R}^2)}^4 = \frac{1}{4} \langle PU^{1/2}, U^{1/2} \rangle$$

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Suppose that $u$ solves the heat equation and write $U = u \otimes u$. We have

$$\|e^{is\Delta} u^{1/2}\|_{L^4_s, \mathbb{R} \times \mathbb{R}^2}^4 = \frac{1}{4} \langle PU^{1/2}, U^{1/2} \rangle = \frac{1}{4} \int_0 \int_{(\mathbb{R}^2)^2} U(\rho x)^{1/2} U(x)^{1/2} \, dx \, d\rho = \frac{1}{4} \int_0 \int_{(\mathbb{R}^2)^2} U_\rho(x)^{1/2} U(x)^{1/2} \, dx \, d\rho,$$

where $U_\rho(x) = U(\rho x)$. 
Suppose that $u$ solves the heat equation and write $U = u \otimes u$. We have

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where $U_\rho(x) = U(\rho x)$. Since $U$, $U_\rho$ solve the heat equation, the theorem follows from the previously established monotonicity of

$$t \mapsto \int_{(\mathbb{R}^2)^2} U_\rho(x)^{1/2} U(x)^{1/2} \, dx$$

for each $\rho$. 
Suppose that \( u \) solves the heat equation and write \( U = u \otimes u \). We have

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where \( U_\rho(x) = U(\rho x) \). Since \( U, U_\rho \) solve the heat equation, the theorem follows from the previously established monotonicity of

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\]

for each \( \rho \). Moreover,

\[
\frac{d}{dt} \| e^{is\Delta} u^{1/2} \|_{L^4_{s,x}(\mathbb{R} \times \mathbb{R}^2)}^4 = \frac{1}{16} \int_{\mathcal{O}} \int (\mathbb{R}^2)^2 \left| \frac{\nabla U}{U} - \frac{\nabla U_\rho}{U_\rho} \right|^2 \, dx \, d\rho.
\]
Suppose that \( u \) solves the heat equation and write \( U = u \otimes u \). We have

\[
\| e^{is\Delta} u^{1/2} \|_{L^4_s(x)}^4 = \frac{1}{4} \langle PU^{1/2}, U^{1/2} \rangle = \frac{1}{4} \int_0^T \int_{\mathbb{R}^2} U(\rho x)^{1/2} U(x)^{1/2} \, dx \, d\rho
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where \( U_\rho(x) = U(\rho x) \). Since \( U, U_\rho \) solve the heat equation, the theorem follows from the previously established monotonicity of

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\]

**Shortcomings:** Very rigid - does not appear to extend to other equations, other flows, Sobolev norms, non-integer exponents etc.
An alternative monotone quantity
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An even more elementary result is the following:
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**Theorem (B-Bez-Iliopoulou)**

If $(p, q, d)$ is Schrödinger admissible, $q \in 2\mathbb{N}$ and $q \mid p$ then

$$t \mapsto C_{p,q} \| e^{t\Delta} f \|_2^p - \| e^{is\Delta} e^{t\Delta} f \|_{L_t^p L_x^q}^q$$

is nonincreasing (and indeed completely monotone).
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If \((p, q, d)\) is Schrödinger admissible, \(q \in 2\mathbb{N}\) and \(q|p\) then

\[ t \mapsto C_{p,q} \| e^{i\Delta t} f \|^p_2 - \| e^{i\mathcal{A} t} e^{i\Delta} f \|^q_{L^p_x L^q_x} \]

is nonincreasing (and indeed completely monotone).

Proof. We know that if \(F = f \otimes f\) then

\[
\mathcal{F}(f) := \frac{1}{4} \| f \|^4_2 - \| e^{i\mathcal{A}} f \|^4_{L^4_{s,x}} = \frac{1}{4} (\| F \|^2_2 - \langle PF, F \rangle) = \frac{1}{4} \| F - PF \|^2_2
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since \(P\) is an orthogonal projection.
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since \(P\) is an orthogonal projection. Furthermore, if \(U = u \otimes u\) solves the heat equation then so does \(PU\) (since the isometries \(\rho\) commute with \(\Delta\)). Thus \(U - PU\) solves the heat equation, and so

\[
\frac{d}{dt} \mathcal{F}(u) = \frac{1}{2} \Re \int_{(\mathbb{R}^2)^2} \overline{(U - PU)} \Delta (U - PU) = -\|\nabla (U - PU)\|_2^2 \leq 0.
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\]

**Apparent shortcomings:** Appears to benefit from similarly fortuitous circumstances as previous result.
An alternative argument

We know that

$$\| e^{is\Delta} f \|_{L^4_{s,x}(\mathbb{R} \times \mathbb{R}^2)}^4 = \frac{1}{4} \langle \hat{P} \hat{F}, \hat{F} \rangle$$
An alternative argument

We know that

$$\| e^{is\Delta} f \|_{L^4_{s,x} (\mathbb{R} \times \mathbb{R}^2)}^4 = \frac{1}{4} \langle \hat{P} \hat{F}, \hat{F} \rangle = \int_{(\mathbb{R}^2)^2} \int_{(\mathbb{R}^2)^2} \hat{F}(\xi) \hat{F}(\eta) d\Sigma_\xi(\eta) d\xi,$$

where $d\Sigma_\xi(\eta) = \delta(\xi_1 + \xi_2 - \eta_1 - \eta_2) \delta(|\xi_1|^2 + |\xi_2|^2 - |\eta_1|^2 - |\eta_2|^2) d\eta$. 
An alternative argument

We know that

\[ \left\| e^{is\Delta} f \right\|_{L^4_{s,x}(\mathbb{R} \times \mathbb{R}^2)}^4 = \frac{1}{4} \langle P \hat{F}, \hat{F} \rangle = \int_{(\mathbb{R}^2)^2} \int_{(\mathbb{R}^2)^2} \hat{F}(\xi)\hat{F}(\eta) d\Sigma_\xi(\eta) d\xi, \]

where \( d\Sigma_\xi(\eta) = \delta(\xi_1 + \xi_2 - \eta_1 - \eta_2)\delta(|\xi_1|^2 + |\xi_2|^2 - |\eta_1|^2 - |\eta_2|^2) d\eta. \) Next observe that

\[ \int_{(\mathbb{R}^2)^2} d\Sigma_\xi(\eta) = \frac{1}{4} P1(\xi) \equiv \frac{1}{4}. \]
An alternative argument

We know that

$$\| e^{is\Delta f} \|_{L^4_t L^4_x(\mathbb{R} \times \mathbb{R}^2)}^4 = \frac{1}{4} \langle P \hat{F}, \hat{F} \rangle = \int_{(\mathbb{R}^2)^2} \int_{(\mathbb{R}^2)^2} \hat{F}(\xi)\overline{\hat{F}(\eta)}d\Sigma_\xi(\eta)d\xi,$$

where $d\Sigma_\xi(\eta) = \delta(\xi_1 + \xi_2 - \eta_1 - \eta_2)\delta(|\xi_1|^2 + |\xi_2|^2 - |\eta_1|^2 - |\eta_2|^2)d\eta$. Next observe that

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An alternative argument

We know that

$$\| e^{is\Delta} f \|_{L^4_{s,x}(\mathbb{R} \times \mathbb{R}^2)}^4 = \frac{1}{4} \langle P \hat{F}, \hat{F} \rangle = \int_{(\mathbb{R}^2)^2} \int_{(\mathbb{R}^2)^2} \hat{F}(\xi)\overline{\hat{F}(\eta)} d\Sigma_\xi(\eta) d\xi,$$

where $d\Sigma_\xi(\eta) = \delta(\xi_1 + \xi_2 - \eta_1 - \eta_2)\delta(|\xi_1|^2 + |\xi_2|^2 - |\eta_1|^2 - |\eta_2|^2) d\eta$. Next observe that

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Using this, along with $d\Sigma_\xi(\eta) d\xi = d\Sigma_\eta(\xi) d\eta$, gives

$$\| f \|_{2}^4 = \| \hat{f} \|_{2}^4 = \| \hat{F} \|_{2}^2 = 4 \int_{(\mathbb{R}^2)^2} \int_{(\mathbb{R}^2)^2} |\hat{F}(\xi)|^2 d\Sigma_\xi(\eta) d\xi$$

$$= 2 \int_{(\mathbb{R}^2)^2} \int_{(\mathbb{R}^2)^2} \left(|\hat{F}(\xi)|^2 + |\hat{F}(\eta)|^2 \right) d\Sigma_\xi(\eta) d\xi.$$
Thus

\[ \mathcal{F}(f) := \frac{1}{4} \left\| f \right\|_2^4 - \left\| e^{is\Delta} f \right\|_{L^4_{s,x}}^4 \]

\[ = \frac{1}{2} \int_{(\mathbb{R}^2)^2} \int_{(\mathbb{R}^2)^2} \left( |\hat{F}(\xi)|^2 + |\hat{F}(\eta)|^2 - 2\hat{F}(\xi)\overline{\hat{F}(\eta)} \right) d\Sigma_\xi(\eta) d\xi \]

\[ = \frac{1}{2} \int_{(\mathbb{R}^2)^2} \int_{(\mathbb{R}^2)^2} |\hat{F}(\xi) - \hat{F}(\eta)|^2 d\Sigma_\xi(\eta) d\xi, \]
Thus

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\[ = \frac{1}{2} \int (\mathbb{R}^2)^2 \int (\mathbb{R}^2)^2 |\hat{F}(\xi) - \hat{F}(\eta)|^2 d\Sigma_\xi(\eta) d\xi, \]

and so

\[ \mathcal{F}(e^{t\Delta} f) = \frac{1}{2} \int (\mathbb{R}^2)^2 \int (\mathbb{R}^2)^2 |e^{-t|\xi|^2} \hat{F}(\xi) - e^{-t|\eta|^2} \hat{F}(\eta)|^2 d\Sigma_\xi(\eta) d\xi \]

\[ = \frac{1}{2} \int (\mathbb{R}^2)^2 \int (\mathbb{R}^2)^2 e^{-2t|\xi|^2} |\hat{F}(\xi) - \hat{F}(\eta)|^2 d\Sigma_\xi(\eta) d\xi, \]

since \(|\xi|^2 = |\xi_1|^2 + |\xi_2|^2 = |\eta_1|^2 + |\eta_2|^2 = |\eta|^2\) on the support of \(d\Sigma_\xi(\eta)\).
Thus

\[ \mathcal{F}(f) := \frac{1}{4} \| f \|_2^4 - \| e^{is\Delta} f \|_{L^4_{s,x}}^4 \]

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Thus \( \mathcal{F}(e^{t\Delta} f) \) is completely monotone.
Generalisations
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Approach extends considerably to an array of sharp Sobolev–Strichartz inequalities;
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Approach extends considerably to an array of sharp Sobolev–Strichartz inequalities; for example

\[ \| (-\Delta)^{\frac{2-d}{4}} e^{is\Delta} f \|^2_{L^2(\mathbb{R} \times \mathbb{R}^d)} \leq \text{OT}(d) \| f \|^4_{L^2(\mathbb{R}^d)}. \]

Here \( d \geq 2, \)

\[ \text{OT}(d) = \frac{2^{-d} \pi^{\frac{2-d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \]

is the optimal constant (Ozawa–Tsutsumi), and is attained on gaussians.
Approach extends considerably to an array of sharp Sobolev–Strichartz inequalities; for example

\[ \| ( -\Delta ) \frac{2-d}{4} | e^{is\Delta} f |^2 \|_{L^2(\mathbb{R} \times \mathbb{R}^d)}^2 \leq \text{OT}(d) \| f \|_{L^2(\mathbb{R}^d)}^4. \]

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is the optimal constant (Ozawa–Tsutsumi), and is attained on gaussians. This is a variant of the classical Sobolev–Strichartz estimate

\[ \| e^{is\Delta} f \|_{L^4(\mathbb{R} \times \mathbb{R}^d)} \lesssim \| f \|_{\dot{H}^{\frac{d-2}{4}}(\mathbb{R}^d)}. \]
Generalisations

Approach extends considerably to an array of sharp Sobolev–Strichartz inequalities; for example

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Theorem (B–Bez–Iliopoulou)

Let \( d \geq 2 \) and \( f \in L^2(\mathbb{R}^d). \) Then

\[ t \mapsto \text{OT}(d)\| e^{t\Delta} f \|_{L^2(\mathbb{R}^d)}^4 - \|(-\Delta)^{\frac{2-d}{4}} e^{is\Delta} e^{t\Delta} f\|^2_{L^2(\mathbb{R} \times \mathbb{R}^d)} \]

is completely monotone (decreasing).
The wave equation

**Strichartz inequalities:** if $p = \frac{2(d+1)}{d-1}$ then

$$\|e^{-isD}f\|_{L^p_s,\mathbb{R}^d} \leq c_d \|f\|_{\dot{H}^{1/2}({\mathbb{R}^d})},$$

where $D = \sqrt{-\Delta}$. 
Strichartz inequalities: if \( p = \frac{2(d+1)}{d-1} \) then

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\]

where \( D = \sqrt{-\Delta} \). Extremisers take form

\[
f(x) = (1 + |x|^2)^{- (d-1)/2} \quad \text{(equiv. } \hat{f}(\xi) = e^{-|\xi|/|\xi|})
\]

for \( d = 2, 3 \), and \( c_2 = (2\pi)^{-1/6}, c_3 = (2\pi)^{-1/4} \)
The wave equation

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f(x) = (1 + |x|^2)^{-\frac{(d-1)}{2}} \quad \text{(equiv. } \hat{f}(\xi) = e^{-|\xi|}/|\xi|)\]

for \( d = 2, 3 \), and \( c_2 = (2\pi)^{-1/6}, c_3 = (2\pi)^{-1/4} \) (Foschi 2006).
Strichartz inequalities: if \( p = \frac{2(d+1)}{d-1} \) then

\[
\| e^{-isD} f \|_{L^p_s, x(\mathbb{R} \times \mathbb{R}^d)} \leq c_d \| f \|_{\dot{H}^{1/2}(\mathbb{R}^d)},
\]

where \( D = \sqrt{-\Delta} \). Extremisers take form

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The wave equation

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– the \( L^2 \) restriction theorem for the cone.
Theorem (B-Bez-Ilioupoulou)

Let \( D = \sqrt{-\Delta} \). If \( p = \frac{2(d+1)}{d-1} \in 2\mathbb{N} \) (i.e. \( d = 2, 3 \))
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Proof in the case $d = 3$. Write

$$\| e^{isD} f \|_{L^4(\mathbb{R} \times \mathbb{R}^3)}^4 = \int_{(\mathbb{R}^3)^2} \int_{(\mathbb{R}^3)^2} \hat{F}(\xi) \hat{F}(\eta) d\Sigma_\xi(\eta) d\xi,$$

where now $F = D^{1/2} f \otimes D^{1/2} f$ and

$$d\Sigma_\xi(\eta) = \frac{\delta(\xi_1 + \xi_2 - \eta_1 - \eta_2) \delta(|\xi_1| + |\xi_2| - |\eta_1| - |\eta_2|)}{|\xi_1|^{1/2} |\xi_2|^{1/2} |\eta_1|^{1/2} |\eta_2|^{1/2}}.$$
Theorem (B-Bez-Ilioupoulou)

Let $\Delta = -\Delta$. If $p = \frac{2(d+1)}{d-1} \in 2\mathbb{N}$ (i.e. $d = 2, 3$) then

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Elementary calculations (see Foschi) reveal that

$$\int_{(\mathbb{R}^3)^2} \frac{|\xi_1|^{1/2} |\xi_2|^{1/2}}{|\eta_1|^{1/2} |\eta_2|^{1/2}} d\Sigma_\xi(\eta) \equiv \frac{1}{2\pi}. \quad (= \mu * \mu(|\xi_1| + |\xi_2|, \xi_1 + \xi_2))$$
Using this, together with $d\Sigma_\xi(\eta)d\xi = d\Sigma_\eta(\xi)d\eta$, we may write

$$
\frac{1}{2\pi} \| f \|_{{H^{1/2}}}^4 = \frac{1}{2\pi} \| \hat{F} \|_2^2 = \int_{(\mathbb{R}^3)^2} \int_{(\mathbb{R}^3)^2} |\hat{F}(\xi)|^2 \frac{|\xi_1|^{1/2} |\xi_2|^{1/2}}{|\eta_1|^{1/2} |\eta_2|^{1/2}} d\Sigma_\xi(\eta) d\xi
$$

$$
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where $\Phi(\xi, \eta) = \frac{|\xi_1|^{1/2} |\xi_2|^{1/2}}{|\eta_1|^{1/2} |\eta_2|^{1/2}}$. 
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where $\Phi(\xi, \eta) = \frac{|\xi_1|^{1/2}|\xi_2|^{1/2}}{|\eta_1|^{1/2}|\eta_2|^{1/2}}$. Thus, since $\Phi(\eta, \xi) = \Phi(\xi, \eta)^{-1}$,

$$\mathcal{F}(f) := \frac{1}{2\pi} \|f\|_{H^{1/2}}^4 - \|e^{-isD}f\|_{L^4}^4$$

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Using the support of $d\Sigma_\xi(\eta)d\eta$ we conclude that

$$
\mathcal{F}(e^{-tD}f) = \frac{1}{2} \int_{(\mathbb{R}^3)^2} \int_{(\mathbb{R}^3)^2} e^{-2t(|\xi_1|+|\xi_2|)} |\Phi(\xi, \eta)^{1/2} \hat{F}(\xi) - \Phi(\eta, \xi)^{1/2} \hat{F}(\eta)|^2 d\Sigma_\xi(\eta)d\xi,
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which is completely monotone.
Remark on the Stein–Tomas restriction theorem
Consider the Fourier extension operator

$$\mathcal{E} g(x, s) := \int_U g(\xi) e^{i(s\phi(\xi) + x \cdot \xi)} d\xi$$

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**Theorem**

*There exists a constant \( c < \infty \) such that the function \( Q : (0, \infty) \rightarrow \mathbb{R} \) given by*

\[ Q(t) = c \| g_t \|^4_{L^2(U)} - \| \mathcal{E} g_t \|^4_{L^4_{x,s}(\mathbb{R}^2 \times \mathbb{R})} \]

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\[ (-1)^k Q^{(k)}(t) \geq \int_{U^4} (\phi(\xi_1) + \phi(\xi_2))^k |g_t \otimes g_t(\xi) - g_t \otimes g_t(\eta)|^2 d\Sigma_\xi(\eta) d\xi \]

where \( d\Sigma_\xi(\eta) = \delta(\phi(\xi_1) + \phi(\xi_2) - \phi(\eta_1) - \phi(\eta_2)) \delta(\xi_1 + \xi_2 - \eta_1 - \eta_2) d\eta \).
Can we remove the arithmetic condition on $p, q$?
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**Theorem**

If $(p, q, d)$ are Schrödinger admissible then there exists a constant $c \geq C_{p,q}$ such that

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**Proof.** Notice that this may be rewritten as the monotonicity of the expression

$$t \mapsto \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} |u(s, t, x)|^q ds \right)^{p/q} dx - c^p \left( \int_{\mathbb{R}^2} |u(0, t, x)|^2 dx \right)^{p/2}$$

where $u : \mathbb{R} \times (0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{C}$ satisfies the equations

$$\frac{\partial u}{\partial t} = i \frac{\partial u}{\partial s} = \Delta_x u.$$
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Taking a Strichartz angle on Kakeya (and more general X-ray) problems means looking at the kinetic transport equation:

$$\partial_s f(s, x, v) + v \cdot \nabla_x f(s, x, v) = 0, \quad f(0, x, v) = f^0(x, v)$$

for \((s, x, v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d\).
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for \((s, x, v) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d\).

It is easy to see that the “velocity average” (aka “macroscopic density”)

$$\rho(f^0)(s, x) := \int_{\mathbb{R}^d} f(s, x, v) dv = \int_{\mathbb{R}^d} f^0(x - vs, v) dv$$

satisfies dispersive and Strichartz estimates very much like \(|u|^2\), where \(u\) solves the Schrödinger equation.
The kinetic transport equation - observations

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$$\|\rho(f^0)\|_{L^q_x L^p_x} \lesssim \|f^0\|_{L^a_x, v},$$

if and only if

$$q > a, \quad p \geq a, \quad \frac{2}{q} = d\left(1 - \frac{1}{p}\right), \quad \frac{1}{a} = \frac{1}{2} \left(1 + \frac{1}{p}\right); \quad \text{ (“transport admissible”).}$$

Taking a Strichartz angle on Kakeya (and more general X-ray) problems means looking at the *kinetic transport equation*:

\[
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quadratic_expression
dq_expression
\]

Due to Castella–Perthame 1996, Keel–Tao 1997, B–Bez–Gutiérrez–Lee 2013; sharp forms when \(p = q = \frac{d+2}{d}\) (Drouot 2010; see also Flock 2013).
From a monotonicity point of view it is natural to consider the sharp inequality of Drouot:

$$\| \rho(f^0) \|_{L^{\frac{d+2}{d}}_{s,x}} \leq C \| f^0 \|_{L^{\frac{d+2}{d+1}}_{x,y}},$$
From a monotonicity point of view it is natural to consider the sharp inequality of Drouot:

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By duality this is equivalent to

\[ \| \rho^*(g) \|_{L^{d+2}_{x,v}} \leq C \| g \|_{L^{d+2}_{s,x}} \]

where

\[ \rho^*(g)(x, v) = \int_{\mathbb{R}} g(s, x + vs) ds, \]

which is space-time X-ray transform.
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$$g(s, x) = \frac{1}{1 + s^2 + |x|^2}$$

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From a monotonicity point of view it is natural to consider the sharp inequality of Drouot:

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$$\partial_t u = \Delta(\varphi(u)),$$

for certain slowly-growing functions $\varphi$, may be suitable...
Consider a more general space-time $k$-plane transform

$$\rho_k^*(g)(x, v) = \int_{\mathbb{R}^k} g(s, x + vs)ds,$$

where $x \in \mathbb{R}^{d+1-k}$ and $v$ is a $(d+1-k) \times k$ matrix.
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where $x \in \mathbb{R}^{d+1-k}$ and $v$ is a $(d + 1 - k) \times k$ matrix.

Christ/Flock: $\|\rho_k^*(g)\|_{L_{x,v}^{d+2}} = C \|T_{k,d+1}(g)\|_{L_{x,v}^{d+2}(\mathcal{G}_{k,d+1})}$ where $T_{k,n}$ is the classical $k$-plane transform on $\mathbb{R}^n$.
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**Observation**

Suppose $d > 1$, $q = k = 2$, $p = \frac{2(d+1)}{d+3}$. If $u$ satisfies

$$\partial_t u = \Delta\left(u^{p/2}\right)$$

then $t \mapsto \mathcal{F}(u(t, \cdot))$ is nonincreasing.
Proof. Just a combination of the fact

\[ \| T_{2,n}(g) \|_{L^2(G_{2,n})}^2 = C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{g(x)g(y)}{|x-y|^{n-2}} \, dx \, dy \]  

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and the HLS monotonicity theorem of Carlen–Carrillo–Loss.
Final remark: multilinear Kakeya as a Strichartz estimate

Suppose $v_1^0, \ldots, v_{d+1}^0$ are fixed nonco-hyperplanar points in $\mathbb{R}^d$. 
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$$\rho_\ell(f_\ell)(s, x) = \int_{\mathbb{R}^d} f_\ell(s, x, v_\ell) d\mu_\ell(v_\ell).$$

Endpoint multilinear Kakeya inequality of Guth:

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \prod_{\ell=1}^{d+1} \rho_\ell(f_\ell)(s, x)^{1/d} dx ds \lesssim \prod_{\ell=1}^{d+1} \|f_\ell\|_{L_{x,v}^1}^{1/d}.$$
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If the $\mu_\ell = \delta_{v_\ell^0}$ then this is the Loomis–Whitney inequality, and choosing heat kernels $H_{t,j}$ appropriately, we find that (see B–Carbery–Christ–Tao 2008)

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t \mapsto \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left( \prod_{j=1}^{d+1} H_{j,t} \ast f_j(s, x, v_j^0) \right)^{1/d} \, dx ds
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is nondecreasing.
Final remark: multilinear Kakeya as a Strichartz estimate

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is nondecreasing. The following theorem is the product of a perturbative analysis of this fact.
Theorem (B–Carbery–Tao 2006)

Suppose $p > 1/d$. Then provided the supports of the measures $\mu_\ell$ are sufficiently small, the quantity

$$Q_p(t) := \int_\mathbb{R} \int_\mathbb{R}^d \left( \prod_{j=1}^{d+1} H_{j,t} \ast \rho_j(f_j)(s,x) \right)^p dx ds \equiv \int_\mathbb{R} \int_\mathbb{R}^d \left( \prod_{j=1}^{d+1} \rho_j(H_{j,t} \ast f_j)(s,x) \right)^p dx ds$$

is approximately monotone in the sense that $Q_p(t) \lesssim Q_p(t')$ whenever $0 < t \leq t'$.

Rescaling we see that

$$Q_p(t^{-2}) = \int_\mathbb{R} \int_\mathbb{R}^d \left( \prod_{j=1}^{d+1} \int_\mathbb{R}^d \int_\mathbb{R}^d e^{-\langle A_j(x-ty_j-sv_j),(x-ty_j-sv_j)\rangle} f_j(y_j,v_j) dy_j d\mu_j(v_j) \right)^p dx ds$$

which reveals a “double transport” ingredient, where $s$ and $t$ take somewhat similar roles prior to the $s$ integration.
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**Question:** is there a suitable diffusion for endpoint multilinear Kakeya?