Lusin’s condition and the distributional determinant for deformations with finite energy

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Abstract. Based on a previous work by the authors on the modelling of cavitation and fracture in nonlinear elasticity, we give an alternative proof of a recent result by Csörnyei, Hencl and Malý on the regularity of the inverse of homeomorphisms in the Sobolev space $W^{1,n-1}$. With this aim, we show that the notion of fracture surface introduced by the authors in their model corresponds precisely to the original notion of cavity surface in the cavitation models of Müller and Spector (1995) and Conti and De Lellis (2003). We also find that the surface energy introduced in the model for cavitation and fracture is related to Lusin’s condition (N) on the non-creation of matter.

A fundamental question underlying this paper is whether $\text{Det } D\mathbf{u} \equiv \det D\mathbf{u}$ necessarily implies that the deformation $\mathbf{u}$ opens no cavities. We show that this is not true unless Müller and Spector’s condition INV for the non-interpenetration of matter is satisfied. Having thus provided an additional justification of its importance, we prove the stability of this condition with respect to weak convergence in the critical space $W^{1,n-1}$. Combining this with the work by Conti and De Lellis, we obtain an existence theory for cavitation in this critical case.

Keywords. Elastic deformations, cavitation, surface energy, created surface, distributional determinant, Lusin’s condition, non-interpenetration of matter.

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1 Introduction

In nonlinear elasticity theory, the elastic energy of a deformation $\mathbf{u} : \Omega \to \mathbb{R}^n$ of a body $\Omega \subset \mathbb{R}^n$ is given by

$$\int_{\Omega} W(\mathbf{x}, D\mathbf{u}(\mathbf{x})) \, d\mathbf{x},$$

(1.1)

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where \( n \) is the space dimension, and \( W : \Omega \times \mathbb{R}^{n \times n} \to \mathbb{R} \cup \{\infty\} \) is the elastic stored-energy function of the material. The seminal paper of Ball [2] proves existence of minimizers of (1.1) in a suitable class of Sobolev functions \( u \), under certain coercivity and polyconvexity assumptions on \( W \). When cavitation is considered, the total energy of a deformation \( u \) will be the sum of the elastic energy (1.1) plus a term accounting for the energy needed to produce the cavities.

Müller and Spector [28] proposed the term \( \text{Per} u(\Omega) \) as an energy due to cavitation. Here \( \text{Per} \) denotes the perimeter of a set, and \( u(\Omega) \) is the image of \( \Omega \) under \( u \) defined in a suitable way. Intuitively, \( \text{Per} u(\Omega) \) measures the area of the cavities created by \( u \) together with the area of \( u(\partial\Omega) \). They pointed out, however, that, in some instances, the term \( \text{Per} u(\Omega) \) fails to detect the area of the created cavities; this is related to the pathological behaviour in which a deformation can be one-to-one a.e., but still interpenetration of matter occurs. To overcome that inconvenience, they defined the topological condition INV (see Definition 2.12), which is related to the invertibility of the deformation. Then they put condition INV as a constraint in the admissible set of deformations, and proved the existence of minimizers within that admissible set. Moreover, they showed that the distributional determinant \( \text{Det} Du \) has the form

\[
\text{Det} Du = (\det Du)\mathcal{L}^n + \sum_{a \in C(u)} c_a \delta_a, \quad (1.2)
\]

where \( C(u) \subset \Omega \) is the countable family of cavitation points of \( u \), and where for each \( a \in C(u) \) the coefficient \( c_a \) is positive and gives the volume of the cavity created at \( a \).

Condition INV expresses, roughly, that a cavity created at one point cannot be filled by material from elsewhere. Its formulation relies on the topological degree, and is set in the \( W^{1,p} \) framework, with \( p > n - 1 \). Later, Conti and De Lellis [7], using degree theory for \( W^{1,n-1} \cap L^\infty \) functions, generalized condition INV and some other aspects of Müller and Spector’s theory to the case of deformations in \( W^{1,n-1} \cap L^\infty \).

An alternative theory for cavitation (which also includes fracture) was given by the authors in [17]. Our idea was to replace the term \( \text{Per} u(\Omega) \) as a surface energy by \( \mathcal{E}(u) \), defined as the supremum, when \( f \in C^1_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n) \) and \( \|f\|_\infty \leq 1 \), of the quantity

\[
\int_\Omega \left[ D_x f(x, u(x)) \cdot \text{cof} Du(x) + \text{div}_y f(x, u(x)) \det Du(x) \right] \, dx.
\]

We thus proved existence of minimizers for the total energy

\[
\int_\Omega W(x, Du(x)) \, dx + \mathcal{E}(u), \quad (1.3)
\]
without imposing any topological condition in the admissible set of deformations. In [18] we also showed that the condition $\mathcal{E}(u) < \infty$ implies an $SBV$ regularity property for the inverse of the deformation $u$, and, based on this result, we introduced a new notion of created surface.

This paper is divided into two parts. In the first part (Sections 3–5) we compare the approach of [17, 18] with related work of Csörgő, Hencl, Koskela, Malý, and Onninen (see [8, 19–22, 29]) on the regularity of the inverse, and with the models for cavitation of Müller and Spector [28] and of Conti and De Lellis [7]. In particular, we show that:

1. the surface energy $\mathcal{E}(u)$ is essentially equivalent to $\text{Per}(u(\Omega))$ for the purpose of analyzing cavitation in the Sobolev setting;
2. our notion of created surface coincides with that of [7, 28], based on Šverák’s definition of topological image [36]; and
3. if a homeomorphism $u : \Omega \to \mathbb{R}^n$ belongs to $W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$, is such that $\det Du > 0$ a.e., and satisfies Lusin’s condition, then one has $u^{-1} \in W^{1,1}(u(\Omega), \mathbb{R}^n)$.

The second part of this paper (Sections 6–8) discusses the invertibility of the deformation and the non-interpenetration of matter. This is motivated by the fact that, based on Mucci’s work [25] on the modelling of fracture within the theory of Cartesian currents, it is possible to obtain Müller and Spector’s characterization of the distributional determinant (see (1.2)) even for deformations that are not one-to-one. However, we show that if condition INV is not satisfied, then the coefficients $c_a$ in formula (1.2) cannot be given the interpretation of cavity volumes, and the distributional determinant does not provide a full description of the cavities created by the deformation. Having justified the importance of condition INV, we finally prove, combining the work of Conti and De Lellis [7] with the result in [17] on the convergence of the geometric images, the existence of minimizers for the functional (1.3) among a class of functions in $W^{1,p}$ satisfying condition INV, in the critical case $p = n - 1$.

## 2 Notation and preliminaries

In this section we set the general notation of the paper and state some important preliminary results.

### 2.1 General notation

We will work in dimension $n$ and tacitly assume that $n \geq 2$. Our basic object will be the deformation, which is a Sobolev map $u : \Omega \to \mathbb{R}^n$ satisfying certain con-
ditions. Throughout the paper, \( \Omega \) is a bounded open set of \( \mathbb{R}^n \). Vector-valued and matrix-valued quantities will be written in boldface. Coordinates in the reference configuration will generically be denoted by \( \mathbf{x} \), while coordinates in the deformed configuration by \( \mathbf{y} \).

The closure of a set \( A \) is denoted by \( \overline{A} \), and its boundary by \( \partial A \). Given two open sets \( U, V \) of \( \mathbb{R}^n \), we will say that \( U \) is compactly contained in \( V \) if \( U \) is bounded and \( \overline{U} \subset V \); in this case, we will write \( U \subset\subset V \). The open ball of radius \( r > 0 \) centred at \( \mathbf{x} \in \mathbb{R}^n \) is denoted by \( B(x,r) \), whereas the closed ball is denoted by \( \overline{B}(x,r) \). Unless otherwise stated, a ball will always be an open ball.

The identity matrix is denoted by \( \mathbf{1} \). Given a square matrix \( A \in \mathbb{R}^{n \times n} \), its transpose is denoted by \( A^T \), and its determinant by \( \det A \). The cofactor matrix of \( A \), denoted by \( \text{cof} A \), is the matrix that satisfies
\[
\det A = \mathbf{1} \times A^T \text{cof} A.
\]
The transpose of \( \text{cof} A \) is the adjugate matrix \( \text{adj} A \). If \( A \) is invertible, its inverse is denoted by \( A^{-1} \), and the transpose of its inverse by \( A^{-T} \). The inner (dot) product of vectors and of matrices will be denoted by \( \cdot, \cdot \). The Euclidean norm of a vector \( \mathbf{x} \) is denoted by \( |\mathbf{x}| \), and the associated matrix norm is also denoted by \( |\cdot| \).

Unless otherwise stated, expressions like measurable or a.e. refer to the Lebesgue measure in \( \mathbb{R}^n \), which is denoted by \( \mathcal{L}^n \). We will use expressions like “for all \( \mathbf{x} \in \Omega \) and a.e. \( r \in (0, \text{dist}(\mathbf{x}, \partial \Omega)) \), property (P) holds”. This means that for each \( \mathbf{x} \in \Omega \) there exists an \( \mathcal{L}^1 \)-null set \( N_\mathbf{x} \) such that property (P) holds for all \( r \in (0, \text{dist}(\mathbf{x}, \partial \Omega)) \setminus N_\mathbf{x} \). The \((n-1)\)-dimensional Hausdorff measure will be indicated by \( \mathcal{H}^{n-1} \).

The Lebesgue \( L^p \) and Sobolev \( W^{1,p} \) spaces are defined in the usual way. So are the set of smooth functions \( C^\infty \), of bounded variation \( BV \), and of special bounded variation \( SBV \); see [1], if necessary, for the definitions. The set \( C_c^\infty(\Omega, \mathbb{R}^n) \) denotes the space of \( C^\infty \) functions with compact support in \( \Omega \). We will always indicate the domain and target space, as in, for example, \( L^p(\Omega, \mathbb{R}^n) \), except if the target space is \( \mathbb{R} \), in which case we will simply write \( L^p(\Omega) \). As usual, \( f_A \) denotes \( \frac{1}{\mathcal{L}^n(A)} \int_A f \) for any measurable set \( A \). Sometimes we will use Lebesgue spaces in \((n-1)\)-dimensional sets; for example, if \( \Omega \) is a set with a Lipschitz boundary, then \( L^1(\partial \Omega) \) denotes the Lebesgue \( L^1 \) space on \( \partial \Omega \) with respect to the \( \mathcal{H}^{n-1} \) measure. From the context it will be clear that these spaces are defined with respect to the \( \mathcal{H}^{n-1} \) measure, and not to the \( \mathcal{L}^n \) measure, so we will not indicate it explicitly.

With \( \langle \cdot, \cdot \rangle \) we will indicate the duality product between a measure and a continuous function, or between a distribution and a smooth function. The identity function in \( \mathbb{R}^n \) is denoted by \( \text{id} \).

The divergence operator is denoted by \( \text{Div} \) in the reference configuration, and by \( \text{div} \) in the deformed configuration. More precisely, the expression \( \text{Div} \phi \) is used for functions \( \phi \) defined on \( \Omega \) (i.e., on the \( \mathbf{x} \) variables), while \( \text{div} \mathbf{g} \) is used for functions \( \mathbf{g} \) defined on the target space (so the differentiation is with respect to \( \mathbf{y} \)).
If $\mu$ is a measure on a set $U$ and if $V$ is a $\mu$-measurable subset of $U$, then the restriction of $\mu$ to $V$ is the measure on $U$, denoted by $\mu|_V$, that satisfies $\mu|_V(A) = \mu(A \cap V)$ for all $\mu$-measurable sets $A$. The measure $|\mu|$ denotes the total variation of $\mu$. The support of a measure or of a function is denoted by $\text{spt}$. For each $x \in \Omega$, the measure $\delta_x$ indicates the Dirac delta at $x$.

Given two sets $A, B$ in $\mathbb{R}^n$, we write $A \subsetneq B$ if $\mathcal{H}^{n-1}(A \setminus B) = 0$. We write $A \approx B$ if $A \subsetneq B$ and $B \subsetneq A$. We write $A = B$ a.e. if $\mathcal{L}^n(A \setminus B) = \mathcal{L}^n(B \setminus A) = 0$.

A simple result that will be used in Sections 4 and 7 is the following.

**Lemma 2.1.** For each $k \in \{0, 1, \ldots, \infty\}$, the linear span generated by the set of $f \in C^\infty_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ such that there exist $\phi \in C^\infty_c(\Omega)$ and $g \in C^\infty_c(\mathbb{R}^n, \mathbb{R}^n)$ for which

$$f(x, y) = \phi(x) g(y) \quad \text{for all } (x, y) \in \Omega \times \mathbb{R}^n$$

is dense in $C^k_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$.

### 2.2 Boundary and perimeter

Given a measurable set $A \subset \mathbb{R}^n$, its characteristic function will be denoted by $\chi_A$. The perimeter of $A$ is defined as

$$\text{Per } A := \sup \left\{ \int_A \text{div } g(y) \, dy : g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n), \|g\|_\infty \leq 1 \right\}.$$  

Half-spaces are denoted by

$$H^+(a, v) := \{ x \in \mathbb{R}^n : (x - a) \cdot v \geq 0 \}, \quad H^-(a, v) := H^+(a, -v),$$
for a given $a \in \mathbb{R}^n$ and a nonzero vector $v \in \mathbb{R}^n$. The set of unit vectors in $\mathbb{R}^n$ is denoted by $S^{n-1}$.

Given a measurable set $A \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, the density of $A$ at $x$ is defined as

$$D(A, x) := \lim_{r \searrow 0} \frac{\mathcal{L}^n(B(x, r) \cap A)}{\mathcal{L}^n(B(x, r))}$$

if that limit exists, whereas the upper density is defined as

$$\tilde{D}(A, x) := \limsup_{r \searrow 0} \frac{\mathcal{L}^n(B(x, r) \cap A)}{\mathcal{L}^n(B(x, r))}.$$  

**Definition 2.2.** Let $A$ be a measurable set of $\mathbb{R}^n$. We define the reduced boundary of $A$, and denote it by $\partial^* A$, as the set of points $y \in \mathbb{R}^n$ for which a unit vector $v_A(y)$ exists such that

$$D(A \cap H^-(y, v_A(y)), y) = \frac{1}{2} \quad \text{and} \quad D(A \cap H^+(y, v_A(y)), y) = 0.$$
This \( \nu_A(y) \) is uniquely determined and is called the unit outward normal to \( A \). The essential boundary of \( A \) is denoted by \( \partial^e A \), and defined as the set of \( y \in \mathbb{R}^n \) such that

\[
\bar{D}(A,y) > 0 \quad \text{and} \quad \bar{D}(\mathbb{R}^n \setminus A,y) > 0.
\]

These definitions of boundary, as well as their notation, may differ from other usual definitions, but thanks to Federer’s theorem [15] (see also [1, Theorem 3.61] or [37, Section 5.6]) they coincide \( H^n \)-a.e. with all other usual definitions of reduced (or essential or measure-theoretic) boundary for sets of finite perimeter.

In particular, \( \partial^* A \subset \partial^e A \) for any measurable set \( A \subset \mathbb{R}^n \), and if \( A \) has finite perimeter, then \( \partial^* A \equiv \partial^e A \).

### 2.3 Approximate differentiability, area formulas and geometric image

We will say that the measurable map \( u : \Omega \to \mathbb{R}^n \) is approximately differentiable at \( x_0 \in \Omega \) if there exists \( L \in \mathbb{R}^{n \times n} \) such that for any \( \delta > 0 \),

\[
D \left( \left\{ x \in A \setminus \{x_0\} : \frac{|u(x) - u(x_0) - L(x - x_0)|}{|x - x_0|} \geq \delta \right\}, x_0 \right) = 0.
\]

In this case, \( L \) (which is uniquely determined) is called the approximate differential of \( u \) at \( x_0 \). It is well known that a \( W^{1,1} \) function is approximately differentiable almost everywhere, and that the approximate differential coincides a.e. with the distributional derivative. Note that we do not identify functions that coincide almost everywhere.

If \( u : \Omega \to \mathbb{R}^n \) is a function of locally bounded variation, \( Du \) denotes the distributional derivative of \( u \), which is a Radon measure in \( \Omega \).

We recall now the classic area formula (or change of variables formula) of Federer [15]. In the statement below, \( N(u,A,y) \) denotes the number of preimages of a point \( y \) in the set \( A \) under \( u \). The formulation is taken from [28, Proposition 2.6].

**Proposition 2.3.** Let \( u \in W^{1,1}(\Omega,\mathbb{R}^n) \), and denote the set of approximate differentiability points of \( u \) by \( \Omega_d \). Then, for any measurable set \( A \subset \Omega \) and any measurable function \( \varphi : \mathbb{R}^n \to \mathbb{R} \),

\[
\int_A (\varphi \circ u)|\det Du| \, dx = \int_{\mathbb{R}^n} \varphi(y)N(u,\Omega_d \cap A,y) \, dy
\]

whenever either integral exists. Moreover, if a map \( \psi : A \to \mathbb{R} \) is measurable and \( \tilde{\psi} : u(\Omega_d \cap A) \to \mathbb{R} \) is given by

\[
\tilde{\psi}(y) := \sum_{x \in \Omega_d \cap A \atop u(x) = y} \psi(x),
\]
then $\tilde{\psi}$ is measurable and

$$
\int_A \psi(\varphi \circ u) | \det Du | \, dx = \int_{u(\Omega_d \cap A)} \tilde{\psi} \varphi \, dy, \quad y \in u(\Omega_d \cap A), 
$$

(2.1)

whenever the integral on the left-hand side of (2.1) exists.

As a consequence of Proposition 2.3, if $u \in W^{1,1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ is one-to-one a.e., then $\det Du \in L^1(\Omega)$.

One of the difficulties in the analysis of $W^{1,p}$ maps, with $p < n$, is that the image of sets of measure zero may have positive measure (see, e.g., the examples by Besicovitch [5], Ponomarev [30], and Malý and Martio [24]). The area formula of Proposition 2.3 has been used to overcome this difficulty (e.g., in [7, 28]). More precisely, it has given rise to the notion of the geometric image (or measure-theoretic image, using the expression in [28]) of a measurable set $A \subset \Omega$ under an approximately differentiable map $u : \Omega \to \mathbb{R}^n$. This was defined as $u(A \cap \Omega_d)$ by Müller and Spector [28]; for technical convenience, however, we use the following definition, which is an adaptation of that of Conti and De Lellis [7].

**Definition 2.4.** Let $u \in W^{1,1}(\Omega, \mathbb{R}^n)$ and suppose that $\det Du > 0$ a.e. Define $\Omega_0$ as the set of $x \in \Omega$ for which the following are satisfied:

(i) the approximate differential of $u$ at $x$ exists and equals $Du(x)$,

(ii) there exist $w \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and a compact set $K \subset \Omega$ of density 1 at $x$ such that $u|_K = w|_K$ and $Du|_K = Dw|_K$, and

(iii) $\det Du(x) > 0$.

For any measurable set $A$ of $\Omega$, we define the geometric image of $A$ under $u$ as $u(A \cap \Omega_0)$, and denote it by $\text{im}_G(u, A)$.

Standard arguments, essentially due to Federer [15, Theorems 3.1.8 and 3.1.16] (see also [28, Proposition 2.4] and [7, Remark 2.5]), show that the set $\Omega_0$ in the definition above is of full measure in $\Omega$.

As an immediate consequence of Proposition 2.3, the following result holds.

**Lemma 2.5.** Let $u \in W^{1,1}(\Omega, \mathbb{R}^n)$ satisfy $\det Du > 0$ a.e. The following properties hold:

(i) If $E \subset \Omega$ is an $\mathcal{L}^n$-null set then $\text{im}_G(u, E)$ is an $\mathcal{L}^n$-null set.

(ii) If $F \subset \mathbb{R}^n$ is an $\mathcal{L}^n$-null set, then $\{x \in \Omega : u(x) \in F\}$ is an $\mathcal{L}^n$-null set.

Consequently, it is equivalent (up to $\mathcal{L}^n$-null sets) to define the geometric image of $A$ as $u(A \cap \Omega_d)$ or as $u(A \cap \Omega_0)$. 


We shall use the following result by Müller and Spector [28, Lemma 2.5].

**Lemma 2.6.** Let \( u \in W^{1,1}(\Omega, \mathbb{R}^n) \) satisfy \( \det Du > 0 \) a.e. Let \( \Omega_0 \) be as in Definition 2.4. Take \( x \in \Omega_0 \) and let \( A \subset \Omega \) be measurable. Then

\[
D(\text{im}_G(u, A), u(x)) = 1 \quad \text{whenever} \quad D(A, x) = 1.
\]

Moreover, if \( v \in S^{n-1} \) and we define \( \tilde{v} := (\text{sgn} \det Du(x))(\text{cof} Du(x))v \), then

\[
D(\text{im}_G(u, A) \cap H^+(u(x), \tilde{v}), u(x)) = \frac{1}{2} \quad \text{whenever} \quad D(A \cap H^+(x, v), x) = \frac{1}{2}.
\]

The counterpart of Proposition 2.3 above for surfaces is the \((n-1)\)-dimensional change of variables formula for approximately differentiable maps, due to Federer [15, Corollary 3.2.20]. Before stating it, we recall briefly the notion of tangential approximate differentiability and some notation from multilinear algebra.

**Definition 2.7** (cf. [15, Definition 3.2.16]). Let \( S \subset \mathbb{R}^n \) be a \( C^1 \) differentiable manifold of dimension \( n-1 \), and let \( x_0 \in S \). Let \( T_{x_0}S \) be the linear tangent space of \( S \) at \( x_0 \). A map \( u : S \to \mathbb{R}^n \) is said to be \( \mathcal{H}^{n-1}_1 \) \( S \)-approximately differentiable at \( x_0 \) if there exists \( L \in \mathbb{R}^{n \times n} \) such that for all \( \delta > 0 \),

\[
\lim_{r \searrow 0} \frac{1}{r^{n-1}} \mathcal{H}^{n-1}_1 \left( \left\{ x \in S \cap B(x_0, r) : \frac{|u(x) - u(x_0) - (L(x-x_0))|}{|x-x_0|} \geq \delta \right\} \right) = 0.
\]

The linear map \( L|_{T_{x_0}S} : T_{x_0}S \to \mathbb{R}^n \) is uniquely determined and called the tangential approximate derivative of \( u \) at \( x_0 \). We denote it by \( \nabla u(x_0) \).

It is well known that if \( u \) is a Sobolev function, then \( u \) is approximately differentiable a.e., and \( Du = \nabla u \) a.e. In this case, the notation \( Du \) will be preferred.

If \( L : V \to \mathbb{R}^n \) is a linear map, where \( V \) is an \((n-1)\)-dimensional subspace of \( \mathbb{R}^n \) (such as \( T_{x_0}S \)), the transformation \( \Lambda_{n-1}L : \Lambda_{n-1}V \to \mathbb{R}^n \) is defined by

\[
(\Lambda_{n-1}L)(a_1 \wedge \cdots \wedge a_{n-1}) = La_1 \wedge \cdots \wedge La_{n-1}, \quad a_1, \ldots, a_{n-1} \in V.
\]

Here \( \wedge \) denotes the exterior product between vectors in \( \mathbb{R}^n \). The space \( \Lambda_{n-1}\mathbb{R}^n \), which consists of all alternating \((n-1)\)-tensors on \( \mathbb{R}^n \), is identified with \( \mathbb{R}^n \). We are interested in the following property of \( \Lambda_{n-1}L \). Since the one-dimensional subspace \( \Lambda_{n-1}V \) can be identified with \( \{\lambda v : \lambda \in \mathbb{R}\} \), where \( v \) is one of the two unit vectors normal to \( V \), the linear transformation \( \Lambda_{n-1}L \) is determined by the value of \( (\Lambda_{n-1}L)v \). This value can be computed as

\[
(\Lambda_{n-1}L)v = (\text{cof} \tilde{L})v, \quad (2.2)
\]

provided \( \tilde{L} : \mathbb{R}^n \to \mathbb{R}^n \) is a linear map that extends \( L \).
Proposition 2.8 (cf. [15, Corollary 3.2.20]). Let $S \subset \mathbb{R}^n$ be a $C^1$ differentiable manifold of dimension $n - 1$. Further, let $u : S \to \mathbb{R}^n$ be $\mathcal{H}^{n-1}$-measurable and $\mathcal{H}^{n-1}$-almost differentiable in $\mathcal{H}^{n-1}$-almost all $S$, and denote the set of points of $\mathcal{H}^{n-1}$-approximate differentiability of $u$ by $S_d$. Then, for any $\mathcal{H}^{n-1}$-measurable subset $A \subset S$,

$$
\int_A |\Lambda_{n-1} \nabla u(x)| \, d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^n} \nabla(u, S_d \cap A, y) \, d\mathcal{H}^{n-1}(y)
$$

whenever either integral exists.

Using the standard technique of approximating nonnegative functions by an increasing sequence of simple functions, we obtain the following result.

Proposition 2.9. Let $S \subset \Omega$ be an orientable $C^1$ differentiable manifold of dimension $n - 1$ oriented by the unit vector field $v$, and let $u \in W^{1,1}(\Omega, \mathbb{R}^n)$ satisfy $\det Du > 0$ a.e. Let $\Omega_0$ be the set given in Definition 2.4. Suppose that there is a set $S_d \subset \Omega_0 \cap S$ such that $\mathcal{H}^{n-1}(S \setminus S_d) = 0$, and such that for every $x \in S_d$ the restriction $u|_S$ is $\mathcal{H}^{n-1}$-almost differentiable at the point $x$, and $\nabla(u|_S)(x) = Du(x)|_{T_x S}$. Suppose, further, that $\text{cof} Du \in L^1(S, \mathbb{R}^{n \times n})$. Then, for every bounded and $\mathcal{H}^{n-1}$-measurable $g : \mathbb{R}^n \to \mathbb{R}^n$, and any $\mathcal{H}^{n-1}$-measurable subset $A \subset S$,

$$
\int_A g(u(x)) \cdot (\text{cof} Du(x))v(x) \, d\mathcal{H}^{n-1}(x) = \int_{u(S_d \cap A)} g(y) \cdot \tilde{v}(y) \, d\mathcal{H}^{n-1}(y),
$$

provided that the integral on the left-hand side of (2.3) exists, and where

$$
\tilde{v}(y) := \sum_{x \in S_d \cap A \atop u(x) = y} \frac{(\text{cof} Du(x))v(x)}{|(\text{cof} Du(x))v(x)|}, \quad y \in u(S_d \cap A).
$$

When we consider the restriction of Sobolev maps $u : \Omega \to \mathbb{R}^n$ to smooth hypersurfaces, in order to use Propositions 2.8 or 2.9, it is necessary to determine whether such an restriction is tangentially approximately differentiable, and whether the tangential approximate derivative can be obtained from the distributional derivative $Du$. This property indeed holds on ‘almost every’ hypersurface, as we will see in Section 2.6.

2.4 Degree, topological image and condition INV in $W^{1,n-1} \cap L^\infty$

In this subsection we recall the main definitions and properties of the degree for $W^{1,n-1} \cap L^\infty$ maps of Brezis and Nirenberg [6]. This degree was used by Conti and De Lellis [7] when considering the problem of modelling cavitation in neo-
Hookean materials. In doing so, they extended part of the theory of cavitation of Müller and Spector [28] developed in the $W^{1,p}$ case for $p > n - 1$.

The following proposition states the main properties of the degree.

**Proposition 2.10.** Let $U \subset \subset \mathbb{R}^n$ be a nonempty open set with a $C^1$ boundary, and suppose that $u \in W^{1,n-1}(\partial U, \mathbb{R}^n) \cap L^\infty(\partial U, \mathbb{R}^n)$. Then there exists a unique $L^1(\mathbb{R}^n)$ function, denoted $\deg(u, \partial U, \cdot)$, such that

$$
\int_{\partial U} g(u(x)) \cdot \Lambda_{n-1}(Du(x)) v(x) \, d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^n} \deg(u, \partial U, y) \, \text{div} \, g(y) \, dy
$$

for all $g \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, where $v$ denotes the unit exterior normal to $U$. Moreover, it satisfies the following properties:

(i) $\deg(u, \partial U, \cdot)$ is in $BV(\mathbb{R}^n)$ and takes only integer values.

(ii) $\deg(u, \partial U, y) = 0$ for a.e. $y \in \mathbb{R}^n \setminus B(0, \|u\|_{\infty})$.

(iii) If $u$ is continuous, then $\deg(u, \partial U, \cdot)$ coincides with the Brouwer degree a.e. in $\mathbb{R}^n \setminus u(\partial U)$.

The concept of topological image was introduced by Šverák [36] (see also [28]); we follow here the extension made by Conti and De Lellis [7, Definition 3.5].

**Definition 2.11.** Let $U \subset \subset \mathbb{R}^n$ be a nonempty open set with a $C^1$ boundary, and suppose that $u \in W^{1,n-1}(\partial U, \mathbb{R}^n) \cap L^\infty(\partial U, \mathbb{R}^n)$. We define the topological image of $U$ under $u$ as

$$
im_T(u, U) := \{ y \in \mathbb{R}^n : D(A_{u,U}, y) = 1 \},$$

where $A_{u,U} := \{ y \in \mathbb{R}^n : \deg(u, \partial U, y) \neq 0 \}$.

It is easy to check that the topological image of $U$ under $u$ does not depend on the representative of $u$.

Note that if we are given $u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$, then for every $x \in \Omega$ and a.e. $r \in (0, \text{dist}(x_0, \partial \Omega))$, the restriction $u|_{\partial B(x,r)}$ belongs to the space $W^{1,n-1}(\partial B(x, r), \mathbb{R}^n) \cap L^\infty(\partial B(x, r), \mathbb{R}^n)$. Condition INV (originally stated in [28] for maps in $W^{1,p}$ with $p > n - 1$) is defined as follows.

**Definition 2.12** ([7, Definition 3.6]). Let $u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$. We say that $u$ satisfies condition INV provided that for every point $x_0 \in \Omega$ and a.e. $r \in (0, \text{dist}(x_0, \partial \Omega))$, the following conditions hold:

(i) $u(x) \in \nim_T(u, B(x_0, r))$ for a.e. $x \in B(x_0, r)$.

(ii) $u(x) \notin \nim_T(u, B(x_0, r))$ for a.e. $x \in \Omega \setminus B(x_0, r)$.
A result [7, Lemma 3.9] (see also [28, Lemma 3.4]) that will be used throughout the paper is that if \( u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n) \) satisfies condition INV and \( \det Du > 0 \) a.e., then \( u \) is one-to-one a.e.

2.5 The distributional determinant

We present the definition of distributional determinant (see [2] or [27]).

**Definition 2.13.** Let \( u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n) \). The distribution \( \text{Det} \, D\!u \) is defined as

\[
\langle \text{Det} \, D\!u, \phi \rangle := -\frac{1}{n} \int_\Omega u(x) \cdot (\text{cof} \, D\!u(x)) D\phi(x) \, dx, \quad \phi \in C_c^\infty(\Omega),
\]

and called the distributional determinant of \( u \). If

\[
\sup \{ \langle \text{Det} \, D\!u, \phi \rangle : \phi \in C_c^\infty(\Omega), \ \text{spt} \phi \subset K, \ \|\phi\|_\infty \leq 1 \} < \infty
\]

for each compact set \( K \) contained in \( \Omega \), then \( \text{Det} \, D\!u \) can be extended uniquely to a Radon measure in \( \Omega \), and this extension will also be called \( \text{Det} \, D\!u \). If this is the case, the set \( C(u) \) is defined as

\[
C(u) := \{ x \in \Omega : \text{Det} \, D\!u(\{ x \}) \neq 0 \}.
\]

It is immediate to check that if \( \text{Det} \, D\!u \) is a measure, then the set \( C(u) \) is countable (that is, it is either countably infinite, finite or empty).

2.6 A class of ‘good’ open sets

Given a Sobolev map \( u \), the restriction of \( u \) to ‘almost all’ hypersurfaces enjoys several desirable properties. In this section we single out a class of open sets whose boundary is such a hypersurface.

We start by invoking the following property of the boundary of \( C^2 \) open sets. The result is well known, and a proof can be found in, e.g., [14, Theorem 16.25.2] (see also [36, p. 112] and [28, p. 48]).

**Proposition 2.14.** Let \( U \subset \Omega \) be a nonempty open set with a \( C^2 \) boundary, and denote the exterior unit normal by \( \nu : \partial U \to \mathbb{R}^n \). Then:

(i) There exists \( \delta > 0 \) such that the map \( w : \partial U \times (-\delta, \delta) \to \Omega \) given by

\[
w(x, t) = x - tv(x), \quad x \in \partial U, \ t \in \mathbb{R},
\]

is a \( C^1 \) diffeomorphism between \( \partial U \times (-\delta, \delta) \) and

\[
N(\partial U, \delta) := \{ x \in \Omega : \text{dist}(x, \partial U) < \delta \}.
\]
(ii) The function \( d : \Omega \to \mathbb{R} \) given by

\[
d(x) := \begin{cases} 
\text{dist}(x, \partial U) & \text{if } x \in U, \\
0 & \text{if } x \in \partial U, \\
-\text{dist}(x, \partial U) & \text{if } x \in \Omega \setminus \bar{U},
\end{cases}
\]

is continuous in \( \Omega \) and of class \( C^2 \) in \( N(\partial U, \delta) \). Moreover, for every \( x \in \partial U \) and \( t \in (-\delta, \delta) \),

\[
Dd(x - tv(x)) = -\nu(x).
\]

(iii) For each \( t \in (-\delta, \delta) \), the set \( U_t := \{x \in \Omega : d(x) > t\} \) is open and compactly contained in \( \Omega \), and has a \( C^2 \) boundary.

We will sometimes apply Proposition 2.14 when \( U = B(x_0, r) \) for some point \( x_0 \in \mathbb{R}^n \) and \( r > 0 \). In that case, its proof becomes trivial, and in fact one can replace the interval \( (-\delta, \delta) \) with \( (-\infty, r) \), and \( N(\partial U, \delta) \) with \( \mathbb{R}^n \setminus \{x_0\} \). Thus, whenever we apply Proposition 2.14 to a ball, we will be tacitly using this slightly stronger result.

**Definition 2.15.** For each nonempty open set \( U \subset \subset \Omega \) with a \( C^2 \) boundary, let \( \delta, d \) and \( U_t \) be as in Proposition 2.14. Let \( u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n) \) be such that \( \det Du > 0 \) a.e. We define \( \mathcal{U}_u \) as the family of nonempty open sets \( U \subset \subset \Omega \) with a \( C^2 \) boundary that satisfy the following conditions:

\begin{enumerate}[(C1)]
\item \( u|_{\partial U} \in W^{1,n-1}(\partial U, \mathbb{R}^n) \cap L^\infty(\partial U, \mathbb{R}^n) \) and \( (\text{cof } Du)|_{\partial U} \in L^1(\partial U, \mathbb{R}^{n \times n}) \).
\item \( \mathcal{H}^{n-1}(\partial U \setminus \Omega_0) = 0 \), where \( \Omega_0 \) is the set of Definition 2.4, and \( D(u|_{\partial U})(x) \) coincides with the orthogonal projection of \( Du(x) \) onto the space \( T_x \partial U \) for \( \mathcal{H}^{n-1} \)-a.e. \( x \in \partial U \).
\item \( \lim_{\varepsilon \searrow 0} \int_0^\varepsilon \int_{\partial U_t} |\text{cof } Du| \, d\mathcal{H}^{n-1} - \int_{\partial U} |\text{cof } Du| \, d\mathcal{H}^{n-1} \, dt = 0 \).
\item For every \( g \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n) \),

\[
\lim_{\varepsilon \searrow 0} \int_0^\varepsilon \int_{\partial U_t} g(u(x)) \cdot (\text{cof } \nabla u(x)) \nu_t(x) \, d\mathcal{H}^{n-1}(x) \right.
\]

\[
\left. - \int_{\partial U} g(u(x)) \cdot (\text{cof } \nabla u(x)) \nu(x) \, d\mathcal{H}^{n-1}(x) \right) \, dt = 0,
\]

where \( \nu_t \) denotes the unit outward normal to \( U_t \) for each \( t \in (-\delta, \delta) \), and \( \nu \) the unit outward normal to \( U \).
\end{enumerate}

Finally, for every \( x \in \Omega \) we denote by \( R_x \) the set of \( r \in (0, \text{dist}(x, \partial \Omega)) \) for which \( B(x, r) \in \mathcal{U}_u \) and conditions (i) and (ii) of Definition 2.12 are satisfied.
The following result guarantees that there are enough sets in $\mathcal{U}_u$, and radii in $R_x$.

**Lemma 2.16.** Suppose $u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ and $\det Du > 0$ a.e., and let $U \subset \subset \Omega$ be a nonempty open set with a $C^2$ boundary. Let $\delta$, $d$ and $U_t$ be as in Proposition 2.14. Then, $U_t \in \mathcal{U}_u$ for a.e. $t \in (-\delta, \delta)$. If, in addition, $u$ satisfies condition INV, then $r \in R_x$ for every $x \in \Omega$ and a.e. $r \in (0, \text{dist}(x, \partial\Omega))$.

**Proof.** Properties (C1) and (C2) follow from the coarea formula, while properties (C3) and (C4) follow from Lebesgue’s theorem; see [18, Lemmas 4 and 12] if necessary.

Bringing together [7, Lemmas 3.8, 3.10 and 3.12] with the ideas in [28, Section 9] (see also Lemma A.1 and [35, Appendix]), one can prove the following.

**Proposition 2.17.** Suppose $u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ satisfies condition INV and $\det Du > 0$ a.e. Let $U, V \in \mathcal{U}_u$. Then the following properties hold:

(i) $\text{im}_G(u, U) \subset \text{im}_T(u, U)$ and $\text{im}_G(u, \partial U) \subset \mathbb{R}^n \setminus \text{im}_T(u, U)$.

(ii) $\text{im}_T(u, U) \cap \text{im}_T(u, V) = \emptyset$ if $U \cap V = \emptyset$.

(iii) $\text{im}_T(u, U) \subset \text{im}_T(u, V)$ if $U \subset V$.

(iv) $\deg(u, \partial U, \cdot) = \chi_{\text{im}_T(u, U)}$ a.e.

(v) $\det Du(U) = \mathcal{L}^n(\text{im}_T(u, U))$.

(vi) $\text{im}_T(u, U)$ has finite perimeter.

(vii) $\partial^* \text{im}_T(u, U) \cong \text{im}_G(u, \partial U)$.

**Proof.** Given $g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, let $\Lambda_g$ denote the distribution on $\Omega$ given by

$$
\langle \Lambda_g, \phi \rangle := -\int_\Omega g(u(x)) \cdot (\text{cof } Du(x)) D\phi(x) \, dx, \quad \phi \in C^\infty_c(\Omega). \tag{2.4}
$$

Suppose that $\text{div } g \geq 0$ in the ball $B(0, \|u\|_\infty)$. Then, proceeding as in the proof of [35, Theorem A.1] (and using Definition 2.15 (C4) and Proposition 2.10), it is possible to obtain that $\Lambda_g$ is a nonnegative Radon measure, and that

$$
\Lambda_g(U) = \int_{\mathbb{R}^n} \text{div } g(y) \deg(u, \partial U, y) \, dy \tag{2.5}
$$

for every $U \in \mathcal{U}_u$.

Let $x \in \Omega$. From (2.5) and [7, Lemmas 3.10 and 4.3] we observe that

$$
\Lambda_g(B(x, r)) = \int_{\mathbb{R}^n} (\text{div } g) \chi_{\text{im}_T(u, B(x, r))} \, dy \leq \|\text{div } g\|_\infty \det Du(B(x, r))
$$
for every $r \in R_x$. Therefore, $\Lambda_g$ is absolutely continuous with respect to $\text{Det} D\mathbf{u}$. Using this, and applying Besicovitch’s covering theorem to $\mathcal{E}^n + \Lambda_g$, we can find, for every $U \in \mathcal{U}_\mathbf{u}$, a disjoint family $\{\bar{B}(\mathbf{b}_k, r_k)\}_{k \in \mathbb{N}}$ of closed balls such that $U$ coincides with $\bigcup_{k \in \mathbb{N}} B(\mathbf{b}_k, r_k)$ up to a null set, and such that

$$
\int_{\mathbb{R}^n} \text{div} \, g(y) \, \text{deg}(\mathbf{u}, \partial U, y) \, dy = \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} (\text{div} \, g) (y) \chi_{\text{im}_T(\mathbf{u}, B(\mathbf{b}_k, r_k))} (y) \, dy
$$

for every $g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ with $\text{div} \, g \geq 0$ in $B(0, \|\mathbf{u}\|_\infty)$. As a consequence, we obtain

$$
\text{deg}(\mathbf{u}, \partial U, \cdot) = \sum_{k=1}^{\infty} \chi_{\text{im}_T(\mathbf{u}, B(\mathbf{b}_k, r_k))} \quad \text{a.e.}
$$

Since the sets $\text{im}_T(\mathbf{u}, B(\mathbf{b}_k, r_k))$ are disjoint (by Lemma A.1), $\text{deg}(\mathbf{u}, \partial U, \cdot)$ assumes only the values 0 and 1, and, up to a null set, $\text{im}_T(\mathbf{u}, U)$ coincides with $\bigcup_{k \in \mathbb{N}} \text{im}_T(\mathbf{u}, B(\mathbf{b}_k, r_k))$. Once we have this, it is clear that the proof of [7, Lemma 4.3] can be reproduced to obtain (v); also, by virtue of [7, Lemma 3.8], it is clear that (i) is satisfied. Properties (vi)–(vii) can be obtained as in [7, Lemma 3.10] and properties (ii)–(iii) as in Lemma A.1. This completes the proof. 

We conclude this section by recalling the notion of topological image of a point (see [36, p. 115], [28, p. 33] or [7, Definition 3.13]).

**Definition 2.18.** Let $\mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ satisfy $\det D\mathbf{u} > 0$ a.e., and let $\mathbf{x} \in \Omega$. The topological image of $\mathbf{x}$ under $\mathbf{u}$, denoted by $\text{im}_T(\mathbf{u}, \mathbf{x})$, is defined as

$$
\text{im}_T(\mathbf{u}, \mathbf{x}) := \bigcap_{\mathbf{u} \in \mathcal{U}_\mathbf{u}} \text{im}_T(\mathbf{u}, U).
$$

Thanks to Proposition 2.17, if $\mathbf{u}$ satisfies condition INV, the above definition coincides with that in [7, 28, 36], namely, $\text{im}_T(\mathbf{u}, \mathbf{x}) := \bigcap_{r \in R_x} \text{im}_T(\mathbf{u}, B(\mathbf{x}, r))$.

## 3 Distributional determinant and the formation of cavities

One of the fundamental concepts introduced by Ball [2] in his existence theory for nonlinear elasticity is that of the distributional determinant $\text{Det} D\mathbf{u}$ of a deformation $\mathbf{u}$ (see Definition 2.13). This concept plays an important role because of its continuity properties, and because the equation $\text{Det} D\mathbf{u} = \det D\mathbf{u}$ is satisfied for sufficiently regular deformations (see [10, 12, 16, 23, 27]). Ball pointed out, in contrast, that there are realistic deformations for which this equation does not hold, for
example, a deformation creating a hole. This observation led to his mathematical
type for radial cavitation [3].

Some of the most attractive features of the existing models for cavitation
in the non-radially symmetric setting, namely the models of Müller and Spector [28]
and of Sivaloganathan and Spector [32], are based on the original idea of Ball
that the distributional determinant can be used to determine the location of the
singularities of the deformation where cavities are originated. Müller and Spector
[28, Section 8] prove, for example, that their admissible deformations \( u \) are such
\[
\text{Det} \ D_u = (\text{det} \ D_u) \mathcal{L}^n + \sum_{a \in C(u)} c_a \delta_a,
\]
for a countable family \( C(u) \subseteq \Omega \), and where \( c_a > 0 \) for all points \( a \in C(u) \). Each
\( a \in C(u) \) is a cavitation point for \( u \), and \( c_a \) provides the volume of the cavity at \( a \).
Sivaloganathan and Spector, on the other hand, exploited those results in order to
obtain a weak continuity property of the determinant, and to model the phenom-
non of cavitation at prescribed flaw points (see [32–35]).

In [17] we presented an \( SBV \) existence theory in nonlinear elasticity that allows
for cavitation and the formation of cracks along \((n-1)\)-dimensional surfaces. An
important object in that study was the functional \( \widetilde{E}_u \) (see Definition 3.1), which
constitutes a generalization of the (singular part of the) distributional determinant,
and provides information on the surface created by the deformation \( u \). Just as
\( \text{Det} \ D_u - \text{det} \ D_u \) gives information on the volume and location of the cavities,
we showed in [18] that \( \widetilde{E}_u \) can be used to measure the area of the created sur-
face in the deformed configuration, and to locate the singularities in the reference
configuration that give rise to that surface. In contrast to the \( SBV \) situation, where
the functional \( \widetilde{E}_u \) provides a richer description of the fracture process than \( \text{Det} \ D_u \),
Theorem 3.2 shows that in the Sobolev case, it is sufficient to know the structure of
the distributional determinant so as to study the creation of surface. In other words,
cavities are the only fracture surfaces that \( W^{1,n-1} \) deformations can create.

The main conclusion of the theorem (that is, point (iv)) was obtained by Conti
and De Lellis [7, Theorem 5.1] for deformations of finite surface energy, by using a
result of Dacorogna and Moser [9]. Indeed, their assumption is that \( \text{im}_G(u, B(x, r)) \)
has finite perimeter for all \( x \in \Omega \) and all \( r \in R_x \), and we will see in Theorem 4.6
that this is, essentially, a finite-energy assumption. The proof presented here, based
on [28, Lemma 8.1], uses more elementary techniques, and in particular does not
require the energy to be finite or the geometric images to have finite perimeter. By
removing the finite-energy assumption, the result applies not only to the models
in [28] and [7], but also, for example, to that of Sivaloganathan and Spector [32].
A second motivation for not requiring that assumption is that this will allow us,
in Section 5, to apply the results of [18] on the regularity of inverses to the case of homeomorphisms in $W^{1,n-1}$, and thus to compare those results with the recent works [8, 19–22, 29].

We recall from [17, 18] the definition of $\tilde{e}_u$ and introduce the associated set-function $\tilde{\mu}_u$.

**Definition 3.1.** Let $u \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ satisfy $Du \in L^1(\Omega)$.

(a) For each $\phi \in C^1_c(\Omega)$ and $g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, define

$$
\tilde{e}_u(\phi, g) := \int_\Omega \left[ g(u(x)) \cdot \text{cof} Du(x) \, D\phi(x) + \phi(x) \, \text{div} g(u(x)) \, \det Du(x) \right] dx.
$$

(b) Define

$$
\tilde{e}(u) := \sup \{ \tilde{e}_u(\phi, g) : \phi \in C^1_c(\Omega), g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n), \|\phi\|_\infty \leq 1, \|g\|_\infty \leq 1 \}.
$$

(c) If $g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, let $\tilde{e}_u(\cdot, g)$ denote the distribution over $\Omega$ defined as $\langle \tilde{e}_u(\cdot, g), \phi \rangle := \tilde{e}_u(\phi, g)$ for all $\phi \in C^\infty_c(\Omega)$. If

$$
\sup \{ \tilde{e}_u(\phi, g) : \phi \in C^\infty_c(\Omega), \text{spt} \phi \subset K, \|\phi\|_\infty \leq 1 \} < \infty \quad (3.1)
$$

for each compact $K \subset \Omega$, then $\tilde{e}_u(\cdot, g)$ is in fact a Radon measure over $\Omega$.

(d) If (3.1) holds for each compact $K \subset \Omega$ and each $g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, we define, for every Borel set $E \subset \Omega$,

$$
\tilde{\mu}_u(E) := \sup \{ \tilde{e}_u(E, g) : g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n), \|g\|_\infty \leq 1 \}.
$$

**Theorem 3.2.** Suppose $u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ satisfies condition INV and is such that $\det Du > 0$ a.e. Assume $\text{Det} Du$ is a measure that can be represented as

$$
\text{Det} Du = (\det Du) \mathcal{L}^n + \sum_{a \in C(u)} c_a \delta_a,
$$

where $c_a \in [-\infty, \infty]$ for each $a \in C(u)$. Then the following properties are satisfied:

(i) $c_a = \text{Det} Du(\{a\}) = \mathcal{L}^n(\text{im}_T(u, a)) > 0$ for each $a \in C(u)$.

(ii) $\sum_{a \in C(u)} c_a < \infty$. 
Lusin’s condition and distributional determinant

(iii) \( \text{im}_T(u, \mathbf{a}) \cap \text{im}_T(u, \mathbf{b}) = \emptyset \) for any \( \mathbf{a}, \mathbf{b} \in C(u) \) with \( \mathbf{a} \neq \mathbf{b} \). For each set \( U \in \mathcal{U}_u \),

\[
\text{im}_T(u, U) = \text{im}_G(u, U) \cup \bigcup_{\mathbf{a} \in C(u) \cap U} \text{im}_T(u, \mathbf{a}) \quad \text{a.e.}
\]

Moreover, \( \text{im}_G(u, \Omega) \cap \text{im}_T(u, \mathbf{a}) = \text{im}_G(u, \{\mathbf{a}\}) \) for each \( \mathbf{a} \in C(u) \).

(iv) For every \( g \in C_c^1(\mathbb{R}^n, \mathbb{R}^n) \),

\[
\sum_{\mathbf{a} \in C(u)} \left| \int_{\text{im}_T(u, \mathbf{a})} \text{div} \ g(y) \ dy \right| < \infty,
\]

\( \tilde{\mathcal{E}}_u(\cdot, g) \) is a finite signed measure, and

\[
\tilde{\mathcal{E}}_u(\cdot, g) = -\sum_{\mathbf{a} \in C(u)} \left( \int_{\text{im}_T(u, \mathbf{a})} \text{div} \ g(y) \ dy \right) \delta_\mathbf{a}
\]

in the sense of measures.

(v) For any Borel set \( E \subset \Omega \),

\[
\tilde{\mu}_u(E) = \text{Per} \left( \bigcup_{\mathbf{a} \in C(u) \cap E} \text{im}_T(u, \mathbf{a}) \right).
\]

Proof. For each \( \mathbf{a} \in C(u) \), clearly \( c_\mathbf{a} = \text{Det} \, D_\mathbf{u}(\{\mathbf{a}\}) \). Moreover, by Proposition 2.17 (iii)–(v) and the remark after Definition 2.18, we have that

\[
\text{Det} \, D_\mathbf{u}(\{\mathbf{a}\}) = \mathcal{L}^n \left( \bigcap_{r \in R_\mathbf{a}} \text{im}_T(u, B(\mathbf{a}, r)) \right) = \mathcal{L}^n (\text{im}_T(u, \mathbf{a})).
\]

That the family \( \{\text{im}_T(u, \mathbf{a})\}_{\mathbf{a} \in C(u)} \) is disjoint is a consequence of Lemma A.1.

Fix \( U \in \mathcal{U}_u \). By Definition 2.18, \( \text{im}_T(u, \mathbf{a}) \subset \text{im}_T(u, U) \) for all \( \mathbf{a} \in C(u) \cap U \), whereas by Proposition 2.17 (i), we get \( \text{im}_G(u, \mathbf{a}) \subset \text{im}_T(u, U) \). Finally, by Proposition 2.17 (v), Proposition 2.3 and (i), we have that

\[
\mathcal{L}^n (\text{im}_T(u, U)) = \mathcal{L}^n (\text{im}_G(u, U)) + \sum_{\mathbf{a} \in C(u) \cap U} \mathcal{L}^n (\text{im}_T(u, \mathbf{a})). \tag{3.2}
\]

Now fix \( \mathbf{a} \in C(u) \). By Proposition 2.17 (i), \( \text{im}_G(u, \{\mathbf{a}\}) \subset \text{im}_T(u, \mathbf{a}) \), whereas by Proposition 2.17 (i) and [7, Lemma 3.9],

\[
\text{im}_T(u, \mathbf{a}) \cap \text{im}_G(u, \Omega) \subset \bigcap_{r \in R_\mathbf{a}} [\text{im}_G(u, \Omega) \setminus \text{im}_G(u, \Omega \setminus B(\mathbf{a}, r))] = \bigcap_{r \in R_\mathbf{a}} \text{im}_G(u, B(\mathbf{a}, r)) = \text{im}_G(u, \{\mathbf{a}\}).
\]

This proves (iii).
Let $D. Henao and C. Mora-Corral

For each $t \in (0,1)$ let $\psi_t \in C_c(0,t)$ be such that $\psi \geq 0$, $\psi' \leq 0$, and $\int_{B(0,1)} \psi(|x|) \, dx = 1$. For each $t \in (0,1)$, let $\psi_t \in C_c((0,t))$ be given by $\psi_t(r) := t^{-n} \psi(r/t)$, and $\varphi_t \in C_c(B(0,t))$ by $\varphi_t(x) := \psi_t(|x|)$. Then, for any $\phi \in C_c(\Omega)$,

$$\langle \Lambda_g, \phi \rangle = \lim_{t \searrow 0} \left( - \int_\Omega g(u(x)) \cdot (\cof D u(x)) \left( \int_\Omega \phi(z) D \varphi_t(x - z) \, dz \right) \, dx \right),$$

where, for each $t \in (0, dist(spt \phi, \partial \Omega))$, we have defined

$$h_t(z) := - \int_\Omega g(u(x)) \cdot (\cof D u(x)) D \varphi_t(x - z) \, dx, \quad z \in spt \phi.$$

By virtue of the coarea formula, $h_t(z)$ can be written as

$$h_t(z) = - \int_0^t \psi_t'(r) \int_{\partial B(z,r)} g \circ u \cdot (\cof D u) H_{n-1} \, dr.$$  

For all $a \in C(u)$, set

$$c^{(g)}_a := \int_{int (u,a)} \text{div} g(y) \, dy.$$  

Then, applying (iii) and Propositions 2.10, 2.17 (iv), and 2.3, we obtain

$$\int_{\partial B(z,r)} g(u(x)) \cdot (\cof D u(x)) H_{n-1}(x)$$

$$= \int_{B(z,r)} (\text{div} g)(u(x)) \det D u(x) \, dx + \sum_{a \in C(u) \cap B(z,r)} c^{(g)}_a$$

for every $r \in R_z$. The function $G \in L^\infty((0,t))$ defined by

$$G(r) := \int_{B(z,r)} (\text{div} g) \circ u \, det D u \, dx = \int_0^r \int_{\partial B(z,\rho)} (\text{div} g) \circ u \, det D u \, dH_{n-1} \, d\rho$$

is absolutely continuous and its derivative is given by

$$G'(r) = \int_{\partial B(z,r)} (\text{div} g)(u(x)) \det D u(x) \, dH_{n-1}(x), \quad a.e. \ r \in (0,t).$$

Thus, integrating by parts, and using that $G(0) = 0$ and $\psi(1) = 0$, we have

$$- \int_0^t \psi_t'(r) \int_{B(z,r)} (\text{div} g)(u(x)) \det D u(x) \, dx \, dr$$

$$= \int_\Omega \varphi_t(x - z)(\text{div} g)(u(x)) \det D u(x) \, dx.$$  

**Author's Note:** The text is continued in subsequent pages.
As for the last term in (3.6), we first observe that, thanks to (iii),
\[ \sum_{a \in C(u)} |c_a^{(g)}| \leq \int_{\mathbb{R}^n} |\text{div } g(y)| \, dy < \infty. \]  
(3.8)

Then, by the dominated convergence theorem we have that
\[ \int_0^t \psi_t'(r) \sum_{a \in C(u) \cap B(z,r)} c_a^{(g)} \, dr = \sum_{a \in C(u)} c_a^{(g)} \int_0^t \psi_t'(r) \chi_{B(z,r)}(a) \, dr. \]  
(3.9)

It is easy to see that \( \int_0^t \psi_t'(r) \chi_{B(z,r)}(a) \, dr = -\varphi_t(a - z) \). Hence, by (3.4), (3.6), (3.7) and (3.9),
\[ h_t(z) = \int_{\Omega} \varphi_t(x - z)(\text{div } g)(u(x)) \det D u(x) \, dx + \sum_{a \in C(u)} c_a^{(g)} \varphi_t(a - z). \]

Combining this with (3.3) we obtain
\[ \langle A_g, \phi \rangle = \int_{\Omega} \phi(x)(\text{div } g)(u(x)) \det D u(x) \, dx + \sum_{a \in C(u)} c_a^{(g)} \phi(a). \]

As \( \sum_{a \in C(u)} |c_a^{(g)}| < \infty \), this shows that \( \tilde{\mathcal{E}}_u(\cdot, g) \) is a finite Radon measure. By Riesz’ representation theorem and the density of smooth functions, this proves (iv).

Property (ii) is a particular case of (iv) when \( g \) coincides with \( \frac{1}{n} \text{id} \) in the ball \( B(0, \|u\|_{\infty}) \).

Finally, (iii) and (iv) yield
\[ \tilde{\mathcal{E}}_u(E, g) = -\int_{\bigcup_{a \in C(u) \cap E} \text{im}_T(u,a)} \text{div } g(y) \, dy \]
for every Borel set \( E \subset \Omega \) and every \( g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n) \). Thus, (v) follows. \( \square \)

We note that the proof of Theorem 3.2 has similarities with that of [28, Theorem 9.1]. Regarding Theorem 3.2 (iii), the natural expectation is that cavities cannot be formed at regular points, and, hence, that
\[ C(u) \cap \Omega_0 = \emptyset \quad \text{and} \quad \text{im}_G(u, \{a\}) = \emptyset \quad \text{for all} \quad a \in C(u). \]

Proving this, however, does not seem to be immediate. Be it as it may, this part of the theorem can safely be regarded as expressing that there is no matter inside the cavities.

The following equality (which, with a different language, is implicit in the proofs of [28, Theorem 9.1], [7, Lemma 4.3], [35, Theorem A.1] and [17, The-
Lemma 3.3. Let $u \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ satisfy $\det D u \in L^1(\Omega)$ and
\[
\sup \{ \bar{E}_u(\phi, g) : \phi \in C^1_c(\Omega), \ g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n), \ \text{spt} \phi \subset U, \ \|\phi\|_{\infty} \leq 1, \ \|g\|_{\infty} \leq 1 \} < \infty.
\]
for all $U \in \mathcal{U}_u$. Then, for every $U \in \mathcal{U}_u$ and every $g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$,
\[
\bar{E}_u(U, g) = -\int_{\partial U} g(u(x)) \cdot (\text{cof} D u(x)) \nu U(x) d\mathcal{H}^{n-1}(x)
+ \int_U (\text{div} g)(u(x)) \det D u(x) dx.
\]

4 A study of the created surface

As mentioned in the Introduction, one of the main goals of this paper is to compare [17, 18], where the authors formulate a new model allowing for cavitation and fracture, with the previous works on cavitation by Müller and Spector [28] and Conti and De Lellis [7]. In the latter models, the lower semicontinuity of the energy functional is recovered by adding $\text{Per} \ \text{im}_G(u, \Omega)$ as a surface energy term; the cavity surfaces in the deformed configuration are obtained by studying the singular part of the distributional determinant, and by considering Šverák’s [36] notion of the topological image of a point. In [17, 18], in contrast, the expression for the surface energy is given by the functional $E(u)$ of Definition 4.1, and the created surface $\Gamma(u)$ is identified by studying the set of jump discontinuities of the inverse (see Definition 4.3). In this section we show that in the Sobolev case, both approaches are equivalent. In particular, we prove that $\Gamma(u) \cong \bigcup_{a \in C(u)} \partial^* \text{im}_T(u, a)$ (Theorem 4.8), and that the conditions $\text{Per} \ \text{im}_G(u, \Omega) < \infty$ and $\bar{E}(u) < \infty$ impose the same control on the creation of new surface (Theorem 4.6).

Along with the definition of $\bar{E}(u)$, we recall the definition of its counterpart for open subsets $U \subset \subset \Omega$, the measure $\mu_u$ (see [18, Section 8]).

Definition 4.1. Let $u \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ satisfy $\det D u \in L^1(\Omega)$.

(a) For each $f \in C^1_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$, define
\[
\mathcal{E}_u(f) := \int_{\Omega} \left[ D_x f(x, u(x)) \cdot \text{cof} D u(x) + \text{div}_y f(x, u(x)) \det D u(x) \right] dx.
\]

(b) Define $\mathcal{E}(u) := \sup \{ \mathcal{E}_u(f) : f \in C^1_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n), \ \|f\|_{\infty} \leq 1 \}$. 


(c) For each open set $U \subset \Omega$, define

$$\mu_u(U) := \sup \{E_u(f) : f \in C_c^1(\Omega \times \mathbb{R}^n, \mathbb{R}^n), \|f\|_\infty \leq 1, \text{spt } f \subset U \times \mathbb{R}^n\}.$$ 

If $\mu_u(U) < \infty$ for each open set $U \subset \subset \Omega$, then $\mu_u$ can be extended uniquely to a Radon measure in $\Omega$, which will still be called $\mu_u$.

We now state our definition of created surface. As explained in [18, Section 5], some portions of the surface created by a deformation $u$ might become ‘invisible’, or undetectable as parts of the reduced boundary of $\text{im}_G(u, \Omega)$, if they come in contact with each other. We distinguish, therefore, between the visible and invisible parts of the created surface. The precise definition of these sets is based on the notions of approximate limit and of approximate jump set, which are now recalled.

**Definition 4.2.** Let $A \subset \mathbb{R}^n$ be a measurable set, and $u : A \rightarrow \mathbb{R}^n$ be a measurable function. Let $x_0 \in \mathbb{R}^n$.

(a) Let $y_0 \in \mathbb{R}^n$ and suppose $F \subset \mathbb{R}^n$ is a measurable set with $\mathcal{D}(A \cap F, x_0) > 0$. If $D(\{x \in A \cap F : |u(x) - y_0| \geq \delta\}, x_0) = 0$ for every $\delta > 0$, we write

$$\text{ap lim}_{x \rightarrow x_0} u(x) = y_0.$$ 

(b) We say that $x_0$ is an approximate jump point of $u$ if $D(A, x_0) = 1$ and

$$\text{ap lim}_{x \rightarrow x_0} u(x) = a, \quad \text{ap lim}_{x \rightarrow x_0} u(x) = b$$

for some $a, b \in \mathbb{R}^n$, $a \neq b$, and some $v \in S^{n-1}$. The points $a$ and $b$ are called the lateral traces of $u$ at $x_0$ with respect to $v$, and are denoted by $u^+(x_0)$ and $u^-(x_0)$, respectively. The set of approximate jump points of $u$ is called the jump set of $u$, and is denoted by $J_u$.

**Definition 4.3** (cf. [18, Section 5]). Let $u \in W^{1,1}(\Omega, \mathbb{R}^n)$ be one-to-one a.e. and satisfy $\det \mathcal{D}u > 0$ a.e. Let $\Omega_0$ be the set of Definition 2.4.

(a) The inverse $u^{-1} : \text{im}_G(u, \Omega) \rightarrow \mathbb{R}^n$ of $u$ is defined as the function that associates, to each $y \in \text{im}_G(u, \Omega)$, the only point $x \in \Omega_0$ satisfying $u(x) = y$.

(b) The visible surface created by $u$, denoted by $\Gamma^V(u)$, is defined as the set of $y_0 \in \mathbb{R}^n$ for which there exists $v \in S^{n-1}$ satisfying the following conditions:

- (i) $D(\text{im}_G(u, \Omega) \cap H^-(y_0, v), y_0) = \frac{1}{2}$. 

(ii) The lateral trace
\[(u^{-1})^-(y_0) = \limap_{y \to y_0, y \in H^-(y_0, v)} u^{-1}(y) \] (4.1)
exists and lies in \( \Omega \).

(iii) \( D(\text{im}_G(u, U) \cap H^+(y_0, v), y_0) = 0 \) for every open set \( U \subset \Omega \).

(c) For any open set \( U \subset \Omega \), the invisible surface created by \( u \) in \( U \) is defined as
\[ \Gamma_I(u, U) := \{ y \in J_{u^{-1}} : (u^{-1})^+(y) \in U \text{ and } (u^{-1})^-(y) \in U \} \]
We also define \( \Gamma_I(u) := \Gamma_I(u, \Omega) \).

(d) The surface created by \( u \), denoted by \( \Gamma(u) \), is \( \Gamma(u) := \Gamma_V(u) \cup \Gamma_I(u) \).

In the above definition it is implicit that necessarily \( D(\text{im}_G(u, \Omega), y) = 1 \) for every \( y \in \Gamma_I(u) \). In addition, \( u^{-1} \) is well defined by virtue of [18, Lemma 2] (or, alternatively, by [7, Lemma 3.9]), which states that \( u|_{\Omega_0} \) is one-to-one.

The following proposition gives a well-defined orientation vector \( \nu_{u^{-1}} \) for the surface \( \Gamma(u) \).

**Proposition 4.4** (cf. [18, Lemma 9]). Let \( u \in W^{1,1}(\Omega, \mathbb{R}^n) \) be one-to-one a.e. and satisfy \( \det Du > 0 \) a.e. Then the sets \( \Gamma_V(u) \), \( \Gamma_I(u) \) and \( \Gamma(u) \) of Definition 4.3 are Borel. Moreover, there exist Borel maps \( \nu_{u^{-1}} : \Gamma(u) \to S^{n-1} \), \( (u^{-1})^- : \Gamma(u) \to \Omega \), and \( (u^{-1})^+ : \Gamma_I(u) \to \Omega \) satisfying the following:

(i) For every \( y_0 \in \Gamma_V(u) \), the orientation vector \( \nu_{u^{-1}}(y_0) \) coincides with the vector \( v \) of Definition 4.3 (b), and the value of \( (u^{-1})^-(y_0) \) is given by (4.1).

(ii) For every \( y \in \Gamma_I(u) \), the points \( (u^{-1})^-(y) \) and \( (u^{-1})^+(y) \) are the lateral traces of \( u^{-1} \) at \( y \) with respect to \( \nu_{u^{-1}}(y) \).

The set \( \Lambda(u, U) \) of the following definition consists of those points in the reference configuration responsible for the creation of new surface (see [18, Section 8]).

**Definition 4.5** (cf. [18, Definition 21]). Let \( u \in W^{1,1}(\Omega, \mathbb{R}^n) \) be one-to-one a.e. and such that \( \det Du > 0 \) a.e. Consider the maps \( (u^{-1})^+ \) and \( (u^{-1})^- \) of Proposition 4.4. For each \( U \subset \Omega \), the set \( \Lambda(u, U) \) is defined as
\[ \Lambda(u, U) := (u^{-1})^+([y \in \Gamma_I(u) : (u^{-1})^+(y) \in U]) \]
\[ \cup (u^{-1})^+([y \in \Gamma_I(u) : (u^{-1})^- (y) \in U]) \]
\[ \cup (u^{-1})^+([y \in \Gamma_V(u) : (u^{-1})^- (y) \in U]). \]
Before stating Theorem 4.6, let us discuss briefly in what sense the energies $\mathcal{E}(u)$ and $\text{Perim}_G(u, \Omega)$ can be expected to be equivalent. While the first of these energies gives only the area of the surface created by $u$, the perimeter of the image accounts both for $\partial^* \text{im}_G(u, \Omega) \setminus u(\partial\Omega)$ (the new boundary) and $u(\partial\Omega)$ (the previously existing boundary). We would like to conclude, therefore, not that the two surface energies are equal, but rather that $\text{Perim}_G(u, \Omega) = \mathcal{E}(u) + \mathcal{H}^{n-1}(u(\partial\Omega))$. It is not clear, however, how to define $u(\partial\Omega)$ precisely. Moreover, pathological deformations might exist for which $\mathcal{H}^{n-1}(u(\partial\Omega)) = \infty$, even if $\mathcal{E}(u)$ is finite. In order to overcome this difficulty, we restrict our attention to subdomains $U$ in the class $\mathcal{U}_u$ of ‘good’ open sets (see Definition 2.15). As Theorem 4.6 (iv) shows, on these sets the desired conclusion can indeed be established.

Let us comment, finally, on Theorem 4.6 (ii). The functional $\tilde{\mathcal{E}}$ of Definition 3.1 was proposed in [17] as a tentative surface energy. This functional gives correctly the area of the created surface for a large class of deformations creating cracks and cavities, such as those of the example in [28, Section 11] (see [17, Section 3]). It was shown, however, that $\tilde{\mathcal{E}}$ could not be regarded as an appropriate surface energy [17, Section 6]. In Theorem 4.6 we prove that when the formation of cracks is not allowed, the situation changes and, in fact, $\mathcal{E}$ and $\tilde{\mathcal{E}}$ coincide.

**Theorem 4.6.** Suppose that $u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ satisfies INV and $\det Du > 0 \text{ a.e.}$ Then the following assertions are equivalent:

1. $\text{Perim}_G(u, U) < \infty$ for all $U \in \mathcal{U}_u$.
2. For all $U \in \mathcal{U}_u$, 
   \[
   \sup \left\{ \tilde{\mathcal{E}}(\phi, g) : \phi \in C^1_c(\Omega), \ g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n), \ \text{spt} \phi \subset U, \ \|\phi\|_\infty \leq 1, \ \|g\|_\infty \leq 1 \right\} < \infty.
   \]
3. $\mu_u(U) < \infty$ for all $U \in \mathcal{U}_u$.
4. $\det Du = (\det Du)\mathcal{L}^n + \sum_{a \in C(u)} c_a \delta_a$, where $c_a \in (0, \infty)$ for all $a \in C(u)$, and, for each $U \in \mathcal{U}_u$, 
   \[
   \sum_{a \in C(u) \cap U} \text{Perim}_T(u, a) < \infty.
   \]

If any of the conditions (1)–(4) holds, then

1. $\tilde{\mu}_u = \mu_u = \sum_{a \in C(u)} \text{Perim}_T(u, a) \delta_a$.
2. $\mathcal{E}(u) = \tilde{\mathcal{E}}(u) = \sum_{a \in C(u)} \text{Perim}_T(u, a)$. 

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For each \( f \in C^1_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n) \),
\[
\mathcal{E}_u(f) = - \sum_{a \in C(u)} \int_{\partial^- \text{im}_T(u, a)} f(a, y) \cdot \nu_{\text{im}_T(u, a)}(y) \, d\mathcal{H}^{n-1}(y).
\]

For every \( U \in \mathcal{U}_u \),
\[
\text{Per}(\text{im}_G(u, U)) = \mu_u(U) + \mathcal{H}^{n-1}(\text{im}_G(u, \partial U)).
\]

**Remark 4.7.**
1. In Theorem 4.6 (i)–(ii), the term \( \sum_{a \in C(u)} \text{Per im}_T(u, a) \) is a series of positive numbers whose sum may be infinite.
2. Condition (i) implies that the set function \( \tilde{\mu}_u \) of Definition 3.1 is a positive Radon measure in \( \Omega \). This is due to the special situation considered (of deformations satisfying condition INV and having finite surface energy), and is not a direct consequence of the definition of \( \tilde{\mu}_u \) itself, or of Theorem 3.2 (v).
3. At first sight, condition (1) might seem stronger than \( \text{Per im}_G(u, \Omega) < \infty \), since the former requires that
\[
\mathcal{H}^{n-1}(\text{im}_G(u, \partial U)) < \infty \quad \text{for all} \quad U \in \mathcal{U}_u
\] (see (iv)). However, by Proposition 2.17 (vi)–(vii), equation (4.2) is always satisfied, regardless of whether the surface energy of \( u \) is finite. In fact,
\[
\text{Per im}_G(u, \Omega) < \infty
\]
is stronger than (1), since it can be proved that
\[
\partial^- \text{im}_G(u, U) \subset \partial^- \text{im}_T(u, U) \cup \partial^- \text{im}_G(u, \Omega).
\]
4. Some of the conclusions in the previous theorem were established by Conti and De Lellis (cf. [7, Theorem 5.1]). Here we present an alternative proof, based not on the result of Dacorogna and Moser [9] by means of which a divergence field can be written as a Jacobian, but on Theorem 3.2.

Theorem 4.6 is intimately related to the second main result of this section, Theorem 4.8, where different notions of created surface (all of which have played an important role in one or more of the works [7, 17, 18, 28]) are shown to be equivalent.

**Theorem 4.8.** Suppose that \( u \in W^{1, n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n) \) satisfies INV and \( \det Du > 0 \) a.e. Assume that any of conditions (1)–(4) of Theorem 4.6 is satisfied.
Then the following hold:

(i) For every $U \in \mathcal{U}_u$, the following sets are $\mathcal{H}_-^{n-1}$-equivalent:

\[
A_1 := \bigcup_{a \in C(u) \cap U} \partial^* \text{im}_T(u, a), \quad A_2 := \partial^* \bigcup_{a \in C(u) \cap U} \text{im}_T(u, a),
\]

\[
A_3 := \partial^* \text{im}_G(u, \Omega) \cap \text{im}_T(u, U), \quad A_4 := \partial^* \text{im}_G(u, U) \cap \text{im}_T(u, U),
\]

\[
A_5 := \partial^*(\text{im}_T(u, U) \setminus \text{im}_G(u, U)), \quad A_6 := \partial^*(\text{im}_T(u, U) \setminus \text{im}_G(u, \Omega)),
\]

\[
A_7 := \partial^* \text{im}_G(u, U) \setminus \text{im}_G(u, \partial U).
\]

(ii) $\Gamma(u) \cong \Gamma_V(u) \cong \bigcup_{a \in C(u)} \partial^* \text{im}_T(u, a)$ and for $\mathcal{H}_-^{n-1}$-a.e. $y \in \Gamma(u)$,

\[
(u^{-1})^{-1}(y) \in C(u).
\]

Moreover, for each $a \in C(u)$ and $\mathcal{H}_-^{n-1}$-a.e. $y \in \partial^* \text{im}_T(u, a)$, we have that

\[
\nu_{u^{-1}}(y) = \nu_{\text{im}_G(u, \Omega)}(y) = -\nu_{\text{im}_T(u, a)}(y).
\]

(iii) For every $U \in \mathcal{U}_u$,

\[
C(u) \cap U \subset \Lambda(u, U) \quad \text{and} \quad \mu(u)(\Lambda(u, U) \setminus (C(u) \cap U)) = 0.
\]

The remainder of this section is devoted to the proof of Theorems 4.6 and 4.8. We begin with the following auxiliary lemma, the proof of which is elementary.

**Lemma 4.9.** Let $A, B \subset \mathbb{R}^n$ be measurable. The following assertions hold:

(i) Let $C := \{y \in \mathbb{R}^n : D(A, y) = 1\}$. Then $C = \{y \in \mathbb{R}^n : D(C, y) = 1\}$.

(ii) If $A \subset B$, then $\partial^- A \subset \{y \in \mathbb{R}^n : \tilde{D}(B, y) > 0\}$.

The following result, whose proof is due to De Lellis [11], will play an important role, especially in proving Theorem 4.8.

**Lemma 4.10.** Let $u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ satisfy INV and $\det Du > 0$ a.e. Then, for each $x_1, x_2 \in \Omega$ with $x_1 \neq x_2$, we have that

\[
\mathcal{H}_-^{n-1}(\partial^- \text{im}_T(u, x_1) \cap \partial^- \text{im}_T(u, x_2)) = 0.
\]

**Proof.** Let $r_i \in R_{x_i}, i = 1, 2$, be such that $B(x_1, r_1) \cap B(x_2, r_2) = \emptyset$. Clearly, we get $\text{im}_T(u, x_i) \subset \text{im}_T(u, B(x_i, r_i))$ for $i = 1, 2$. Using Lemma 4.9 and Proposition 2.17 (vi), it is easy to see that

\[
\partial^- \text{im}_T(u, x_i) \subset \text{im}_T(u, B(x_i, r_i)) \cup \partial^* \text{im}_T(u, B(x_i, r_i)), \quad i = 1, 2.
\]

The conclusion follows by arguing as in the proof of Lemma A.1. □
The following result was stated, without proof, as one of the conclusions in [7, Theorem 4.2]. We present a proof below, due to De Lellis [11], for the convenience of the reader.

**Proposition 4.11.** Suppose \( u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n) \) satisfies condition INV and is such that \( \det Du > 0 \) a.e. Then

\[
\sum_{a \in C(u)} \text{Per im}_T(u, a) \leq \text{Per im}_G(u, \Omega).
\]

**Proof.** Using Lemmas 2.6 and Proposition 2.17 (vii), it is easy to prove (see, e.g., [7, p. 535], or the proof of Theorem 4.8) that for each \( x \in \Omega \), and every \( r \in R_x \),

\[
\partial^- (\text{im}_T(u, B(x, r)) \setminus \text{im}_G(u, \Omega)) \subset \partial^- \text{im}_G(u, \Omega) \cap \text{im}_T(u, B(x, r)).
\]

Consider a finite number \( a_1, \ldots, a_M \in C(u) \) of distinct cavity points, and denote the set \( \bigcap_{i=1}^M R_{a_i} \) by \( S \). Then, on the one hand, for every \( r \in S \),

\[
\partial^- \left( \bigcup_{i=1}^M \text{im}_T(u, B(a_i, r)) \setminus \text{im}_G(u, \Omega) \right) \subset \partial^- \text{im}_G(u, \Omega) \cap \bigcup_{i=1}^M \text{im}_T(u, B(a_i, r)),
\]

and, hence,

\[
\liminf_{r \searrow 0} \left( \text{Per} \left( \bigcup_{i=1}^M \text{im}_T(u, B(a_i, r)) \setminus \text{im}_G(u, \Omega) \right) \right) \leq \text{Per im}_G(u, \Omega).
\]

On the other hand,

\[
\bigcap_{r \in S} \left( \bigcup_{i=1}^M \text{im}_T(u, B(a_i, r)) \setminus \text{im}_G(u, \Omega) \right) = \bigcup_{i=1}^M \text{im}_T(u, a_i) \setminus \text{im}_G(u, C(u)),
\]

as can be proved easily using Theorem 3.2 (iii). Hence, by the lower semicontinuity of the perimeter (and the fact that \( \text{im}_G(u, C(u)) \) is countable), we obtain

\[
\text{Per} \left( \bigcup_{i=1}^M \text{im}_T(u, a_i) \right) \leq \text{Per im}_G(u, \Omega).
\]

By virtue of Lemma 4.10, and since \( M \) is arbitrary, the conclusion follows. \( \square \)

**Proof of Theorem 4.6.** The proof is divided into several implications.

(1) implies (4). The implication follows by applying [7, Theorem 4.2] and Proposition 4.11, for each \( U \in \mathcal{U}_u \), to \( u|_U \in W^{1,p}(U, \mathbb{R}^n) \cap L^\infty(U, \mathbb{R}^n) \).
(4) implies (iii). By Theorem 3.2 (iv) and the divergence theorem for sets of finite perimeter, we have that

$$\tilde{\mathcal{E}}_u(\phi, g) = - \sum_{a \in C(u)} \int_{\partial^* \text{im}_T(u, a)} \phi(a) g(y) \cdot \nu_{\text{im}_T(u, a)}(y) \, d\mathcal{H}^{n-1}(y) \quad (4.3)$$

for every $\phi \in C^1_c(\Omega)$ and $g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$. Combining this with Lemma 2.1, the continuity of $\tilde{\mathcal{E}}_u(f)$ with respect to $f$ in the $C^1_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ norm, the fact that $\tilde{\mathcal{E}}_u(\phi, g) = \mathcal{E}_u(\phi g)$, and the fact that $\sum_{a \in C(u) \cap U} \mathcal{H}^{n-1}(\partial^* \text{im}_T(u, a)) < \infty$ for every $U \in \mathcal{U}_u$ containing the set $\{x \in \Omega : (x, y) \in \text{sp} f \text{ for some } y \in \mathbb{R}^n\}$, we conclude (iii).

(4) and (iii) imply (i). First we prove that $\tilde{\mu}_u = \sum_{a \in C(u)} \text{Per}_{\text{im}_T(u, a)} \delta_a$. Let $E \subset \Omega$ be a Borel set, and let $U \in \mathcal{U}_u$. By Theorem 3.2 (iv) and the divergence theorem for sets of finite perimeter, we have that

$$\tilde{\mathcal{E}}_u(E \cap U, g) = - \sum_{a \in C(u) \cap E \cap U} \int_{\partial^* \text{im}_T(u, a)} g(y) \cdot \nu_{\text{im}_T(u, a)}(y) \, d\mathcal{H}^{n-1}(y)$$

for every $g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$. It follows that

$$\tilde{\mu}_u(E \cap U) \leq \sum_{a \in C(u) \cap E \cap U} \text{Per}_{\text{im}_T(u, a)}. \quad (4.4)$$

On the other hand, in light of Lemma 4.10 we may write

$$\tilde{\mathcal{E}}_u(E \cap U, g) = - \int_{\bigcup_{a \in C(u) \cap E \cap U} \partial^* \text{im}_T(u, a)} g(y) \cdot \nu(y) \, d\mathcal{H}^{n-1}(y) \quad (4.5)$$

for all $g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, where

$$\nu(y) \text{ denotes } \nu_{\text{im}_T(u, a)}(y) \text{ whenever } y \in \partial^* \text{im}_T(u, a) \text{ for some } a \in C(u). \quad (4.6)$$

Since

$$\mathcal{H}^{n-1} \bigcup_{a \in C(u) \cap E \cap U} \partial^* \text{im}_T(u, a)$$

is a finite Radon measure in $\mathbb{R}^n$ (by virtue of (4)), we can apply Lusin’s theorem to obtain that

$$\int_{\bigcup_{a \in C(u) \cap E \cap U} \partial^* \text{im}_T(u, a)} g(y) \cdot \nu(y) \, d\mathcal{H}^{n-1}(y) \leq \|g\|_\infty \tilde{\mu}_u(E \cap U)$$

for every bounded Borel function $g : \mathbb{R}^n \to \mathbb{R}^n$. In particular, setting $g(y) = \nu(y)$ for any $y \in \bigcup_{a \in C(u) \cap U} \partial^* \text{im}_T(u, a)$ at which $\nu(y)$ is well defined, and $g(y) = 0$
otherwise, and using Lemma 4.10 once again, we obtain the reverse of inequality (4.4). Therefore,

\[ \overline{\mu_u} \preceq \sum_{a \in C(u)} \text{Per } \text{im}_\tau(u, a) \delta_a \preceq U. \]

Since this holds for every \( U \in \mathcal{U}_u \), we conclude that \( \overline{\mu_u} \) is a positive Radon measure in \( \Omega \), and that \( \overline{\mu_u} = \sum_{a \in C(u)} \text{Per } \text{im}_\tau(u, a) \delta_a \), as desired.

Now we prove that \( \overline{\mu_u} = \mu_u \). Let \( U \in \mathcal{U}_u \). From (iii) and the definition of \( \mu_u \) we have that

\[ \mu_u(U) \leq \sum_{a \in C(u) \cap U} \text{Per } \text{im}_\tau(u, a) = \overline{\mu_u}(U). \]

To prove the reverse inequality, choose a sequence of functions \( \{\phi_j\}_{j \in \mathbb{N}} \subset C_c^1(\Omega) \), with \( \|\phi_j\|_\infty \leq 1 \), that converges pointwise to \( \chi_U \). Using Riesz’ representation theorem, and the dominated convergence theorem for Radon measures, we obtain that

\[ \overline{\mathcal{E}}_u(U, g) = \lim_{j \to \infty} \overline{\mathcal{E}}_u(\phi_j, g) = \lim_{j \to \infty} \mathcal{E}_u(\phi_j g) \leq \mu_u(U) \tag{4.7} \]

for every \( g \in C_c^1(\mathbb{R}^n, \mathbb{R}^n) \) with \( \|g\|_\infty \leq 1 \). Hence \( \mu_u(U) \geq \overline{\mu_u}(U) \), as desired.

(4) and (i) imply (ii). From (i) and the definitions of \( \mathcal{E}, \overline{\mathcal{E}}, \overline{\mu_u} \) and \( \mu_u \), it is clear that

\[ \sum_{a \in C(u)} \text{Per } \text{im}_\tau(u, a) = \overline{\mu_u}(\Omega) \leq \overline{\mathcal{E}}(\Omega) \leq \mathcal{E}(\Omega) = \mu_u(\Omega) \]

\[ = \sum_{a \in C(u)} \text{Per } \text{im}_\tau(u, a). \]

(4) and (i) imply (3). It requires no proof.

(3) imply (2). It follows from the definition of \( \mu_u \) and the fact that \( \overline{\mathcal{E}}_u(\phi \cdot g) = \mathcal{E}_u(\phi \cdot g) \) for any \( \phi \in C_c^1(\Omega) \) and \( g \in C_c^1(\mathbb{R}^n, \mathbb{R}^n) \).

(2) imply (1). This implication was established in the proof of [18, Proposition 7].

(1), (4) and (i) imply (iv). Recall from [18, Proposition 8] that \( \text{im}_\tau^*(u, \partial U) \subset \partial^* \text{im}_\tau(u, U) \). We now prove that (1) and (4) imply that the sets \( A_7 \) and \( A_1 \) of Theorem 4.8 are equal up to \( \mathcal{H}^{n-1} \)-null sets. Using Proposition 2.9, Lemma 2.6 and the divergence theorem, we obtain that

\[ \int_{\partial^* \text{im}_\tau(u, U) \setminus \text{im}_\tau(u, \partial U)} g(y) \cdot v_{\text{im}_\tau(u, U)}(y) \, d\mathcal{H}^{n-1}(y) \]

\[ = \int_{\text{im}_\tau(u, U)} \text{div } g(y) \, dy - \int_{\partial U} g(u(x)) \cdot (\text{cof } D u(x)) v_U(x) \, d\mathcal{H}^{n-1}(x) \tag{4.8} \]
for every $g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$. By Lemma 3.3, the expression at the right-hand side coincides with $\mathcal{E}_u(U, g)$. Thus, in light of Theorem 3.2 (iv), the divergence theorem and Lemma 4.10, we have that

$$\int_{A_7} g(y) \cdot v_{\text{img}(u, U)}(y) \, d\mathcal{H}^{n-1}(y) = -\int_{A_1} g(y) \cdot v(y) \, d\mathcal{H}^{n-1}(y),$$

$v(y)$ being defined as in equation (4.6). By approximation and Lusin’s theorem, the above identity is valid for every bounded Borel $g: \mathbb{R}^n \to \mathbb{R}^n$. In particular, setting $g(y) = v(y)$ for every $y \in A_1$ at which $v(y)$ is well defined, and $g(y) = 0$ otherwise, we conclude that $A_1 \subseteq A_7$. The inclusion $A_7 \subseteq A_1$ can be proved analogously. For future reference, we note that the same argument also shows that $v_{\text{img}(u, U)}(y) = -v_{\text{img}(u, a)}(y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \partial^* \text{img}(u, a)$, for all $a \in C(u)$.

Thanks to the inclusion $\text{img}(u, \partial U) \subseteq \partial^* \text{img}(u, U)$, we obtain from the relation $A_1 \equiv A_7$ that

$$\text{Per}(\text{img}(u, U)) = \mathcal{H}^{n-1}\left( \bigcup_{a \in C(u) \cap U} \partial^* \text{img}(u, a) \right) + \mathcal{H}^{n-1}(\text{img}(u, \partial U)).$$

This implies iv), because of Lemma 4.10 and (i).

\begin{proof}[Proof of Theorem 4.8] The proof is divided into several steps.

Proof of $A_7 \equiv A_1$ and $v_{\text{img}(u, U)} = -v_{\text{img}(u, a)}$ These properties were proved in Theorem 4.6.

Proof of $A_3 = A_4$ and $v_{\text{img}(u, \Omega)} = v_{\text{img}(u, U)}$. This is a consequence of Proposition 2.17 (i). Indeed, suppose $y \in A_4$. Since $\text{img}(u, U) \cap H^+(y, v_{\text{img}(u, U)}(y))$ has density 0 at the point $y$, and since $\text{img}(u, U)$ is disjoint with $\text{img}(u, \Omega \setminus U)$, it is clear that $y \in A_3$. Conversely, the inclusion $A_3 \subseteq A_4$ follows from the fact that $\text{img}(u, \Omega) \cap \text{img}(u, U) = \text{img}(u, U)$ a.e., which is a consequence of Theorem 3.2 (iii).

Proof of $A_5 = A_6$. It follows from the fact that

$$\text{img}(u, U) \cap \text{img}(u, \Omega) = \text{img}(u, U) \text{ a.e.}$$

Proof of $A_6 \equiv A_3$. From the definition of reduced boundary and standard properties of sets of finite perimeter, for $\mathcal{H}^{n-1}$-a.e. $y \in A_6$ we have that

$$D(\text{img}(u, U), y) \in \left\{ \frac{1}{2}, 1 \right\}, \quad D((\mathbb{R}^n \setminus \text{img}(u, U)) \cup \text{img}(u, \Omega), y) = \frac{1}{2}$$

and

$$\bar{D}(\mathbb{R}^n \setminus \text{img}(u, \Omega), y) \geq \frac{1}{2}. $$
Since $\partial^* \text{im}_T(u, U) \cong \text{im}_G(u, \partial U)$ (by Proposition 2.17), and since $\text{im}_G(u, \Omega)$ has density 1 at $y$ for every $y \in \text{im}_G(u, \partial U)$ (by Lemma 2.6), it is clear that $\text{im}_T(u, U)$ must have density 1 at $y$, and $\text{im}_G(u, \Omega)$ density $\frac{1}{2}$. This proves $A_6 \subset A_3$. The converse inclusion $A_3 \subset A_6$ requires no proof.

Proof of $A_5 = A_2$. It can be seen easily by using Theorem 3.2 (iii).

Proof of $A_1 \cong A_2$. From Theorem 3.2 (iv), the divergence theorem and Lemma 4.10 we observe that, for every $g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$,

$$\mathcal{E}_u(U, g) = -\int_{A_2} g(y) \cdot \mathbf{v}(y) \, d\mathcal{H}^{n-1}(y),$$

where $\mathbf{v}$ denotes the unit outward normal to $\bigcup_{a \in C(u) \cap U} \text{im}_T(u, a)$, which has finite perimeter because it coincides with $\text{im}_T(u, U) \setminus \text{im}_G(u, U)$. Arguing as in Theorem 4.6 it is possible to obtain that, for every bounded Borel $g : \mathbb{R}^n \to \mathbb{R}^n$,

$$\int_{A_2} g(y) \cdot \mathbf{v}(y) \, d\mathcal{H}^{n-1}(y) = \int_{A_1} g(y) \cdot \mathbf{v}(y) \, d\mathcal{H}^{n-1}(y),$$

where $\mathbf{v}$ is defined in (4.6). The conclusion can be obtained by evaluating the previous identity, first with $g := \mathbf{v}\chi_{A_1}$, then with $g := \mathbf{v}\chi_{A_2}$.

Proof of (ii). First we shall prove that for each point $a \in C(u)$, and $\mathcal{H}^{n-1}$-a.e. $y_0 \in \partial^* \text{im}_T(u, a)$,

$$\lim_{y \to y_0} u^{-1}(y) = a,$$  \hspace{1cm} (4.9)

with $v_0 := v_{\text{im}_T(u, a)}(y_0)$. Let $a \in C(u)$, and let $\{r_j\}_{j \in \mathbb{N}}$ be a decreasing sequence in $R_a$ tending to 0. From the facts that $A_1 \cong A_4$ and $v_{\text{im}_G(u, U)} = -v_{\text{im}_T(u, a)}$ for all $U \in \mathcal{U}$, it is clear that $\mathcal{H}^{n-1}$-a.e. $y_0 \in \partial^* \text{im}_T(u, a)$ belongs to the set $\partial^* \text{im}_G(u, B(a, r_j)) \cap \text{im}_T(u, B(a, r_j))$ for all $j \in \mathbb{N}$. Therefore,

$$D(\text{im}_G(u, B(a, r_j)) \cap H^+(y_0, v_0), y_0) = \frac{1}{2} \quad \text{for all} \quad j \in \mathbb{N}.$$ 

Thus, for every $r > 0$ the set

$$\{y \in \text{im}_G(u, \Omega) \cap H^+(y_0, v_0) : u^{-1}(y) \in B(a, r)\}$$

has density $\frac{1}{2}$ at $y_0$, as claimed.

From (4.9) and [18, Theorem 5], we obtain that $\partial^* \text{im}_T(u, a) \subset \Gamma(u)$ for all $a \in C(u)$. Moreover, since $\partial^* \text{im}_T(u, a) \subset \partial^* \text{im}_G(u, \Omega)$, and since

$$D(\text{im}_G(u, \Omega), y) = 1 \quad \text{for every} \quad y \in \Gamma_I(u)$$

(by definition), we conclude that $\bigcup_{a \in C(u)} \partial^* \text{im}_T(u, a) \subset \Gamma_T(u)$. 

Combining the previous result with Theorem 4.6 (i) and [18, Proposition 11], we find that for every $U \in \mathcal{C}_\mathbf{u}$,

$$\mu_{\mathbf{u}}(U) \geq \mathcal{H}^{n-1}((\Gamma_U(\mathbf{u}, U)) + \mathcal{H}^{n-1}(\{y \in \Gamma_V(\mathbf{u}) : (\mathbf{u}^{-1})^{-1}(y) \in U\})$$

$$\geq \mathcal{H}^{n-1}\left(\bigcup_{\mathbf{a} \in \mathcal{C}(\mathbf{u}) \cap U} \partial^* \operatorname{im}_T(\mathbf{u}, \mathbf{a})\right) = \mu_{\mathbf{u}}(U).$$

This implies that $\mathcal{H}^{n-1}(\Gamma_U(\mathbf{u})) = 0$ (by virtue of [18, Proposition 5 iv]) and that $\Gamma_V(\mathbf{u}) \subset \bigcup_{\mathbf{a} \in \mathcal{C}(\mathbf{u})} \partial^* \operatorname{im}_T(\mathbf{u}, \mathbf{a})$. The proof is completed by noting that we have $\nu_{\mathbf{u}^{-1}} = \nu_{\operatorname{img}(\mathbf{u}, \Omega)}$, by virtue of [18, Lemma 9 iia]).

Proof of (iii). Using Theorem 4.6 (i), we observe that

$$\mu_{\mathbf{u}}(\Lambda(\mathbf{u}, U) \setminus (C(\mathbf{u}) \cap U)) \leq \mu_{\mathbf{u}}(\Omega \setminus C(\mathbf{u})) = 0,$$

and that

$$\sum_{\mathbf{a} \in \mathcal{C}(\mathbf{u}) \cap U \setminus \Lambda(\mathbf{u}, U)} \operatorname{Per}_T(\mathbf{u}, \mathbf{a}) = \mu_{\mathbf{u}}(C(\mathbf{u}) \cap U \setminus \Lambda(\mathbf{u}, U))$$

$$\leq \mu_{\mathbf{u}}\Lambda(U \setminus \Lambda(\mathbf{u}, U)).$$

To obtain (iii), note that $\mu_{\mathbf{u}}\Lambda U = \mu_{\mathbf{u}}\Lambda(\mathbf{u}, U)$, as established in [18, Theorem 7].

5 Regularity of the inverses, Lusin’s condition and homeomorphisms

It was obtained in [18] that the inverse of a one-to-one a.e. map that is approximately differentiable a.e., and has finite surface energy $\mathcal{E}$, belongs to $SBV$. Using that analysis, as well as that of Sections 3 and 4, we prove in this section that the inverse of a $W^{1,n-1} \cap L^\infty$ orientation-preserving homeomorphism satisfying Lusin’s condition is in $W^{1,1}$. Thus, we recover a recent result by Csörnyei, Hencl and Malý [8] under the additional assumption that the deformation is orientation-preserving and satisfies Lusin’s condition. Although our result is weaker, the techniques used here provide an interesting connection between Lusin’s condition and surface energy for $W^{1,n-1} \cap L^\infty$ homeomorphisms.

Recall that a map is said to satisfy Lusin’s condition if the image of a set of zero measure is a set of zero measure. This implies (see, e.g., [31, IV.1.4 Corollary 2]) that the image of a measurable set is measurable.

It was proved in [18, Theorems 2 and 3] that suitable truncations of the inverse of $\mathbf{u}$ belong to $SBV$. In the following proposition we calculate their distributional derivative.
Proposition 5.1. Suppose \( u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n) \) satisfies condition INV and is such that \( \det Du > 0 \) a.e. and \( \mathcal{E}(u) < \infty \). Then for every \( U \in \mathcal{U}_u \), the map \( u_U^{-1} \) defined as

\[
u^{-1}(y) := \begin{cases} u^{-1}(y) & \text{if } y \in \operatorname{im}_G(u, U), \\ 0 & \text{if } y \in \mathbb{R}^n \setminus \operatorname{im}_G(u, U), \end{cases}
\]

belongs to \( SBV(\mathbb{R}^n, \mathbb{R}^n) \), and its distributional derivative is given by

\[
\langle Du_U^{-1}, g \rangle = \int_U \operatorname{adj} Du(x) \cdot g(u(x)) \, dx \\
- \int_{\partial U} x \cdot (g(u(x))) \operatorname{cof} Du(x) v_U(x) \, d\mathcal{H}^{n-1}(x) \\
+ \sum_{a \in C(u) \cap U} \int_{\partial^* \operatorname{im}_T(u, a)} a \cdot (g(y) v_{\operatorname{im}_T(u,a)}(y)) \, d\mathcal{H}^{n-1}(y),
\]

for each \( g \in C_c(\mathbb{R}^n, \mathbb{R}^{n \times n}) \).

Proof. It was proved in [18, Theorems 2 and 3] that \( u_U^{-1} \in SBV(\mathbb{R}^n, \mathbb{R}^n) \).

As in the proof of [18, Theorem 2], we fix \( \varphi \in C^\infty(\mathbb{R}) \) satisfying \( \varphi(t) = 0 \) for \( t \leq 0 \), \( \varphi(t) = 1 \) for \( t \geq 1 \), and \( \varphi' \geq 0 \). There exists \( j_0 \in \mathbb{N} \) such that for each \( j \in \mathbb{N} \), the function \( \eta_j : \Omega \to \mathbb{R} \) defined by

\[
\eta_j(x) := \varphi((j - j_0) \text{dist}(x, \partial U)), \quad x \in \Omega,
\]

is of class \( C^1 \). Moreover, \( \eta_j \to \chi_U \) pointwise in \( \Omega \) as \( j \to \infty \).

For \( j \in \mathbb{N} \) define \( \phi_j \in C^1_c(\Omega, \mathbb{R}^n) \) and \( \psi_j : \mathbb{R}^n \to \mathbb{R}^n \) as \( \phi_j(x) := \eta_j(x)x \) for \( x \in \Omega \), and

\[
\psi_j(y) := \begin{cases} \phi_j(u^{-1}(y)) & \text{if } y \in \operatorname{im}_G(u, \Omega), \\ 0 & \text{if } y \in \mathbb{R}^n \setminus \operatorname{im}_G(u, \Omega). \end{cases}
\]

Let \( \phi_1, \ldots, \phi_n \in C^1_c(\Omega) \) denote the components of \( \phi_j \).

Let \( g \in C^\infty_c(\mathbb{R}^n, \mathbb{R}^{n \times n}) \), and let \( g_1, \ldots, g_n \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \) correspond to the rows of \( g \). By [18, equation (18)],

\[
\langle D\psi_j, g \rangle = \int_\Omega \eta_j(x) g(u(x)) \cdot \operatorname{adj} Du(x) \, dx \\
+ \int_\Omega x \cdot g(u(x))(\operatorname{cof} Du(x)) D\eta_j(x) \, dx - \sum_{\alpha=1}^n \tilde{E}_u(\phi_j^\alpha, g_\alpha).
\]
Lusin’s condition and distributional determinant

We clearly have

\[
\lim_{j \to \infty} \int_{\Omega} \eta_j(x) \mathbf{g}(u(x)) \cdot \text{adj} \, D\mathbf{u}(x) \, dx = \int_{U} \mathbf{g}(u(x)) \cdot \text{adj} \, D\mathbf{u}(x) \, dx.
\]

From the proof of [18, Theorem 2] we infer that \( \langle D\psi_j, \mathbf{g} \rangle \to \langle D\mathbf{u}_U^{-1}, \mathbf{g} \rangle \), as \( j \to \infty \). Moreover, using Definition 2.15 (C4) we obtain that

\[
\lim_{j \to \infty} \int_{\Omega} x \cdot (\mathbf{g}(u(x)) \text{cof} \, D\mathbf{u}(x) D\eta_j(x)) \, dx
\]

\[
= -\int_{\partial U} x \cdot (\mathbf{g}(u(x)) \text{cof} \, D\mathbf{u}(x) v_U(x)) \, d\mathcal{H}^{n-1}(x).
\]

Fix \( \alpha \in \{1, \ldots, n\} \). By Theorem 4.6 (iii),

\[
\tilde{E}_u(\phi_j^\alpha, \mathbf{g}_\alpha) = -\sum_{a \in C(u)} \phi_j^\alpha(a) \int_{\partial^* \text{im}_G(u,a)} \mathbf{g}_\alpha(y) \cdot v_{im_T(u,a)}(y) \, d\mathcal{H}^{n-1}(y).
\]

On the other hand, \( \phi_j^\alpha(a) \to a^\alpha \chi_U(a) \) as \( j \to \infty \), for each \( a \in C(u) \). Moreover, by Theorem 4.6 (ii), we can take limits in the above equation to obtain that

\[
\lim_{j \to \infty} \tilde{E}_u(\phi_j^\alpha, \mathbf{g}_\alpha) = -\sum_{a \in C(u) \cap U} a^\alpha \int_{\partial^* \text{im}_G(u,a)} \mathbf{g}_\alpha(y) \cdot v_{im_T(u,a)}(y) \, d\mathcal{H}^{n-1}(y).
\]

This shows that (5.1) is valid for all \( \mathbf{g} \in C^\infty_c(\mathbb{R}^n, \mathbb{R}^{n \times n}) \). By density, it is also valid for all \( \mathbf{g} \in C_c(\mathbb{R}^n, \mathbb{R}^{n \times n}) \).

The conclusion of Proposition 5.1 is close in spirit to stating that \( u^{-1} \) is a Sobolev function. Indeed, as explained in [18, Section 3], the first term of the right-hand side of (5.1) controls the absolutely continuous part of \( D\mathbf{u}_U^{-1} \); the second term appears because of the artificial jump that \( u^{-1} \) creates at the boundary of \( \text{im}_G(u,U) \); the third term corresponds to the creation of new surface, and expresses that for \( W^{1,n-1} \) deformations the only created surface is due to cavitation. Formula (5.1), thus, suggests that the derivative of \( u^{-1} \) is absolutely continuous with respect to the Lebesgue measure, since the last two terms of the right-hand side of (5.1) should disappear when taking test functions \( \mathbf{g} \) compactly supported in \( \text{im}_G(u,U) \). The difficulty of making this argument work is that \( \text{im}_G(u,U) \) need not be open whenever \( U \subset \Omega \) is open; moreover, \( \text{im}_G(u,U) \) may differ drastically from \( u(U) \) if Lusin’s condition is not satisfied. That is why in Theorem 5.4 we assume that \( u \) is a homeomorphism that satisfies Lusin’s condition. The following two lemmas are auxiliary results for Theorems 5.4 and 5.5.
Lemma 5.2. Suppose that $u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ is a homeomorphism. Then $u$ satisfies condition INV, and $u(U) \subset \text{im}_T(u, U) \subset u(\tilde{U})$ for every nonempty open set $U \subset \subset \Omega$ with a $C^1$ boundary. If, in addition, $\det Du > 0$ a.e., then $\text{im}_T(u, x) = \{u(x)\}$ for every $x \in \Omega$.

Proof. Let $U \subset \subset \Omega$ be a nonempty open set with a $C^1$ boundary. According to Proposition 2.10 (iii), we choose the degree for continuous maps (which is defined in the set $\mathbb{R}^n \setminus u(\partial U)$) as a representative in $\mathbb{R}^n \setminus u(U)$ of the degree for $W^{n-1} \cap L^\infty$ maps (which is defined almost everywhere). Standard properties of the degree for continuous maps (see, e.g., [13, Sections 1.3 and 1.5]) imply that $\deg(u, \partial U, y) \in \{-1, 1\}$ for all $y \in u(U)$, and $\deg(u, \partial U, y) = 0$ for all $y \in \mathbb{R}^n \setminus u(\widetilde{U})$. Since $u(U)$ is open and $u(\widetilde{U})$ is closed, this implies that condition INV holds and that $u(U) \subset \text{im}_T(u, U) \subset u(\widetilde{U})$. This readily shows that $\text{im}_T(u, x) = \{u(x)\}$ for all $x \in \Omega$, provided that $\det Du > 0$ a.e. \hfill \Box

Lemma 5.3. Suppose that $u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ satisfies condition INV and $\det Du > 0$ a.e. Then the following assertions are equivalent:

(i) $\text{im}_T(u, U) = \text{im}_G(u, U)$ a.e., for each $U \in U_u$.

(ii) $\det Du = (\det Du) \mathcal{L}^n$.

(iii) $\mathcal{E}(u) = 0$.

If, in addition, $u$ is a homeomorphism, then any of conditions (i)–(iii) is equivalent to Lusin’s condition.

Proof. We first show that (iii) implies (ii). By Theorems 4.6 and 3.2,

$$\det Du = (\det Du) \mathcal{L}^n + \sum_{a \in C(u)} \mathcal{L}^n(\text{im}_T(u, a)) \delta_a$$

and $\text{Per im}_T(u, a) = 0$ for all $a \in C(u)$. As $\text{im}_T(u, a)$ is bounded for each point $a \in C(u)$, we get, thanks to the isoperimetric inequality, $\mathcal{L}^n(\text{im}_T(u, a)) = 0$.

The proof that (ii) implies (i) follows from Theorem 3.2 (iii).

We now show that (i) implies (iii). By Proposition 2.17 (vi) and Theorem 4.6, $\mathcal{E}(u) = \sum_{a \in C(u)} \text{Per im}_T(u, a)$. By Theorem 3.2, $C(u) = \emptyset$.

If $u$ is a homeomorphism satisfying Lusin’s condition, then condition (i) holds thanks to Lemma 5.2.

Finally, assume that $u$ is a homeomorphism satisfying condition (i). Thanks to Lemma 5.2, for any nonempty open set $U \subset \subset \Omega$ with a $C^1$ boundary, we obtain $u(U) = \text{im}_G(u, U)$ a.e., and, hence, by Proposition 2.3,

$$\int_U \det Du(x) \, dx = \mathcal{L}^n(u(U)).$$  (5.2)
Now define the positive measure $\lambda_1$ in $\Omega$ by
\[ \lambda_1(E) := \int_E \det D\mathbf{u}(x) \, dx, \quad \text{for any measurable set } E \subset \Omega, \]
and the outer measure $\lambda_2$ in $\Omega$ as
\[ \lambda_2(E) := \mathcal{L}^n_\ast(u(E)), \quad \text{for any } E \subset \Omega, \]
where $\mathcal{L}^n_\ast$ denotes the Lebesgue outer measure in $\mathbb{R}^n$. The fact that $\lambda_2$ is indeed an outer measure follows easily from the fact that $u$ is a homeomorphism. By Carathéodory’s criterion, the restriction of $\lambda_2$ to the Borel sets is a Borel measure. Therefore, $\lambda_1$ and $\lambda_2$ are two positive finite Borel measures that, thanks to (5.2), coincide in the open sets with a $C^1$ boundary. Consequently, the density of $\lambda_1$ with respect to $\lambda_2$ is 1 at every point of $\Omega$, and, hence, $\lambda_1$ and $\lambda_2$ coincide in all Borel sets. It follows that $\lambda_2(E) = 0$ for every set $E \subset \Omega$ with zero measure, and, therefore, $u$ satisfies Lusin’s condition.

**Theorem 5.4.** Let $u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ be a homeomorphism satisfying Lusin’s condition, and $\det Du > 0$ a.e. Then $u^{-1} \in W^{1,1}(u(\Omega), \mathbb{R}^n)$,
\[ Du^{-1}(u(x)) = Du(x)^{-1}, \quad \text{a.e. } x \in \Omega, \quad (5.3) \]
and for any measurable set $A \subset \Omega$,
\[ \int_{u(A)} |Du^{-1}(y)| \, dy = \int_A |\operatorname{cof} Du(x)| \, dx. \quad (5.4) \]

**Proof.** Note that $u(\Omega)$ is an open set in virtue of the invariance of domain theorem. Let $g \in C_c^\infty(u(\Omega), \mathbb{R}^{n \times n})$, and let $\bar{g} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^{n \times n})$ denote the extension of $g$ by zero. The set $u^{-1}(\text{spt } g)$ is compact and contained in $\Omega$. Choose $U \in \mathcal{U}_u$ satisfying $u^{-1}(\text{spt } g) \subset U$, which is possible thanks to [18, Lemma 5].

Let $\bar{u}^{-1}_U$ be the function defined in Proposition 5.1. We calculate $Du_{U}^{-1}$ in two different ways. By Lusin’s condition, we have that
\[ \langle Du_{U}^{-1}, \bar{g} \rangle = -\int_{\text{spt } g} u_{U}^{-1}(y) \cdot \operatorname{div} g(y) \, dy = -\int_{u(\Omega)} u^{-1}(y) \cdot \operatorname{div} g(y) \, dy, \]
whereas by Proposition 5.1 (the assumptions of which are satisfied thanks to Lemmas 5.2 and 5.3),
\[ \langle Du_{U}^{-1}, \bar{g} \rangle = \int_{\Omega} \operatorname{adj} Du(x) \cdot g(u(x)) \, dx \]
\[ -\int_{\partial U} x \cdot (g(u(x)) \, \operatorname{cof} Du(x) \nu_U(x)) \, d\mathcal{H}^{n-1}(x). \quad (5.5) \]
As $\text{spt } g$ is compactly supported in $u(U)$, we get $g(u(x)) = 0$ for all $x \in \partial U$. Hence, the second term of the right-hand side of (5.5) is zero. Finally, by Proposition 2.3,

$$
\int_{\Omega} \text{adj } Du(x) \cdot g(u(x)) \, dx = \int_{u(\Omega)} (Du(u^{-1}(y)))^{-1} \cdot g(y) \, dy.
$$

Altogether, we obtain

$$
-\int_{u(\Omega)} u^{-1}(y) \cdot \text{div } g(y) \, dy = \int_{u(\Omega)} (Du(u^{-1}(y)))^{-1} \cdot g(y) \, dy,
$$

which shows that $u^{-1}$ is a Sobolev function and that (5.3) holds.

Formula (5.4) follows from Proposition 2.3, Lusin’s condition and (5.3). □

The assumptions of Theorem 5.4 exclude, of course, cavitation, but if $u$ satisfies, for instance, the assumptions of Theorem 4.6, we may apply Theorem 5.4 to the restriction of $u$ to $\Omega \setminus C(u)$.

The techniques of this paper, which ultimately are based on the machinery of [18], are not enough to remove the assumption of Lusin’s condition from Theorem 5.4, as done in [8]. While the need to include Lusin’s condition as a hypothesis prevents Theorem 5.4 from being more general, it can nevertheless be given a positive interpretation as a regularity result. In fact, the following result shows a connection between Lusin’s condition and surface energy. This gives an additional argument in favour of the use of the surface energy $E$ in the modelling of cavitation (and fracture), since it suggests that deformations with finite surface energy are expected to satisfy Lusin’s condition (cf. [28, Section 10]).

**Theorem 5.5.** Suppose that $u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ is a homeomorphism satisfying $\det Du > 0$ a.e. Then, one of the following options occurs:

(i) $u$ satisfies Lusin’s condition and $E(u) = 0$.

(ii) $u$ does not satisfy Lusin’s condition, $E(u) = \infty$, $Du^{-1} = 0$ a.e. in the set $u(\Omega \setminus \Omega_0)$, and

$$
\mathcal{H}^{n-1}\left(\{y \in \mathbb{R}^n : \tilde{D}(\text{im}_G(u, \Omega), y) > 0, \tilde{D}(u(\Omega \setminus \Omega_0), y) > 0\}\right) = \infty,
$$

where $\Omega_0$ is the set of Definition 2.4.

**Proof.** By a result of Csörnyei, Hencl and Malý [8, Theorem 1.2], $u^{-1}$ belongs to $W^{1,1}_{\text{loc}}(u(\Omega), \mathbb{R}^n)$ and satisfies

$$
\det Du^{-1}(y) \neq 0 \quad \text{for a.e. } y \in u(\Omega) \text{ such that } Du^{-1}(y) \neq 0.
$$

(5.7)
Suppose $\mathcal{E}(u) < \infty$. By Theorems 4.6 and 3.2, $\text{Det } D\mathbf{u}$ is a measure and

$$\text{Det } D\mathbf{u} = (\det D\mathbf{u}) \mathcal{L}^n + \sum_{a\in C(u)} \mathcal{L}^n(\text{im}_T(u, a)) \delta_a.$$ 

Moreover, by Lemma 5.2,

$$\text{im}_T(u, a) = \{u(a)\} \text{ for all } a \in C(u).$$

Therefore, $C(u) = \emptyset$ and, by Lemma 5.3, condition (i) holds.

Suppose $\mathcal{E}(u) = \infty$. By Lemma 5.3, Lusin’s condition does not hold. The set $u(\Omega \setminus \Omega_0)$ is measurable, as so are the sets $u(\Omega)$ (thanks to the invariance of domain theorem) and $u(\Omega_0)$ (as a consequence of Proposition 2.3). Define $F$ as the set of $y \in u(\Omega)$ such that $\det D\mathbf{u}^{-1}(y) \neq 0$, which, thanks to (5.7), coincides a.e. with the set of $y \in u(\Omega)$ such that $D\mathbf{u}^{-1}(y) \neq 0$. An application of Proposition 2.3 to $D\mathbf{u}^{-1}$ shows that

$$\int_{F \setminus u(\Omega_0)} |\det D\mathbf{u}^{-1}(y)| \, dy \leq \mathcal{L}^n(\Omega \setminus \Omega_0) = 0.$$ 

Therefore, $\mathcal{L}^n(F \setminus u(\Omega_0)) = 0$ and, hence, $D\mathbf{u}^{-1} = 0$ a.e. in $u(\Omega \setminus \Omega_0)$.

If $\text{Per}(\text{im}_G(u, U)) < \infty$ for all $U \in \mathcal{U}_u$, then, by Theorem 4.6, $\text{Det } D\mathbf{u}$ is a measure and

$$\infty = \mathcal{E}(u) = \sum_{a\in C(u)} \text{Per } \text{im}_T(u, a).$$

But by Lemma 5.2, $\text{Per}(\text{im}_T(u, a)) = 0$ for all $a \in C(u)$, which is a contradiction. Therefore, there exists $U \in \mathcal{U}_u$ such that $\text{Per}(\text{im}_G(u, U)) = \infty$. Fix such a $U$.

By Proposition 2.17 (vi), $\text{im}_T(u, U)$ has finite perimeter. Moreover, the inclusion

$$\partial^- \text{im}_G(u, U)$$ 

$$\subset \partial^- \text{im}_T(u, U) \cup \{y \in \mathbb{R}^n : \tilde{D}(\text{im}_G(u, U), y) > 0, \tilde{D}(u(\tilde{U} \setminus \Omega_0), y) > 0\}$$

holds thanks to Proposition 2.17 (i) and Lemma 5.2. As $\text{Per}(\text{im}_G(u, U)) = \infty$ and $\text{Per}(\text{im}_T(u, U)) < \infty$, it follows that

$$\mathcal{H}^{n-1}\left(\{y \in \mathbb{R}^n : \tilde{D}(\text{im}_G(u, U), y) > 0, \tilde{D}(u(\tilde{U} \setminus \Omega_0), y) > 0\}\right) = \infty.$$ 

and, hence, (5.6) holds. This concludes the proof.

We point out that examples of Theorem 5.5 (ii) can be found in [30].
6 Singular part of the distributional determinant without invertibility

This section is motivated by a related work of Mucci [25], who addressed the problem of characterizing the singular part of $\text{Det } Du$ as a sum of Dirac masses. We restrict our attention, in particular, to one of the propositions of his paper, [25, Proposition 3.1], which states that the isoperimetric inequality in the proof of [28, Theorem 8.4] is valid in a certain class of deformations allowing for $(n-1)$-dimensional fractures. As pointed out in [26] and [17, Section 7], that statement is false. Nevertheless, if the ideas of Mucci are transported (from the context of Cartesian maps with fractures) to the $W^{1,n-1}$ setting, they do yield a new result in the spirit of (1.2) (see Theorem 6.2).

**Proposition 6.1** (cf. [25, Proposition 3.1]). Let $u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ satisfy $\text{det } Du \in L^1(\Omega)$ and $\mathcal{E}(u) < \infty$. Then, for every $U \in \mathcal{U}_u$ we have that

$$|\tilde{\mathcal{E}}_u(U, g)| \leq c_n \|\text{div } g\|_\infty \tilde{\mu}_u(U) \frac{n}{n-1}, \quad g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n),$$

where $c_n > 0$ is a constant that only depends on $n$.

**Proof.** By virtue of Propositions 2.3 and 2.10, and Lemma 3.3, for every $U \in \mathcal{U}_u$ and every $g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$, we may write

$$\int_{\mathbb{R}^n} \Delta_U(y) \text{div } g(y) \, dy = -\tilde{\mathcal{E}}_u(U, g), \quad (6.1)$$

where the function $\Delta_U : \mathbb{R}^n \rightarrow \mathbb{Z}$ is defined by

$$\Delta_U(y) := \text{deg}(u, \partial U, y) - \sum_{x \in \Omega_d \cap U} \text{sgn}(\text{det } Du(x)), \quad y \in \mathbb{R}^n. \quad (6.2)$$

Here $\Omega_d$ denotes the approximate differentiability set of $u$, as specified in Proposition 2.3. Formula (6.1) shows that the $\text{BV}$ seminorm of $\Delta_U$ is bounded by $\tilde{\mu}_u(U)$, whereas formula (6.2) shows that $\Delta_U$ vanishes a.e. in $\mathbb{R}^n \setminus B(0, \|u\|_\infty)$. Therefore, by the Poincaré inequality, $\Delta_U \in \text{BV}(\mathbb{R}^n)$. Since $\text{BV}(\mathbb{R}^n)$ is continuously embedded in $L^{\frac{n}{n-1}}(\mathbb{R}^n)$ (see, e.g., [1, Theorem 3.47]), there exists a constant $c_n$ (depending only on $n$) such that

$$\int_{\mathbb{R}^n} |\Delta_U(y)| \frac{n}{n-1} \, dy \leq c_n \tilde{\mu}_u(U) \frac{n}{n-1}.$$

Consequently, for any $g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$,

$$|\tilde{\mathcal{E}}_u(U, g)| \leq \|\text{div } g\|_\infty \int_{\mathbb{R}^n} |\Delta_U(y)| \, dy \leq c_n \|\text{div } g\|_\infty \tilde{\mu}_u(U) \frac{n}{n-1},$$
where we have used (6.1) and the fact that $\Delta_U$ is integer-valued. This completes the proof. \hfill \Box

Note that if the $\mathbf{g}$ of Proposition 6.1 coincides with $\frac{1}{n} \mathbf{id}$ in $\mathcal{B}(0, \|\mathbf{u}\|_{\infty})$, we obtain

$$|(\det D\mathbf{u} - (\det D\mathbf{u})\mathcal{L}^n)(U)| \leq c_n \bar{\mu}_u(U)^{\frac{n-1}{n}}$$

as a particular case.

Let us introduce the following notation, which is used in Theorem 6.2. Given a Borel measure $\mu$ on a set $E \subset \mathbb{R}^n$, we define

$$\text{At}(\mu) := \left\{ x \in E : \limsup_{r \searrow 0} \mu(B(x, r) \cap E) > 0 \right\}.$$ 

If $\mu(A) = \mu(A \cap \text{At}(\mu))$ for every Borel set $A \subset E$, we say that $\mu$ is purely atomic. As is well known, if $\mu$ is a Radon measure, then $\text{At}(\mu)$ is countable.

**Theorem 6.2** (cf. [25, Proposition 4.2]). Let $\mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ be such that $\det D\mathbf{u} \in L^1(\Omega)$ and $\mathcal{E}(\mathbf{u}) < \infty$. Then for every $\mathbf{g} \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$, the measure $\mathcal{E}_\mathbf{u}(\cdot, \mathbf{g})$ is purely atomic, and $\text{At}(\mathcal{E}_\mathbf{u}(\cdot, \mathbf{g})) \subset \text{At}(\mu_\mathbf{u})$. Moreover, $\det D\mathbf{u}$ is a measure and

$$\det D\mathbf{u} = (\det D\mathbf{u})\mathcal{L}^n + \sum_{\mathbf{a} \in C(\mathbf{u})} \det D\mathbf{u}(\{\mathbf{a}\})\delta_\mathbf{a}. \quad (6.3)$$

Theorem 6.2 follows by combining Proposition 6.1 with the following lemma, due to Mucci [25, Lemma 4.4], which captures the essence of the second part of the proof of [28, Theorem 8.4] (and of its $W^{1,n-1}$ counterpart in [7, Theorem 4.2]).

**Lemma 6.3.** Let $\mu$ be a vector-valued Radon measure in $\Omega$, and let $\lambda$ be a non-negative Radon measure in $\Omega$. Suppose that there exist $c > 0$ and $\alpha > 1$ such that for each $x \in \Omega$,

$$|\mu(\tilde{B}(x, r))| \leq c \lambda(\tilde{B}(x, r))^\alpha \quad \text{for a.e. } r \in (0, \text{dist}(x, \partial \Omega)).$$

Then $\mu$ is purely atomic, and $\text{At}(\mu) \subset \text{At}(\lambda)$.

Although it is not mentioned in [25], one of the steps in the proof of Lemma 6.3 is to show that $\mu$ is absolutely continuous with respect to $\lambda$. We provide a proof of this fact for the convenience of the reader.

**Proof.** Let $E$ be any Borel set such that $\lambda(E) = 0$. For any $\varepsilon > 0$ there exists an open set $U$ such that $E \subset U \subset \Omega$ and $\lambda(U) \leq \varepsilon$. By the regularity of $\mu$, we can find a decreasing sequence $\{U_j\}_{j \in \mathbb{N}}$ of open sets, all containing $E$ and contained in $U$, such that

$$\mu(E) = \lim_{j \to \infty} \mu(U_j). \quad (6.4)$$

For any Borel \( A \subseteq U \) we have that \( \lambda(A) \leq \lambda(U) \leq \varepsilon \), and, hence, we obtain \( \lambda(A)^\alpha \leq \varepsilon^{\alpha-1} \lambda(A) \). Thus for every \( x \in U \) and a.e. \( r \in (0, \text{dist}(x, \partial U)) \),

\[
|\mu(\bar{B}(x, r))| \leq c \varepsilon^{\alpha-1} \lambda(\bar{B}(x, r)).
\]

Fix \( j \in \mathbb{N} \). Using Besicovitch’s covering theorem, we can find a family \( \mathcal{B}_j \) of disjoint closed balls contained in \( U \) such that \( |\mu(B)| \leq c \varepsilon^{\alpha-1} \lambda(B) \) for all \( B \in \mathcal{B}_j \) and \( (|\mu| + \lambda)(U_j \setminus \bigcup \mathcal{B}_j) = 0 \). Thus,

\[
|\mu(U_j)| \leq \sum_{B \in \mathcal{B}_j} |\mu(B)| \leq c \varepsilon^{\alpha-1} \sum_{B \in \mathcal{B}_j} \lambda(B) = c \varepsilon^{\alpha-1} \lambda(U_j) \leq c \varepsilon^\alpha.
\]

Using (6.4), we obtain that \( |\mu(E)| \leq c \varepsilon^\alpha \). Since \( \varepsilon \) is arbitrary, it follows that \( \mu \) is absolutely continuous with respect to \( \lambda \).

The proof is completed as in [25, Lemma 4.4].

**Proof of Theorem 6.2.** Fix \( g \in C^1_c(\mathbb{R}^n, \mathbb{R}^n) \). Given \( x \in \Omega \), all but countably many \( r \in (0, \text{dist}(x, \partial \Omega)) \) satisfy \( \bar{E}_u(\partial B(x, r), g) = 0 \) and \( \mu_u(\partial B(x, r)) = 0 \). Therefore, by Proposition 6.1 (and the fact that \( \bar{\mu} \leq \mu \), see (4.7)), the measures \( \bar{E}_u(\cdot, g) \) and \( \mu_u \) satisfy the assumptions of Lemma 6.3. Hence, \( \bar{E}_u(\cdot, g) \) is purely atomic, and is concentrated on \( \text{At}(\mu_u) \). The last conclusion of the theorem is obtained by considering a function \( g \) coinciding with \( \frac{1}{n} \text{id} \) in \( B(0, \|u\|_\infty) \).

7 **Det = det does not imply \( \mathcal{E} = 0 \)**

One of the novelties in Theorem 6.2 is that it does not require the deformation \( u \) to be invertible or orientation-preserving. In this section, however, we advocate the use of condition INV [28, Section 3] in the modelling of cavitation, on the grounds that if this condition is removed, then an important part of our understanding of the problem is lost. In particular, the interpretation of equation (1.2) as stating that \( u \) opens a cavity at each \( a \in C(u) \), and that the coefficients \( c_a \) give the volumes of the cavities is no longer valid if condition INV is not satisfied. In order to justify this last assertion, we consider two examples. With the first example (that of cavitation followed by eversion, due to Müller and Spector [28, p. 17, Remark 4]) we show that without condition INV it is impossible to provide a sensible notion of ‘created cavity’. With the second example (see Theorem 7.2) we show that even if the singularities of a deformation do correspond to the formation of cavities, they may escape unnoticed by the distributional determinant. In other words, without
condition INV the characterization of $\text{Det } Du$ in equation (6.3) may not give a full account of the cavitation process. Both situations clearly contrast with what has been established in Theorem 3.2 and in Section 4.

As in [28, p. 17, Remark 4], suppose that $\Omega = B(0, 1) \subset \mathbb{R}^3$, and let the map $u : \Omega \to \mathbb{R}^3$ be given by

$$u(x) = \frac{2 - |x|}{|x|}(x_1, x_2, -x_3), \quad x = (x_1, x_2, x_3) \in \Omega.$$  \hspace{1cm} (7.1)

For this deformation, $\text{Det } Du$ is of the form $\text{Det } Du = (\text{det } Du)^3 + c \delta_0$, but it cannot be said that $u$ has created a cavity. Indeed,

$$\text{im}_G(u, \Omega) = B(0, 2) \setminus \tilde{B}(0, 1) \quad \text{and} \quad u(\partial \Omega) = \partial \Omega.$$  

Hence, if we were to say that $u$ has created a cavity, it would only make sense to say that the cavity is $B(0, 1)$, since this is the region enclosed by the body in the deformed configuration. But the ‘cavity surface’ would then be $\partial \Omega = u(\partial \Omega)$, which is the previously existing boundary (the image of the boundary) and is certainly not a cavity surface. Moreover, it can be seen that $c = -|\mathcal{L}^3(B(0, 2))$, so $c$ is not the volume of the ‘cavity’ $B(0, 1)$, but the volume (up to a sign) of the topological image of $\Omega$, namely $B(0, 2)$. Note also that $c$ is negative (as is $\deg(u, \partial B(0, r), y)$, for any $r \in (0, 1)$ and $y \in \text{im}_T(u, B(0, r))$), in spite of $\text{det } Du$ being positive in $\Omega \setminus \{0\}$.

In light of the results in [7, 28], and of what has been established in the previous sections, the natural expectation is that if we have $u \in W^{1,n-1} \cap L^\infty$ and $\text{Det } Du = \text{det } Du$, then $\tilde{\varepsilon}(u) = \varepsilon(u) = 0$ (that is, that $u$ creates no new surface). While the converse is clearly true (see Proposition 7.1), without condition INV the claim under consideration is false, as Theorem 7.2 shows.

**Proposition 7.1.** Let $q \in [1, \infty]$, let $q'$ be its conjugate exponent, and consider $u \in W^{1,1} \cap L^q$ satisfying

$$\text{det } Du \in L^1(\Omega) \quad \text{and} \quad \text{cof } Du \in L^q(\Omega, \mathbb{R}^n).$$

If $\tilde{\varepsilon}(u) = 0$, then $\text{Det } Du = \text{det } Du$.

**Proof.** Fix $\phi \in C_c^\infty(\Omega)$. For each $M > 0$ define $\Omega_M := \{x \in \Omega : |u(x)| > M\}$. Then

$$\left| \int_{\Omega_M} \left[ \frac{1}{n} u \cdot (\text{cof } Du) D\phi + \phi \text{det } Du \right] dx \right| \leq \frac{1}{n} \|u\|_{L^{q'}(\Omega_M, \mathbb{R}^n)} \|\text{cof } Du\|_{L^q(\Omega_M, \mathbb{R}^{n \times n})} \|D\phi\|_\infty + \|\phi\|_\infty \|\text{det } Du\|_{L^1(\Omega_M)}.$$
Now choose any \( g \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n) \) such that \(|g| \leq \frac{1}{n}|\text{id}|\) everywhere, \( g = \frac{1}{n}\text{id} \) in \( B(0, M) \), and \( \|\text{div} \ g\|_\infty \leq 2 \). Then

\[
\left| \int_{\Omega_M} \left[ (g \circ u) \cdot (\text{cof} \ D u) D \phi + \phi(\text{div} \ g) \circ u \text{ det} \ D u \right] dx \right|
\]

\[
\leq \|g \circ u\|_{L^{q'}(\Omega_M, \mathbb{R}^n)} \|\text{cof} \ D u\|_{L^q(\Omega_M, \mathbb{R}^{n \times n})} \|D \phi\|_\infty
+ 2 \|\phi\|_\infty \|\text{det} \ D u\|_{L^1(\Omega_M)}.
\]

As \( \tilde{\mathcal{E}}(u) = 0 \),

\[
\left| \int_{\Omega} \left[ \frac{1}{n} u \cdot (\text{cof} \ D u) D \phi + \phi \text{ det} \ D u \right] dx \right|
\]

\[
= \left| \int_{\Omega_M} \left[ \left( g \circ u - \frac{1}{n} u \right) \cdot (\text{cof} \ D u) D \phi + \phi((\text{div} \ g) \circ u - 1) \text{ det} \ D u \right] dx \right|
\]

\[
\leq 2 \|u\|_{L^{q'}(\Omega_M, \mathbb{R}^n)} \|\text{cof} \ D u\|_{L^q(\Omega_M, \mathbb{R}^{n \times n})} \|D \phi\|_\infty
+ 3 \|\phi\|_\infty \|\text{det} \ D u\|_{L^1(\Omega_M)}.
\]

Now, by the absolute continuity of the integral with respect to the Lebesgue measure, the quantities \( \|u\|_{L^{q'}(\Omega_M, \mathbb{R}^n)} \), \( \|\text{cof} \ D u\|_{L^q(\Omega_M, \mathbb{R}^{n \times n})} \) and \( \|\text{det} \ D u\|_{L^1(\Omega_M)} \) can be made as small as we wish, by taking \( M \) sufficiently large. Hence

\[
\int_{\Omega_M} \left[ \frac{1}{n} u \cdot (\text{cof} \ D u) D \phi + \phi \text{ det} \ D u \right] dx = 0
\]

and \( \text{Det} \ D u = \text{det} \ D u \).

**Theorem 7.2.** Let \( \Omega \) be the unit ball in \( \mathbb{R}^2 \). There exists a function \( u \) that is one-to-one a.e., and satisfies \( u \in W^{1,p}(\Omega, \mathbb{R}^2) \cap L^\infty(\Omega, \mathbb{R}^2) \) for every \( p \in [1, 2) \), \( \text{det} \ D u > 0 \) a.e., \( \text{Det} \ D u = \text{det} \ D u \), and \( \mathcal{E}(u) > 0 \).

**Proof.** Let the map \( \tilde{u} : \mathbb{R} \to \mathbb{R}^2 \) be given by \( \tilde{u}(\theta) = (\tilde{u}_1(\theta), \tilde{u}_2(\theta)) \), with

\[
\tilde{u}_1(\theta) := \sin \theta + \cos^3 \theta, \quad \tilde{u}_2(\theta) := \cos \theta + \frac{8}{3} \sin \theta \cos^2 \theta, \quad \theta \in \mathbb{R}. \quad (7.2)
\]

Set \( \Gamma := \tilde{u}([0, 2\pi]) \) and define the map \( \nu_\Gamma : \Gamma \to \mathbb{S}^1 \) as

\[
\nu_\Gamma(\tilde{u}(\theta)) := \frac{(\tilde{u}_2'(\theta), -\tilde{u}_1'(\theta))}{|(\tilde{u}_2'(\theta), -\tilde{u}_1'(\theta))|}, \quad \theta \in [0, 2\pi].
\]

Since \( \frac{\partial \tilde{u}}{\partial \theta} \neq 0 \) for all \( \theta \in [0, 2\pi] \), the vector \( \nu_\Gamma \) is well defined and perpendicular to \( \Gamma \) at each point \( \tilde{u}(\theta) \) of this curve (see Figure 1).
Lusin’s condition and distributional determinant

Figure 1. The set $\Gamma$ and the orientation $\nu_\Gamma$ of Theorem 7.2. Regions $A$, $B$, $C$, $D$ and $E$ compose the set $\text{im}_\Gamma(u, 0)$.

Let the angles $0 < \theta_1 < \cdots < \theta_4 < \theta_6 < \cdots < \theta_9 < 2\pi$ satisfy

$$\bar{u}(\theta_1) = \bar{u}(\theta_9), \quad \bar{u}(\theta_2) = \bar{u}(\theta_8), \quad \bar{u}(\theta_3) = \bar{u}(\theta_7), \quad \bar{u}(\theta_4) = \bar{u}(\theta_6).$$

Thus, $\bar{u}(\theta_1), \ldots, \bar{u}(\theta_4)$ are the points of self-intersection of $\Gamma$. Note that the curve forms a positive angle at those points. Take any $\theta_0 \in [0, \theta_1)$ and $\theta_5 \in (\theta_4, \theta_6)$, and define $\theta_{10} := \theta_0 + 2\pi$. Consider the curves $\gamma_i := \bar{u}((\theta_i, \theta_{i+1}))$, $i \in \{0, \ldots, 9\}$. The construction is depicted in Figure 2.

Figure 2. The points $(\cos \theta_i, \sin \theta_i)$, $i \in \{0, \ldots, 10\}$, in the reference configuration are denoted by $p_i$. The figure shows the curves $\gamma_i$, and the points $q_j := \bar{u}(\theta_j)$, $j \in \{1, \ldots, 4\}$, of self-intersection of $\Gamma$.

Thanks to [14, Theorem 16.25.2], there exists $\delta_0 > 0$ such that, for every index $i \in \{0, \ldots, 9\}$, the map

$$w_i : (-\delta_0, \delta_0) \times \gamma_i \to \mathbb{R}^n,$$

$$(t, y) \mapsto y + t\nu_\Gamma(y)$$
is a diffeomorphism onto its image, which is included in \( N(\gamma_i, \delta_0) \). Moreover, if for each \( y \in w_i((\delta_0, \delta_0) \times \gamma_i) \) we denote \((t_1, y_1) := w_i^{-1}(y)\), then we obtain that \( t_1 = \text{dist}(y, \gamma_i) \), and \( y_1 \) is the only point on \( \gamma_i \) at a distance \( t_1 \) from \( y \).

Let \( g : [0, 2\pi] \rightarrow [0, 1] \) be smooth and such that \( g^{-1}(0) = \{\theta_0, \ldots, \theta_9\} \). Define \( u : \Omega \rightarrow \mathbb{R}^2 \), in polar coordinates, as

\[
    u(r, \theta) := \tilde{u}(\theta) + \delta g(\theta) v_\Gamma(\tilde{u}(\theta)), \quad r \in (0, 1), \quad \theta \in [0, 2\pi]. \tag{7.3}
\]

where the value of \( \delta \in (0, \delta_0) \) is to be specified later in the proof. Denote the sets \((0, 1) \times (\theta_i, \theta_{i+1}) \), \( i \in \{0, \ldots, 9\} \), by \( A_i \), and the corresponding circular sectors \( \{(r \cos \theta, r \sin \theta) : (r, \theta) \in A_i\} \) by \( D_i \). Since we have \( u(r, \theta) = w_i(\delta g(\theta), \tilde{u}(\theta)) \) for all \( (r, \theta) \in A_i \), the restriction of \( u \) to \( D_i \) is a diffeomorphism onto its image, which is contained in \( N(\gamma_i, \delta) \). Furthermore, if \( \delta \) is small enough, the sets \( u(D_i), i \in \{0, \ldots, 9\} \), are disjoint (see Figure 3). This proves that \( u \) is continuous, smooth away from the origin, and one-to-one a.e.

We now show that \( u \in W^{1,p}(\Omega, \mathbb{R}^2) \cap L^\infty(\Omega, \mathbb{R}^2) \) for all \( p \in [1, 2) \). The fact that \( \|u\|_\infty \leq \|\tilde{u}\|_\infty + \delta < \infty \) clearly follows from (7.3). In order to show that \( u \in W^{1,p}(\Omega, \mathbb{R}^2) \) for all \( p \in [1, 2) \), recall that the gradient of a function defined in polar coordinates can be written as

\[
    Du = \frac{\partial u}{\partial r} \otimes e_r + \frac{1}{r} \frac{\partial u}{\partial \theta} \otimes e_\theta, \tag{7.4}
\]

where \( e_r, e_\theta \) denote the unit vectors \( e_r := (\cos \theta, \sin \theta) \) and \( e_\theta := (-\sin \theta, \cos \theta) \). Here the expression \( a \otimes b \), with \( a, b \in \mathbb{R}^n \), denotes the tensor product of \( a \) and \( b \), which is the linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) defined as \( v \mapsto a(b \cdot v) \). From (7.3) we have that

\[
    \frac{\partial u}{\partial r} = \delta g(\theta) v_\Gamma(\tilde{u}(\theta)), \quad \frac{\partial u}{\partial \theta} = \frac{d\tilde{u}}{d\theta} + \delta r \frac{d}{d\theta}(g(\theta) v_\Gamma(\tilde{u}(\theta))). \tag{7.5}
\]
The leading term of $|Du|$, therefore, is given by $\frac{1}{r} \frac{\|\tilde{u}\|_{\infty}}{\|\nabla u\|_{\infty}}$. This implies that $Du$ has the desired integrability.

Let $u_1, u_2$ denote the components of $u$. Using formula (7.4) it is easy to find that $(\text{cof } Du)e_r = \frac{1}{r} \left( \frac{\partial u_2}{\partial \theta} - \frac{\partial u_1}{\partial \phi} \right)$. Combining this with (7.5), we obtain

$$
\det Du = (\text{cof } Du)e_r \cdot (Du)e_r = \frac{\delta g(\theta)}{r} \left[ \left| \frac{\partial \tilde{u}}{\partial \theta} \right| + \delta rg(\theta) \det \left( \nu_G, \frac{d\nu_G}{d\theta} \right) \right].
$$

It follows that, if $\delta$ is sufficiently small,

$$
\det Du(x) > 0 \quad \text{for all } x \in \Omega_0 = D_0 \cup \cdots \cup D_{9}.
$$

It remains to show that $\text{Det } Du = \det Du$ and $\mathcal{E}(u) > 0$. Define the map

$$
u^{-1} \circ \text{id} : im_G(u, \Omega) \to \Omega_0 \times \mathbb{R}^n,
$$

$$y \mapsto (u^{-1}(y), y),
$$

where $u^{-1}$ denotes the inverse of $u|\Omega_0$. Then, for every $f \in C_c^1(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$,

$$
\mathcal{E}_u(f) = \lim_{r \searrow 0} \int_{\Omega \setminus B(0, r)} \text{div}(f \circ (u^{-1} \circ \text{id})) \, dy
$$

$$= \lim_{r \searrow 0} \int_{u(\partial B(0, r))} f(u^{-1}(y), y) \cdot v_u(\Omega \setminus B(0, r))(y) \, d\mathcal{H}^{n-1}(y)
$$

$$= -\lim_{r \searrow 0} \int_0^{2\pi} f((r \cos \theta, r \sin \theta), u(r, \theta)) \cdot (\text{cof } Du(r, \theta))e_r(r, \theta) \, d\theta.
$$

Combining this with (7.3), (7.5), and the fact that $(\text{cof } Du)e_r = \frac{1}{r} \left( \frac{\partial u_2}{\partial \theta} - \frac{\partial u_1}{\partial \phi} \right)$, we obtain

$$
\mathcal{E}_u(f) = -\int_\Gamma f(0, y) \cdot v_\Gamma(y) \, d\mathcal{H}^{n-1}(y).
$$

This shows that $\Gamma(u) = \Gamma$, and that $\tilde{\mathcal{E}}(u) = \mathcal{E}(u) = \mathcal{H}^1(\Gamma) > 0$. Finally, substituting $\phi(x)g(y)$ for $f(x, y)$ in formula (7.6), where $\phi \in C_c^\infty(\Omega)$ is arbitrary, and $g \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$ coincides with $\frac{1}{2} \text{id}$ in $B(0, \|u\|_{\infty})$, we obtain

$$
\text{Det } Du - (\det Du)\mathcal{E}^2 = \left( \int_0^{2\pi} (\tilde{u}_1(\theta)\tilde{u}_2'(\theta) - \tilde{u}_2(\theta)\tilde{u}_1'(\theta)) \, d\theta \right) \delta_0
$$

$$= \left( \int_0^{2\pi} \left[ -\frac{5}{6} \sin(2\theta) + \frac{11}{12} \sin(4\theta) + \frac{4}{3} \cos(2\theta) + \frac{1}{3} \cos(4\theta) \right] \, d\theta \right) \delta_0
$$

$$= 0.
$$

This completes the proof. □
We conclude this section with some remarks on Theorem 7.2. The example in (7.3) is inspired in the map $\bar{u} : \mathbb{R}^2 \to \mathbb{R}^2$ given by
\[
\bar{u}(x_1, x_2) := \frac{1}{|x|^3} \left( x_1^3 + x_1^2 x_2 + x_2^3, x_1^2 + \frac{8}{3} x_1^2 x_2 + x_1 x_2^2 \right)
\]
for $x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$. Note, for example, that $\bar{u} N 2$ is the restriction of $Q_u$ to $\partial B(0, 1)$. The map $\bar{u}$ also belongs to the space $W^{1,p}(\Omega, \mathbb{R}^2) \cap L^\infty(\Omega, \mathbb{R}^2)$ for every $p \in [1, 2)$, satisfies (7.6), and is such that $\det D\bar{u} = 0$ and $\bar{e}(\bar{u}) = \bar{e}(\bar{u}) = \mathcal{H}^1(\Gamma)$. The main difference between $u$ and $\bar{u}$ is that $\bar{u}$ is not one-to-one a.e. Nevertheless, since the definition of $\bar{u}$ is more explicit, it might prove more relevant than $u$ for certain purposes.

The reason why $\det D u = \det D\bar{u}$ in Theorem 7.2 is that in some of the regions enclosed by $\Gamma$ (see Figure 1) the orientation vector $\nu_\Gamma$ points into those regions, whereas in other regions $\nu_\Gamma$ points outwards. In light of Proposition 2.10, we find that the degree is $+1$ in the regions $A$, $C$ and $E$, while it is $-1$ in $B$ and $D$. The distributional determinant satisfies
\[
\text{Det } D u - (\det D u) \mathcal{L}^2 = \left( \mathcal{L}^2(A \cup C \cup E) - \mathcal{L}^2(B \cup D) \right) \delta_0,
\]
and $\bar{u}$ was chosen so that $\mathcal{L}^2(A \cup C \cup E) = \mathcal{L}^2(B \cup D)$.

8 Existence of minimizers in $W^{1,n-1}$ with condition INV

In this section, we prove the existence of minimizers for a functional modelling the energy of elastic materials undergoing cavitation, in a class of $W^{1,n-1}$ maps satisfying condition INV. As the counterexample of Conti and De Lellis [7, Section 6] shows, modelling the phenomenon of cavitation in $W^{1,n-1}$ is not straightforward, since the sequence of cofactors $\{\text{cof } D u_j\}_{j \in \mathbb{N}}$ may not be equiintegrable for a minimizing sequence $\{u_j\}_{j \in \mathbb{N}}$. This translates into the topological degree, the distributional determinant and the surface energy $\mathcal{E}$ being discontinuous with respect to the weak convergence. This obstacle disappears as soon as the equiintegrability of the cofactors is guaranteed through a suitable coercivity assumption on the elastic stored-energy function. However, the existence of minimizers still requires that condition INV is closed under the weak limit, which we prove by using the result on convergence of the geometric images obtained in [17].

The following property shows the continuity of the degree.

**Lemma 8.1.** Let $U \subset \mathbb{R}^n$ be a nonempty open set with a $C^1$ boundary. For each $j \in \mathbb{N}$, let $u_j, u \in W^{1,n-1}(\partial U, \mathbb{R}^n) \cap L^\infty(\partial U, \mathbb{R}^n)$. Suppose that
\[
\sup_{j \in \mathbb{N}} \|u_j\|_{L^\infty(\partial U, \mathbb{R}^n)} < \infty
\]
and
\[ u_j \to u \text{ in } \mathcal{H}^{n-1} \text{-a.e. in } \partial U, \quad (\Lambda_{n-1} D u_j) v_U \rightharpoonup (\Lambda_{n-1} D u) v_U \text{ in } L^1(\partial U, \mathbb{R}^n), \]
as \( j \to \infty \). Then \( \deg(u_j, \partial U, \cdot) \) tends to \( \deg(u, \partial U, \cdot) \) a.e. and weakly* in the space \( BV(\mathbb{R}^n) \), as \( j \to \infty \).

**Proof.** According to Proposition 2.10,
\[
\int_{\partial U} g(u_j(x)) \cdot \Lambda_{n-1}(D u_j(x)) v_U(x) \, d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^n} \deg(u_j, \partial U, y) \, \text{div} \, g(y) \, dy
\]
for all \( j \in \mathbb{N} \) and \( g \in C^1(\mathbb{R}^n, \mathbb{R}^n) \). Therefore, the total variation of \( \deg(u_j, \partial U, \cdot) \) is bounded by \( \sup_{j \in \mathbb{N}} \| \Lambda_{n-1} D u_j \|_{L^1(\partial U, \mathbb{R}^n)} \). By the Poincaré inequality (see, e.g., [1, Theorem 3.47]), \( \sup_{j \in \mathbb{N}} \| \deg(u_j, \partial U, \cdot) \|_{BV(\mathbb{R}^n)} < \infty \). Therefore, there exists \( \phi \in BV(\mathbb{R}^n) \) such that for a subsequence (not relabelled),
\[ \deg(u_j, \partial U, \cdot) \rightharpoonup \phi \text{ in } BV(\mathbb{R}^n) \quad \text{and} \quad \deg(u_j, \partial U, \cdot) \to \phi \text{ a.e. as } j \to \infty. \]

Passing to the limit in (8.1), we obtain
\[
\int_{\partial U} g(u(x)) \cdot \Lambda_{n-1}(D u(x)) v_U(x) \, d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^n} \phi(y) \, \text{div} \, g(y) \, dy.
\]
for all \( g \in C^1(\mathbb{R}^n, \mathbb{R}^n) \). For an explanation on how to pass to the limit, see, e.g., [32, Lemma 6.7], where the result is proved for the Lebesgue measure, but the proof is also valid for any finite Radon measure (such as \( \mathcal{H}^{n-1} \subseteq \partial U \)). By the uniqueness of the degree (see Proposition 2.10), we obtain that \( \phi = \deg(u, \partial U, \cdot) \) a.e., and that the whole sequence \( \{\deg(u_j, \partial U, \cdot)\}_{j \in \mathbb{N}} \) tends to \( \phi \).

The following result shows the weak continuity of the cofactors on almost all hypersurfaces.

**Lemma 8.2.** For each \( j \in \mathbb{N} \), let \( u_j, u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \) satisfy
\[ \text{cof } D u_j \rightharpoonup \text{cof } D u \text{ in } L^1(\Omega, \mathbb{R}^{n \times n}), \quad \text{as } j \to \infty. \]
Let \( U \subset \subset \Omega \) be a nonempty open subset with a \( C^2 \) boundary. Let \( \delta, \delta \) and \( U_t \) be as in Proposition 2.14. Then, for a.e. \( t \in (-\delta, \delta) \) there exists a subsequence \( \{j_k\}_{k \in \mathbb{N}} \) such that
\[ (\text{cof } D u_{j_k}) v_t \rightharpoonup (\text{cof } D u) v_t \text{ in } L^1(\partial U_t, \mathbb{R}^n), \quad \text{as } k \to \infty, \]
where \( v_t \) is the unit exterior normal to \( U_t \).
Proof. First we prove that for each \( \mathbf{v} \in W^{1, n-1}(\Omega, \mathbb{R}^n) \) there is a set \( N_{\mathbf{v}} \subset (-\delta, \delta) \) of measure zero such that for all \( t \in (-\delta, \delta) \setminus N_{\mathbf{v}} \) and all \( \varphi \in C^\infty(\Omega, \mathbb{R}^n) \),

\[
\int_{\partial U_t} \varphi(x) \cdot (\text{cof } D\mathbf{v}(x)) \mathbf{v}_t(x) \, d\mathcal{H}^{n-1}(x) = \int_{U_t} D\varphi(x) \cdot \text{cof } D\mathbf{v}(x) \, dx. \tag{8.2}
\]

Indeed, if \( \mathbf{v} \in C^\infty(\Omega, \mathbb{R}^n) \), then (8.2) is true for all \( t \in (-\delta, \delta) \), and follows from the identity \( \text{Div } \text{cof } D\mathbf{v} = 0 \) and the divergence theorem. Let \( \mathbf{v} \in W^{1, n-1}(\Omega, \mathbb{R}^n) \), and let \( \{\mathbf{v}_j\}_{j \in \mathbb{N}} \) be a sequence in \( C^\infty(\Omega, \mathbb{R}^n) \) converging to \( \mathbf{v} \) in \( W^{1, n-1}(\Omega, \mathbb{R}^n) \). Then, \( \text{cof } D\mathbf{v}_j \to \text{cof } D\mathbf{v} \) in \( L^1(\Omega, \mathbb{R}^{n \times n}) \) as \( j \to \infty \), and by the coarea formula, \( \text{cof } D\mathbf{v}_j \to \text{cof } D\mathbf{v} \) in \( L^1(\partial U_t, \mathbb{R}^{n \times n}) \) as \( j \to \infty \), for a.e. \( t \in (-\delta, \delta) \). This shows formula (8.2).

Thanks to (8.2), for a.e. \( t \in (-\delta, \delta) \) the following equalities hold for all \( j \in \mathbb{N} \) and all \( \varphi \in C^\infty(\Omega, \mathbb{R}^n) \):

\[
\int_{\partial U_t} \varphi(x) \cdot (\text{cof } D\mathbf{u}_j(x)) \mathbf{v}_t(x) \, d\mathcal{H}^{n-1}(x) = \int_{U_t} D\varphi(x) \cdot \text{cof } D\mathbf{u}_j(x) \, dx \tag{8.3}
\]

and

\[
\int_{\partial U_t} \varphi(x) \cdot (\text{cof } D\mathbf{u}(x)) \mathbf{v}_t(x) \, d\mathcal{H}^{n-1}(x) = \int_{U_t} D\varphi(x) \cdot \text{cof } D\mathbf{u}(x) \, dx. \tag{8.4}
\]

Fix such a \( t \), let \( \varphi_t \in C^\infty(\partial U_t, \mathbb{R}^n) \), and let \( \varphi \in C^\infty(\Omega, \mathbb{R}^n) \) be an extension of \( \varphi_t \). Applying (8.3) and (8.4), we obtain that

\[
\lim_{j \to \infty} \int_{\partial U_t} \varphi_t \cdot (\text{cof } D\mathbf{u}_j) \mathbf{v}_t \, d\mathcal{H}^{n-1}(x) = \int_{U_t} D\varphi \cdot \text{cof } D\mathbf{u} \, dx
\]

\[
= \int_{\partial U_t} \varphi_t \cdot (\text{cof } D\mathbf{u}) \mathbf{v}_t \, d\mathcal{H}^{n-1}(x).
\]

By density of \( C^\infty(\partial U_t, \mathbb{R}^n) \) in \( C(\partial U_t, \mathbb{R}^n) \), this shows that

\[
(\text{cof } D\mathbf{u}_j) \mathbf{v}_t \xrightarrow{*} (\text{cof } D\mathbf{u}) \mathbf{v}_t \quad \text{as } j \to \infty
\]

in the sense of measures. Therefore, the proof will be finished as soon as we show that for a.e. \( t \in (-\delta, \delta) \), there is a subsequence \( \{j_k\}_{k \in \mathbb{N}} \) such that \( \{\text{cof } D\mathbf{u}_{j_k}\}_{k \in \mathbb{N}} \) is equiintegrable in \( L^1(\partial U_t, \mathbb{R}^{n \times n}) \).

By equiintegrability, there exists \( \psi : [0, \infty) \to [0, \infty) \), increasing, continuous, and superlinear at infinity, such that

\[
\sup_{j \in \mathbb{N}} \int_{\Omega} \psi(|\text{cof } D\mathbf{u}_j(x)|) \, dx < \infty. \tag{8.5}
\]
For each \( j \in \mathbb{N} \), let \( \theta_j, \theta : (-\delta, \delta) \to [0, \infty] \) be the functions defined by
\[
\theta_j(t) := \int_{\partial U_t} \psi(|\text{cof } Du_j(x)|) \, d\mathcal{H}^{n-1}(x), \quad t \in (-\delta, \delta),
\]
and \( \theta := \liminf_{j \to \infty} \theta_j \). By Fatou’s lemma, Fubini’s theorem and (8.5),
\[
\int_{-\delta}^{\delta} \theta(t) \, dt \leq \liminf_{j \to \infty} \int_{-\delta}^{\delta} \theta_j(t) \, dt \leq \sup_{j \in \mathbb{N}} \int_{\Omega} \psi(|\text{cof } Du_j(x)|) \, dx < \infty.
\]
In particular, \( \theta(t) < \infty \) for a.e. \( t \in (-\delta, \delta) \). Therefore, for a.e. \( t \in (-\delta, \delta) \) there exists a subsequence \( \{j_k\}_{k \in \mathbb{N}} \) such that \( \lim_{k \to \infty} \theta_{j_k}(t) < \infty \), and, hence, the sequence \( \{\text{cof } Du_{j_k}\}_{k \in \mathbb{N}} \) is equiintegrable in \( L^1(\partial U_t, \mathbb{R}^{n \times n}) \). This completes the proof. \( \square \)

In order to show the stability of condition \( \text{INV} \) under the weak limit, we present a useful criterion for condition \( \text{INV} \) to hold.

**Lemma 8.3.** Let \( u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n) \) satisfy \( \det Du > 0 \) a.e. Then, the map \( u \) satisfies condition \( \text{INV} \) if and only if for each point \( x_0 \in \Omega \) and a.e. \( r \in (0, \text{dist}(x_0, \partial \Omega)) \), the following hold:

(a) \( \deg(u, \partial B(x_0, r), u(x)) = 1 \) for a.e. \( x \in B(x_0, r) \).

(b) \( \deg(u, \partial B(x_0, r), u(x)) = 0 \) for a.e. \( x \in \Omega \setminus B(x_0, r) \).

**Proof.** Suppose that condition \( \text{INV} \) holds, and take a point \( x_0 \in \Omega \). Then, for a.e. radii \( r \in (0, \text{dist}(x_0, \partial \Omega)) \), conditions (i)–(ii) of Definition 2.12 are satisfied, and \( \deg(u, \partial B(x_0, r), \cdot) = \chi_{\text{im}_T(u, B(x_0, r))} \) a.e. (see Proposition 2.17 (iv)). Fix such a radius \( r \), and call \( B := B(x_0, r) \). For each \( i = 0, 1 \), define \( F_i \) as the set of \( y_0 \in \mathbb{R}^n \) such that
\[
D \left( \{y \in \mathbb{R}^n : \deg(u, \partial B, y) = i\}, y_0 \right) = 1.
\]
Note that \( F_1 = \text{im}_T(u, B) \) and that, by Lebesgue’s theorem, \( F_0 \cup F_1 = \mathbb{R}^n \) a.e. Lemma 2.5 then shows that condition (ii) of Definition 2.12 implies condition (b), and that condition (i) implies condition (a).

The remaining parts of the proof are shown similarly. \( \square \)

The following shows the stability of condition \( \text{INV} \) under the weak limit.

**Proposition 8.4.** For each \( j \in \mathbb{N} \), let \( u_j \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n) \) satisfy condition \( \text{INV} \) and \( \det Du_j > 0 \) a.e. Let \( u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n) \) satisfy \( \det Du > 0 \) a.e. Suppose
\[
\sup_{j \in \mathbb{N}} \mathcal{E}(u_j) < \infty \quad \text{and} \quad \sup_{j \in \mathbb{N}} \|u_j\|_{L^\infty(\Omega, \mathbb{R}^n)} < \infty.
\]
Assume that there exists $\theta \in L^1(\Omega)$ such that $\theta > 0$ a.e. and 

$$u_j \rightarrow u \text{ a.e., } \text{cof } Du_j \rightarrow \text{cof } Du \text{ in } L^1(\Omega, \mathbb{R}^{n \times n}), \text{ det } Du_j \rightarrow \theta \text{ in } L^1(\Omega)$$ 
as $j \rightarrow \infty$. Then $u$ satisfies INV.

Proof. For each $j \in \mathbb{N}$, let $\Omega_j$ (respectively, $\Omega_0$) be the set of Definition 2.4 corresponding to $u_j$ (respectively, $u$). By [7, Lemma 3.9] and [17, Theorem 2 ii]), there exists a measurable set $\Omega_I \subset \bigcap_{j \in \mathbb{N}} \Omega_j \cap \Omega_0$ of full measure in $\Omega$ such that for all $j \in \mathbb{N}$, the maps $u_j$ and $u$ are one-to-one in $\Omega_I$.

Let $x_0 \in \Omega$. For a.e. $r \in (0, \text{dist}(x_0, \partial \Omega))$, there exists a subsequence $\{j_k\}_{k \in \mathbb{N}}$ such that the following properties hold:

(a) $B(x_0, r) \in \mathcal{C}_u$ for all $j \in \mathbb{N}$.
(b) $u_j \rightarrow u \text{ }H^{n-1}\text{-a.e. in } \partial B(x_0, r)$ as $j \rightarrow \infty$, and 

$$\sup_{j \in \mathbb{N}} \|u_j\|_{L^\infty(\partial B(x_0, r), \mathbb{R}^n)} < \infty.$$ 

(c) $(\text{cof } Du_{j_k})v_{B(x_0, r)} \rightarrow (\text{cof } Du)v_{B(x_0, r)}$ in $L^1(\partial B(x_0, r), \mathbb{R}^n)$ as $k \rightarrow \infty$.
(d) $\chi_{\text{img}(u_{j_k}, B(x_0, r))} \rightarrow \chi_{\text{img}(u, B(x_0, r))}$ a.e., as $k \rightarrow \infty$.

Indeed, property (a) follows from Lemma 2.16, property (b) from the coarea formula, property (c) from Lemma 8.2, whereas property (d) was proved in [17, Theorem 2 iii]). Fix such an $r$ and denote $B := B(x_0, r)$.

Let $E$ be the set of $x \in \Omega_I$ such that the following properties are satisfied:

(i) $\deg(u_{j_k}, \partial B, u(x)) \rightarrow \deg(u, \partial B, u(x))$ as $k \rightarrow \infty$.
(ii) $\chi_{u_{j_k}(\Omega_I \cap B)}(u(x)) \rightarrow \chi_{u}\Omega_I(\cap B)(u(x))$ as $k \rightarrow \infty$.
(iii) For all but finitely many $k \in \mathbb{N}$, either 

$$\deg(u_{j_k}, \partial B, u(x)) = 1 \quad \text{and} \quad u(x) \in u_{j_k}(\Omega_I \cap B)$$ 
or

$$\deg(u_{j_k}, \partial B, u(x)) = 0 \quad \text{and} \quad u(x) \in u_{j_k}(\Omega_I \setminus B).$$

Let $x \in E \cap B$, and assume, for a contradiction, that $\deg(u, \partial B, u(x)) \neq 1$. By (i) and (iii), we have that $u(x) \in u_{j_k}(\Omega_I \setminus B)$ for infinitely many $j \in \mathbb{N}$. By the injectivity of $u_j$ and (ii), we conclude that $u(x) \notin u(\Omega_I \cap B)$, which is a contradiction.

Analogously, if an $x \in E \setminus B$ satisfies $\deg(u, \partial B, u(x)) \neq 0$, then we obtain $u(x) \in u(\Omega_I \cap B)$, which contradicts the injectivity of $u$. Thus, in virtue of Lemma 8.3, the proof will be finished as soon as we show that the set $E$ has full measure in $\Omega$. 

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Property (i) holds for a.e. \( x \in \Omega_I \) because of (b)–(c) and Lemmas 8.1 and 2.5. Property (ii) holds for a.e. \( x \in \Omega_I \) because of (d) and Lemma 2.5. Property (iii) holds for a.e. \( x \in \Omega_I \) because of (a), (ii), Proposition 2.17 (parts (i) and (iv)) and Lemma 2.5. This concludes the proof.

The existence result, which for simplicity is stated for the case \( n = 3 \), is as follows.

**Theorem 8.5.** Suppose \( \Omega \) is a bounded open set of \( \mathbb{R}^3 \) with a (strongly) Lipschitz boundary. Let \( \Gamma_D \subset \partial \Omega \) be a 2-rectifiable set with \( \mathcal{H}^2(\Gamma_D) > 0 \). Further, suppose \( b : \Gamma_D \to \mathbb{R}^3 \) is a measurable map. Let \( K \subset \mathbb{R}^3 \) be compact. Define \( A \) as the set of \( u \in W^{1,2}(\Omega, \mathbb{R}^3) \) such that

\[
\det Du(x) > 0 \quad \text{and} \quad u(x) \in K \text{ for a.e. } x \in \Omega, \ u \text{ satisfies INV, and } u|_{\Gamma_D} = b,
\]

the equality on \( \Gamma_D \) being in the sense of traces. Let \( \mathcal{R} := \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \times (0, \infty) \).

Let the function \( W : \Omega \times \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \to \mathbb{R} \) satisfy the following properties:

(i) There exists a function \( \Phi : \Omega \times \mathbb{R}^3 \times \mathcal{R} \to \mathbb{R} \) such that for a.e. \( x \in \Omega \) and every \( y \in \mathbb{R}^3 \), the function \( \Phi(x, y, \cdot) \) is convex and

\[
W(x, y, F) = \Phi(x, y, (F, \text{cof } F, \det F))
\]

for all \( F \in \mathbb{R}^{3 \times 3} \) such that \( \det F > 0 \).

(ii) The function \( \Phi(x, \cdot, \cdot) : \mathbb{R}^3 \times \mathcal{R} \to \mathbb{R} \) is continuous for a.e. \( x \in \Omega \), and \( \Phi(\cdot, y, D) : \Omega \to \mathbb{R} \) is measurable for every \( (y, D) \in \mathbb{R}^{3 \times 3} \times \mathcal{R} \).

(iii) There exist \( a \in L^1(\Omega) \), a positive constant \( c > 0 \), an increasing function \( h_1 : (0, \infty) \to [0, \infty) \) and a convex function \( h_2 : (0, \infty) \to \mathbb{R} \) such that

\[
\lim_{t \to \infty} \frac{h_1(t)}{t} = \lim_{t \to \infty} \frac{h_2(t)}{t} = \lim_{t \searrow 0} h_2(t) = \infty
\]

and

\[
W(x, y, F) \geq a(x) + c|F|^2 + h_1(|\text{cof } F|) + h_2(\det F)
\]

for a.e. \( x \in \Omega \), all \( y \in \mathbb{R}^3 \) and all \( F \in \mathbb{R}^{3 \times 3} \) with \( \det F > 0 \).

Assume that \( \mathcal{A} \neq \emptyset \), and define \( I : \mathcal{A} \to \mathbb{R} \cup \{\infty\} \) as

\[
I(u) := \int_{\Omega} W(x, u(x), Du(x)) \, dx + \mathcal{E}(u), \quad u \in \mathcal{A}.
\]

Then there exists a minimizer of \( I \) in \( \mathcal{A} \).

**Proof.** If \( I \) is identically \( +\infty \), the result is trivial. Assume otherwise and note that (iii) implies that \( I \) is bounded below.

Let \( \{u_j\}_{j \in \mathbb{N}} \) be a minimizing sequence for \( I \) in \( \mathcal{A} \). Assumption (iii) implies that the sequence \( \{Du_j\}_{j \in \mathbb{N}} \) is bounded in \( L^2(\Omega, \mathbb{R}^3) \), whereas \( \{\text{cof } Du_j\}_{j \in \mathbb{N}} \) and
\{\det D u_j\}_{j \in \mathbb{N}} are equiintegrable. Thus, by the boundary condition and the Poincaré inequality, we obtain that there exist \( u \in W^{1,2}(\Omega, \mathbb{R}^3) \), \( \vartheta \in L^1(\Omega, \mathbb{R}^{3 \times 3}) \) and \( \theta \in L^1(\Omega) \) such that, for a subsequence (not relabelled),

\[
\begin{align*}
  u_j \rightharpoonup u \quad & \text{in } W^{1,2}(\Omega, \mathbb{R}^3), \\
  \operatorname{cof} D u_j \rightharpoonup \vartheta \quad & \text{in } L^1(\Omega, \mathbb{R}^{3 \times 3}), \\
  \det D u_j \rightharpoonup \theta \quad & \text{in } L^1(\Omega).
\end{align*}
\]

as \( j \to \infty \). Clearly, \( \theta(x) \geq 0 \) and \( u(x) \in K \) for a.e. \( x \in \Omega \). If \( \theta \) were zero in a set \( A \) of positive measure, then we would have (for a subsequence) \( \det D u_j \to 0 \) a.e. in \( A \); hence by assumption (iii), we obtain \( h_2(\det D u_j) \to \infty \) a.e. in \( A \), as \( j \to \infty \). Again by assumption (iii) and Fatou’s lemma, we get \( I(u_j) \to \infty \) as \( j \to \infty \), which is a contradiction. Therefore, \( \theta > 0 \) a.e.

As \( \operatorname{cof} D u_j \) converges weakly to \( \vartheta \) in \( L^1 \), and \( \sup_{j \in \mathbb{N}} \| D u_j \|_{L^2} < \infty \), by a standard result on weak continuity of minors (see, e.g., [4, Theorem 4.11]), we obtain \( \vartheta = \operatorname{cof} D u \) a.e. Thanks to [17, Theorem 3], we have that \( \theta = \det D u \) a.e. and \( \mathcal{E}(u) \leq \liminf_{j \to \infty} \mathcal{E}(u_j) \). By Proposition 8.4, \( u \) satisfies INV. Since the boundary condition is also preserved under the limit, we conclude that \( u \in \mathcal{A} \).

Thanks to the lower semicontinuity theorem of [4, Theorem 5.4],

\[
\int_{\Omega} W(x, u(x), D u(x)) \, dx \leq \liminf_{j \to \infty} \int_{\Omega} W(x, u_j(x), D u_j(x)) \, dx.
\]

This shows that \( u \) is a minimizer of \( I \) in \( \mathcal{A} \).

In the proof of Theorem 8.5, we have used [17, Theorem 3] to prove that \( \det D u_j \to \det D u \) in \( L^1(\Omega) \) as \( j \to \infty \). We could have used instead [17, Theorem 2] to show first that \( \det D u_j \to |\det D u| \) in \( L^1(\Omega) \) as \( j \to \infty \), and then argue as in [28, Theorem 4.2] to show that \( \det D u > 0 \) a.e.

A corrected proof of [7, Lemma 3.12]

In the proof of some of the results of the paper (e.g., in Theorem 3.2 or Lemma 4.10) we have invoked the following result by Conti and De Lellis [7, Lemma 3.12].

**Lemma A.1.** Let \( u \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n) \). Suppose that condition INV holds and that \( \det D u > 0 \) a.e. Then, for any \( a, b \in \Omega \) and any \( r \in R_a, \ s \in R_b \),

(i) \( \operatorname{im}_T(u, B(a, r)) \cap \operatorname{im}_T(u, B(b, s)) = \emptyset \) if \( B(a, r) \cap B(b, s) = \emptyset \).

(ii) \( \operatorname{im}_T(u, B(a, r)) \subset \operatorname{im}_T(u, B(b, s)) \) if \( B(a, r) \subset B(b, s) \).

In this section we provide a corrected proof of this lemma, as some minor modifications to the proof in [7] are necessary. In order to do this, we prove first the fol-
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lowing preliminary result, which corresponds to [7, Lemma A.1] (we have added the hypothesis $\mathcal{H}^{n-1}(\partial^* A \cap \partial^* C) = 0$).

**Lemma A.2.** Let $A, C \subset \mathbb{R}^n$ be two sets of finite perimeter, and denote by $A^*$ and $C^*$ the sets of points with density 1 with respect to $A$ and $C$, respectively. Suppose that

(i) $A^* \neq \mathbb{R}^n$ or $C^* \neq \mathbb{R}^n$.

(ii) $\mathcal{H}^{n-1}(\partial^* A \cap C^*) = \mathcal{H}^{n-1}(\partial^* C \cap A^*) = \mathcal{H}^{n-1}(\partial^* A \cap \partial^* C) = 0$.

Then $\mathcal{L}^n(A \cap C) = 0$ and $A^* \cap C^* = \emptyset$.

**Proof.** Since $A$ and $C$ have finite perimeter, it follows that $A \cap C$ has finite perimeter and

$$\partial^*(A \cap C) \subseteq (\partial^* A \cap C^*) \cup (\partial^* C \cap A^*) \cup (\partial^* A \cap \partial^* C).$$

Using (ii), we get $\mathcal{H}^{n-1}(\partial^*(A \cap C)) = 0$. This implies that $\mathcal{L}^n(A \cap C) = 0$, by virtue of (i) and the isoperimetric inequality. As a consequence, $A^* \cap C^* = \emptyset$. □

**Proof of Lemma A.1.** We only prove (i), as (ii) is obtained analogously. Define $A := \text{im}_T(u, B(a, r))$ and $C := \text{im}_T(u, B(b, s))$. Thanks to Proposition 2.17 (vii), we have $\partial^* A \cong \text{im}_G(u, \partial B(a, r))$ and $\partial^* C \cong \text{im}_G(u, \partial B(b, s))$. Thus, $\partial^* C \cap A$ and $\partial^* A \cap C$ are $\mathcal{H}^{n-1}$-null sets (by virtue of [7, Lemma 3.8]). Since $u$ is one-to-one in $\Omega_0$ (by [7, Lemma 3.9]), we also have that $\mathcal{H}^{n-1}(\partial^* A \cap \partial^* C) = 0$ (it follows from the definition of geometric image). The proof is thus completed, in light of Lemma A.2. □

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