Sharp global estimates for local and nonlocal porous medium-type equations in bounded domains

Matteo Bonforte, Alessio Figalli, and Juan Luis Vázquez

Abstract

This paper provides a quantitative study of nonnegative solutions to nonlinear diffusion equations of porous medium-type of the form \( \partial_t u + Lu^m = 0 \), \( m > 1 \), where the operator \( L \) belongs to a general class of linear operators, and the equation is posed in a bounded domain \( \Omega \subset \mathbb{R}^N \). As possible operators we include the three most common definitions of the fractional Laplacian in a bounded domain with zero Dirichlet conditions, and also a number of other nonlocal versions. In particular, \( L \) can be a power of a uniformly elliptic operator with \( C^1 \) coefficients. Since the nonlinearity is given by \( u^m \) with \( m > 1 \), the equation is degenerate parabolic.

The basic well-posedness theory for this class of equations has been recently developed in [13, 14]. Here we address the regularity theory: decay and positivity, boundary behavior, Harnack inequalities, interior and boundary regularity, and asymptotic behavior. All this is done in a quantitative way, based on sharp a priori estimates. Although our focus is on the fractional models, our results cover also the local case when \( L \) is a uniformly elliptic operator, and provide new estimates even in this setting.

A surprising aspect discovered in this paper is the possible presence of non-matching powers for the long-time boundary behavior. More precisely, when \( L = (-\Delta)^s \) is a spectral power of the Dirichlet Laplacian inside a smooth domain, we can prove that:

- when \( 2s \geq 1 - 1/m \), for large times all solutions behave as \( \text{dist}^{1/m} \) near the boundary;
- when \( 2s < 1 - 1/m \), different solutions may exhibit different boundary behavior.

This unexpected phenomenon is a completely new feature of the nonlocal nonlinear structure of this model, and it is not present in the elliptic case.

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1 Introduction

In this paper we address the question of obtaining a priori estimates, positivity, boundary behavior, regularity, and Harnack inequalities for a suitable class of weak solutions of nonlinear nonlocal diffusion equations of the form:

$$\partial_t u + L F(u) = 0 \quad \text{posed in } Q = (0, \infty) \times \Omega,$$

where $$\Omega \subset \mathbb{R}^N$$ is a bounded domain with $$C^{1,1}$$ boundary, $$N \geq 1$$, and $$L$$ is a linear operator representing diffusion of local or nonlocal type, the prototype example being the fractional Laplacian (the class of admissible operators will be precisely described below). Although our arguments hold for a rather general class of nonlinearities $$F : \mathbb{R} \rightarrow \mathbb{R}$$, for the sake of simplicity we shall focus on the model case $$F(u) = u^m$$ with $$m > 1$$.

The use of nonlocal operators in diffusion equations reflects the need to model the presence of long-distance effects not included in evolution driven by the Laplace operator, and this is well documented in the literature. The physical motivation and relevance of the nonlinear diffusion models with nonlocal operators has been mentioned in many references, see for instance [3, 12, 13, 39, 40, 47]. Because $$u$$ usually represents a density of particle, all data and solutions are supposed to be nonnegative. Since the problem is posed in a bounded domain we need boundary conditions that we assume of Dirichlet type.

This kind of problems has been extensively studied when $$L = -\Delta$$ and $$F(u) = u^m$$, $$m > 1$$, in which case the equation becomes the classical Porous Medium Equation [15, 21, 22, 44]. Here, we are interested in treating nonlocal diffusion operators, in particular fractional Laplacian operators. However, as we shall see, our arguments are general enough to provide new interesting results even in the local setting. Note that, since we are working on a bounded domain, the concept of fractional Laplacian admits several non-equivalent versions, the best known being the Restricted Fractional Laplacian (RFL), the Spectral Fractional Laplacian (SFL), and the Censored Fractional Laplacian (CFL); see Section 2 for more details.

The case of the SFL operator with $$F(u) = u^m$$, $$m > 1$$, has been already studied by the first and the third author in [13, 14]. In particular, in [14] the authors presented a rather abstract setting where they were able to treat not only the usual fractional Laplacians but also a large number of variants that will be listed below for the reader’s convenience. Besides, rather general increasing nonlinearities $$F$$ were allowed. The basic questions of existence and uniqueness of suitable solutions for this problem were solved in [14] in the class of ‘weak dual solutions’, an extension of the concept of solution introduced in [13] that has proved to be quite flexible and efficient. A number of a priori estimates (absolute bounds and smoothing effects) were also derived in that generality.

Since these basic facts are settled, here we focus our attention on the finer aspects of the theory, such as decay estimates, sharp boundary estimates, and various kinds of local and global Harnack type inequalities. The latter will be derived by combining quantitative sharp upper and lower bounds up to the boundary. Such upper and lower bounds will be formulated in terms of the first eigenfunction $$\Phi_1$$ of $$L$$, that under our assumptions will satisfy $$\Phi_1 \asymp \text{dist}(\cdot, \partial \Omega)^\gamma$$ for a certain characteristic power $$\gamma \in (0, 1]$$ that depends on the particular operator we consider. Typical values are $$\gamma = s$$, $$\gamma = 1$$, and $$\gamma = s - 1/2$$ for $$s > 1/2$$, cf. Subsection 3.3. Thanks to these bounds, we are able to prove both interior and boundary regularity, and to find the large-time asymptotic behavior of solutions.

Let us indicate here some notation of general use. The symbol $$\infty$$ will always denote $$+\infty$$. 

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Given $a, b$, we use the notation $a \asymp b$ whenever there exist universal constants $c_0, c_1 > 0$ such that $c_0 b \leq a \leq c_1 b$. We also use the symbols $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. We will always consider bounded domains $\Omega$ with boundary of class $C^2$. In the paper we use the short form ‘solution’ to mean ‘weak dual solution’, unless differently stated.

**Overview of the main results.** A basic principle in the paper is that the sharp boundary estimates depend not only on $\mathcal{L}$ but also on the behavior of the nonlinearity $F(u)$ near $u = 0$, i.e., in our case, on the exponent $m > 1$. We show that the behavior strongly depends on the fundamental parameter $\sigma := 1 \wedge \frac{2sm}{s(m-1)}$, and on whether this parameter is equal to 1 or not. As we shall see later, $\sigma$ encodes the interplay between the “elliptic scaling power” $2s/(m-1)$, the “eigenfunction power” $\gamma$, and the “nonlinearity power” $m$.

The elliptic analysis performed in the companion paper [7] combined with some standards arguments implies that, in all cases,

$$
\left\| t^{\frac{1}{m-1}} u(t, \cdot) - S \right\|_{L^\infty(\Omega)} \to 0,
$$

where $S \asymp \Phi_1^{\sigma/m}$ is the solution of the elliptic problem (see Theorems 4.4 and 11.1 below). This fact and the results in [6] prompted us to look for estimates of the form

$$
c_0 \left( 1 \wedge \frac{t}{t_s} \right)^{m-1} \frac{\Phi_1^{\sigma/m}(x_0)}{t^{m-1}} \leq u(t, x_0) \leq c_1 \frac{\Phi_1^{\sigma/m}(x_0)}{t^{m-1}} \quad \text{for all } t > 0, \ x_0 \in \Omega,
$$

where $c_0, c_1$ are positive constants independent of $u$, while $t_s > 0$ depends on the initial datum. Indeed, these bounds hold for the separate-variables solutions (see (4.3) and Theorem 4.4) and in the case of the RFL [6]. In addition, the results in the classical local case [2, 44] show that, in these nonlinear problems, bounds of these form are expected to hold for large times. However, it came as a complete surprise to us that, in this nonlocal nonlinear setting, there exist solutions to the parabolic problem that do not satisfy the bounds (1.2). More precisely, as we explain now, the general behavior of solutions depend both on the value of $\sigma$ and on whether we are in the spectral or in the non-spectral setting. We remark that, in the local case $s = 1$, the spectral and non-spectral setting coincide.

- The “non-spectral case”. As we shall see later, in this case $\sigma = 1$ independently of $s$ and $m$. Whenever $s < 1$ (i.e., in the nonlocal case), we prove that the two-sided estimate in (1.2) is valid for all nontrivial solutions $u \geq 0$. Note that $\sigma = 1$ both in the limit $m \to 1$ independently of $s \in (0, 1]$ (i.e., linear diffusion, either fractional or standard) and for $s \to 1$ (i.e., standard diffusion). When $s = 1$, we show that the upper bound (1.3) holds for all $t > 0$, while the lower bound is valid only for $t \geq t_s$. Note that this waiting time is inevitable in the local case $s = 1$, due to the finite speed of propagation (cf. [2, 45]).

- The “spectral case”. Since the spectral and non-spectral setting coincide for $s = 1$, it is enough to consider the case $s < 1$. In this case we prove that the following non-matching bounds hold:

$$
c_0 \left( 1 \wedge \frac{t}{t_s} \right)^{m-1} \frac{\Phi_1(x_0)}{t^{m-1}} \leq u(t, x_0) \leq c_1 \frac{\Phi_1^{\sigma/m}(x_0)}{t^{m-1}}.
$$

It is remarkable that both the lower and the upper bounds are actually *sharp* in many situations. Indeed, for some suitable choices of $s$ and $m$, one can find initial data such that $u(t) \lesssim \Phi_1$ for short times. On the other hand, separate-variable solutions saturate the upper bound.
Concerning the long time behavior, we need to distinguish between the case $\sigma = 1$ and $\sigma < 1$. Indeed, in the case $\sigma = 1$ we can improve the lower bound in (1.3) and show that the matching bounds in (1.2) hold for all $t \geq t_*$. On the other hand, when $\sigma < 1$, we can find initial data for which the upper bound in (1.3) is not sharp. This shows that, depending on the initial data, there are several possible long-time behaviors near the boundary. Note that, if one looks for universal bounds independent of the initial condition, Figures 2-3 below seem to suggest that the bounds provided by (1.3) are optimal for all times.

As already mentioned above, such exceptional behaviors came as a surprise to us, since the solution to the corresponding “elliptic setting” $LS^m = S$ satisfies $S \sim \Phi_{1/m}^\sigma$ without exceptions, hence separate-variable solutions always satisfy (1.2) (see (4.3) and Theorem 4.4). This led us to a deeper analysis here with respect to the elliptic case, with a number of new tools and results that we explain in the next paragraphs.

After discovering this strange boundary behavior, we looked for numerical confirmations. These have been given to us by the authors of [20], who exploited the analytical tools developed in this paper to support our results by means of accurate numerical simulations. We include here some of these simulations, by courtesy of the authors. In all the figures we shall consider the Spectral Fractional Laplacian, so that $\gamma = 1$ (see Section 2 for more details).

We take $\Omega = (-1,1)$, and we consider as initial datum the compactly supported function $u_0(x) = e^{4 - \frac{1}{|x-1/2|^{1/2}}} \chi_{|x|<1/2}$ appearing in the left of Figure 1. In all the other figures, the solid line represents either $\Phi_{1/m}^\sigma$ or $\Phi_{1-2s}^1$, while the dotted lines represent $t^{1/m-1} u(t)$ for different values of $t$, where $u(t)$ is the solution starting from $u_0$. These choices are motivated by Theorem 9.1 and Proposition 9.3. Since the map $t \mapsto t^{1/m-1} u(t,x)$ is nondecreasing for all $x \in \Omega$ (cf. (2.3) in [14]), the lower dotted line corresponds to an earlier time with respect to the higher one.

Figure 1: On the left, the initial condition $u_0$. On the right, the solid line represents $\Phi_{1/m}^1$, and the dotted lines represent $t^{1/m-1} u(t)$ at $t = 1$ and $t = 5$. The parameters are $m = 2$ and $s = 1/2$, hence $\sigma = 1$. While $u(t)$ appears to behave as $\Phi_1 \sim \text{dist}(\cdot, \partial \Omega)$ for very short times, already at $t = 5$ it exhibits the matching boundary behavior predicted by Theorem 9.1.
In both pictures, the solid line represents \( \Phi_1^{1/m} \). On the left, the dotted lines represent \( t^{\frac{1}{m-1}}u(t) \) at \( t = 30 \) and \( t = 150 \), with parameters \( m = 4 \) and \( s = 3/4 \) (hence \( \sigma = 1 \)). In this case \( u(t) \) appears to behave as \( \Phi_1 \approx \text{dist}(\cdot, \partial\Omega) \) for quite some time, and only around \( t = 150 \) it exhibits the matching boundary behavior predicted by Theorem 9.1. On the right, the dotted lines represent \( t^{\frac{1}{m-1}}u(t) \) at \( t = 150 \) and \( t = 600 \) with parameters \( m = 4 \) and \( s = 1/5 \) (hence \( \sigma = 8/15 < 1 \)). In this case \( u(t) \) seems to exhibit a linear boundary behavior even after long time (this linear boundary behavior is proved in Theorems 7.1-7.2 for short times, and it is a universal lower bound for all times by Theorem 7.1). The second picture may lead one to conjecture that, in the case \( \sigma < 1 \) and \( u_0 \lesssim \Phi_1 \), the behavior \( u(t) \approx \Phi_1 \) holds for all times. However, as shown in Figure 3, there are cases when \( u(t) \gg \Phi_1^{1-2s} \) for large times.

Comparing Figures 2 and 3, it seems that when \( \sigma < 1 \) there is no hope to find a universal behavior of solutions for large times. In particular, the bound provided by (1.3) seems to be optimal.

After this short overview, we now give a more detailed list of the results obtained in this paper.

**Sharp lower and upper bounds.** First, in all cases, whenever \( s < 1 \) we prove that the solution becomes strictly positive inside the domain at positive times. This is called infinite speed of propagation, a property that does not hold in the limit \( s = 1 \) for any \( m > 1 \) \([43]\) (in that case, finite speed of propagation holds and a free boundary appears). Previous results on this
infinite speed of propagation can be found in [6, 10]. We recall that infinite speed of propagation is typical of the evolution with nonlocal operators representing long-range interactions, but it is not true for the standard porous medium equation, hence a trade-off takes place when both effects are combined; all our models fall on the side of infinite propagation, but we recall that finite propagation holds for a related nonlocal model called “nonlinear porous medium flow with fractional potential pressure”, cf. [17].

Our proof is based on a new quantitative lower estimate, Theorem 7.1, where the lower bound appearing in the left-hand side of (1.3) is proven. As shown in Theorem 7.2, at least for short times the result is optimal in the spectral case when \( \sigma < 1 \). On the other hand, Theorem 8.1 shows that, in the non-spectral case, the power 1 can be improved to \( \sigma/m \) as in (1.2).

In conclusion, for all our class of operators, we find universal lower bounds (the exponent depending on the kind of operators) for small times.

Concerning the large time behavior, in the non-spectral case the global positivity with matching powers as in (1.3) is obtained in Theorem 5.1. In addition, Theorem 9.1 proves that, for large times and when \( \sigma = 1 \), the lower estimate of (1.2) holds both for the spectral fractional Laplacian and in the local case \( s = 1 \). On the other hand, when \( \sigma < 1 \), we can show that the lower bound may be false (see Propositions 9.2 and 9.3). As suggested by Figure 2, it seems plausible that the lower bound provided by Theorem 7.1 is optimal.

A final remark concerning equations with the standard Laplacian and its variants: since \( \sigma = 1 \) when \( s = 1 \) (independently of \( m > 1 \)), our results give a sharp behavior in the local case after a “waiting time”. Although this is well-known for the classical porous medium equation, our results apply also to the case uniformly elliptic operator in divergence form with \( C^1 \) coefficients, and yield new results in this setting. Actually one can check that, even when the coefficients are merely measurable, many of our results are still true and they provided universal upper and lower estimates. At least to our knowledge, such general results are completely new.

Harnack inequalities and regularity. The lower and upper bounds immediately imply Harnack inequalities. Then, by a variant of the techniques used in [6], we can show interior Hölder regularity. In addition, if the kernel representing the operator satisfies some suitable continuity assumptions, we show that solutions are classical in the interior, are Hölder continuous up to the boundary if the upper and lower bound have matching powers. We refer to Section 12 for more details.

Sharp asymptotic behavior. Another important consequence of our quantitative Global Harnack inequalities is the sharp asymptotic behavior. Indeed, exploiting the techniques in [10], we can prove a sharp asymptotic behavior for our solutions when the upper and lower bound have matching powers. Such sharp results hold true for a quite general class of local and nonlocal operators. For more details and comments we refer to Section 11.

Method and generality. Our work is part of a current effort aimed at extending the theory of evolution equations of parabolic type to general nonlocal operators, in particular operators with rough kernels, that have been studied by various authors (see for instance [28, 33, 43]). Our approach is different from many others: indeed, even if the equation is nonlinear, we concentrate on the properties of the inverse operator \( \mathcal{L}^{-1} \) (more precisely, on its kernel given by the Green function \( \mathcal{K} \)), rather than on the operator \( \mathcal{L} \) itself. Once this setting is well-established and good linear estimates for the Green function are available, the calculations and estimates are very general. Hence, the method is applicable to a very large class of
equations, both for Elliptic and Parabolic problems, as well as to more general nonlinearities than \( F(u) = u^m \) (see also related comments in the works [13, 10, 14]).

2 Main examples of operators and sharp boundary behavior

When working in the whole \( \mathbb{R}^N \), the fractional Laplacian admits different definitions that can be shown to be all equivalent. On the other hand, when we deal with bounded domains, there are at least three different operators in the literature, that we will call the Restricted (RFL), the Spectral (SFL), and the Censored (CFL) Fractional Laplacian. The distinction has been clear in the probabilistic literature for years, but not so much in the analysis literature. Let us present the statement and results for the three model cases, and we refer to Section 3.3 for further examples. Here, we collect the sharp results about the boundary behavior, namely the Global Harnack inequalities from Theorems 10.1, 10.3, and 10.4.

As explained later, all the results in the “non-spectral case” will apply to the RLF and the CLF, while the “spectral case” applies to the SFL.

The parameters \( \gamma \) and \( \sigma \). The strong difference between the various operators \( \mathcal{L} \) is reflected in the different boundary behavior of their nonnegative solutions. We will often use the exponent \( \gamma \), that represents the boundary behavior of the first eigenfunction \( \Phi_1 \propto \text{dist}(\cdot, \partial \Omega)^\gamma \), see [7].

In the parabolic case, solutions corresponding to different operators have different boundary behavior (as we expect for the linear case by the Green function estimates), but we can appreciate also the nontrivial interplay with the nonlinearity, that is reflected in the new parameter

\[
\sigma = 1 \wedge \frac{2sm}{\gamma(m - 1)}. \]

In all our estimate there is a “critical time” that separates two different regimes: the short and the long time behavior. This critical time \( t^* \) is given by a weighted \( L^1 \) norm as follows:

\[
t^* := \frac{\kappa^*}{\|u_0\|_{L^1_{\text{loc}}(\Omega)}^{m-1}},
\]

where \( \kappa^* > 0 \) is a universal constant. We can now state our main results in the model cases.

The RFL. We define the fractional Laplacian operator acting on a bounded domain by using the integral representation on the whole space in terms of a hypersingular kernel, namely

\[
(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x) - g(z)}{|x-z|^{N+2s}} \, dz,
\]

where \( c_{N,s} > 0 \) is a normalization constant, and we “restrict” the operator to functions that are zero outside \( \Omega \). We denote such operator by \( \mathcal{L} = (-\Delta)^s \), and call it the restricted fractional Laplacian\(^1\) (RFL). The initial and boundary conditions associated to the fractional diffusion equation (1.1) read \( u(t,x) = 0 \) in \( (0,\infty) \times \mathbb{R}^N \setminus \Omega \) and \( u(0,\cdot) = u_0 \). As explained in [10], such boundary conditions can also be understood via the Caffarelli-Silvestre extension, see [16].

\(^1\)In the literature this is often called the fractional Laplacian on domains, but this simpler name may be confusing when the spectral fractional Laplacian is also considered, cf. [13]. As discussed in this paper, there are other natural versions.
sharp expression of the boundary behavior for RFL has been investigated in [11]. We refer to [10] for a careful construction of the RFL in the framework of fractional Sobolev spaces, and [4] for a probabilistic interpretation.

Let us present the results. Since in this case we have \( \gamma = s < 2s \), we see that \( \sigma = 1 \) for all \( 0 < s < 1 \), and Theorem [10.1] shows the sharp boundary behavior for all times, namely for all \( t > 0 \) and a.e. \( x \in \Omega \) we have

\[
\kappa \left( 1 \wedge \frac{t}{t^*} \right) \frac{m - 1}{m} \frac{\text{dist}(x, \partial \Omega)^{s/m}}{t^{m-1}} \leq u(t, x) \leq \frac{\Pi \text{dist}(x, \partial \Omega)^{s/m}}{t^{m-1}}. \tag{2.2}
\]

Moreover, solutions are classical in the interior and we prove sharp Hölder continuity up to the boundary. These regularity results have been first obtained in [6]; we give here different proofs valid in the more general setting of this paper. See Section 12 for further details.

The SFL. Starting from the classical Dirichlet Laplacian \( \Delta_\Omega \) on the domain \( \Omega \), the so-called spectral definition of the fractional power of \( \Delta_\Omega \) may be defined via a formula in terms of the semigroup associated to the Laplacian, namely

\[
(-\Delta_\Omega)^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{t\Delta_\Omega} g(x) - g(x)) \frac{dt}{t^{1+s}} = \sum_{j=1}^{\infty} \lambda_j^s \hat{g}_j \varphi_j(x), \tag{2.3}
\]

where \( (\lambda_j, \varphi_j), j = 1, 2, \ldots \), is the normalized spectral sequence of the standard Dirichlet Laplacian on \( \Omega \), \( \hat{g}_j = \int_\Omega g(x) \varphi_j(x) \, dx \), and \( \| \varphi_j \|_{L^2(\Omega)} = 1 \). We denote this operator by \( \mathcal{L} = (-\Delta_\Omega)^s \), and call it the spectral fractional Laplacian (SFL) as in [15]. The initial and boundary conditions associated to the fractional diffusion equation (1.1) read \( u(t, x) = 0 \) on \( (0, \infty) \times \partial \Omega \) and \( u(0, \cdot) = u_0 \). Such boundary conditions can also be understood via the Caffarelli-Silvestre extension, see [10].

In this paper, we show that this operator admits a kernel \( K \) such that

\[
(-\Delta_\Omega)^s g(x) = c_{N,s} \text{ P.V.} \int_\Omega [g(x) - g(z)] K(x, z) \, dz, \tag{2.4}
\]

and we find the sharp behavior of \( K \) (see Lemma 3.1 or [7] for further details). As we shall see, in our parabolic setting, the degeneracy of the kernel is responsible for a peculiar change of the boundary behavior of the solutions for small and large times.

In this case we have \( \gamma = 1 \), therefore \( \sigma \) can be less than 1, depending on the values of \( s \) and \( m \). Then the lower bounds change both for short and large times, and they strongly depend on \( \sigma \). More precisely, Theorem 10.4 shows that for all \( t > 0 \) and all \( x \in \Omega \) we have

\[
\kappa \left( 1 \wedge \frac{t}{t^*} \right) \frac{m - 1}{m} \frac{\text{dist}(x, \partial \Omega)^{\sigma/m}}{t^{m-1}} \leq u(t, x) \leq \frac{\Pi \text{dist}(x, \partial \Omega)^{\sigma/m}}{t^{m-1}}. \tag{2.5}
\]

Such lower behavior is somehow minimal, in the sense that it holds in all cases, but it is sharp for small times and for initial data satisfying \( u_0 \lesssim \Phi_1 \), see Theorem 7.2. The question then becomes what happens for large times.

Let \( S(x) \propto \text{dist}(x, \partial \Omega)^{\sigma/m} \) be the solution of the associated elliptic problem, see Section 4 (recall that in this case \( \Phi_1 \propto \text{dist}(-, \partial \Omega) \)). Since \( t^{m-1} u(t, x) \) converges in \( L^\infty(\Omega) \) to \( S(x) \) as
t \to \infty \text{ (see Section 11), one may expect that, for } t \text{ large enough, the lower bound in (2.5) could be improved by replacing dist}(x, \partial \Omega) \text{ with dist}(x, \partial \Omega)^{\sigma/m}. \text{ This is shown to be true for } \sigma = 1 \text{ (cf. Theorem 9.1) but it is false for } \sigma < 1 \text{ (cf. Proposition 9.2). In particular, there are solutions whose boundary behavior does not match the one of the separate-variables solutions } U_T(t, x) := (t + T)^{-\frac{1}{m-1}} S(x).

It is interesting that, in the spectral case, one can appreciate the interplay between the “elliptic scaling power” $2s/(m-1)$ related to the invariance of the equation $L S^m = S$ under the scaling $S(x) \mapsto \lambda^{-2s/(m-1)} S(\lambda x)$, the “eigenfunction power” $\gamma (= 1)$, and the “nonlinearity power” $m$, made clear through the parameter $\sigma/m$. Also in this case, thanks to the strict positivity in the interior, we can show interior space-time regularity of solutions, as well as sharp boundary Hölder regularity for large times whenever upper and lower bounds match.

The CFL. In the simplest case, the infinitesimal operator of censored stochastic processes has the form

$$L f(x) = \text{P.V.} \int_{\Omega} \frac{f(x) - f(y)}{|x-y|^{N+2s}} \, dy, \quad \text{with } \frac{1}{2} < s < 1. \quad (2.6)$$

This operator has been introduced in [5] (see also [19] and [14] for further details and references).

In this case $\gamma = s - 1/2 < 2s$, hence $\sigma = 1$ for all $1/2 < s < 1$, and Theorem 10.1 shows that for all $t > 0$ and $x \in \Omega$ we have

$$\kappa \left( 1 \wedge \frac{t}{t^*} \right)^{-\frac{m-1}{m}} \frac{\text{dist}(x, \partial \Omega)^{(s-1/2)/m}}{t^{1/m-1}} \leq u(t, x) \leq \kappa \frac{\text{dist}(x, \partial \Omega)^{(s-1/2)/m}}{t^{1/m-1}}.$$

Again, we have interior space-time regularity of solutions, as well as sharp boundary Hölder regularity for all times.

The local case. In the case $L = -\Delta$ we have $\gamma = s = 1$, hence $\sigma = 1$. Since we are dealing with the classical PME, so there is finite speed of propagation, and global positivity is false for small times. Hence, in this case $t^*$ is an estimate the time that it takes to the solution to become everywhere positive, and the lower bound given by Theorem 9.1 reads as follows:

$$u(t, x) \geq \kappa \frac{\text{dist}(x, \partial \Omega)^{1/m}}{t^{1/m-1}} \quad \text{for all } t \geq t^* \text{ and all } x \in \Omega.$$

Observe that this lower boundary behavior is sharp for large times, because of the matching upper bounds of Theorem 5.1. As mentioned at the beginning of Section 9, when $L = -\Delta$ the result has been proven in [2, 44] with different proofs. However, our arguments are very flexible and work also when the Laplacian is replaced by a linear elliptic equation $-\partial_i (a_{ij} \partial_j)$ with $C^1$ coefficients (see Section 3.3 for more details).

3 General class of operators and their kernels

The interest of the theory developed here lies in the sharpness of the results and the wide range of applicability. We have just discussed in Section 2 the most relevant examples appearing in the literature, and more are listed at the end of this section. Actually, our theory applies to a general class of operators with definite assumptions, and this is what we explain next.
Here we present the properties that have to be assumed on the class of admissible operators. Some of them already appeared in [14]. However, to further develop our theory, more hypotheses need to be introduced. In particular, while [14] only uses the properties of the Green function, here we shall make some assumptions also on the kernel of $L$ (whenever it exists). Note that assumptions on the kernel $K$ of $L$ are needed for the positivity results, because we need to distinguish between the local and nonlocal cases. The study of the kernel $K$ is performed in Subsection 3.2.

### 3.1 Basic assumptions on $L$

The linear operator $L : \text{dom}(L) \subseteq L^1(\Omega) \to L^1(\Omega)$ is assumed to be densely defined and sub-Markovian, more precisely, it satisfies (A1) and (A2) below:

- **(A1)** $L$ is $m$-accretive on $L^1(\Omega)$;
- **(A2)** If $0 \leq f \leq 1$ then $0 \leq e^{-tL}f \leq 1$.

Under these assumption, in [14], the first and the third author proved existence, uniqueness, weighted estimates, and smoothing effects.

#### Assumptions on $L^{-1}$

In order to prove our quantitative estimates, we need to be more specific about the operator $L$. Besides satisfying (A1) and (A2), we will assume that it has a left-inverse $L^{-1} : L^1(\Omega) \to L^1(\Omega)$ with a kernel $K$ such that

$$L^{-1}[f](x) = \int_{\Omega} K(x, y) f(y) \, dy,$$

where $K$ satisfies at least one of the following estimates, for some $s \in (0, 1]$:

- There exists a constant $c_{1, \Omega} > 0$ such that, for a.e. $x, y \in \Omega$:
  $$0 \leq K(x, y) \leq c_{1, \Omega} |x - y|^{-(N-2s)}.$$  
  \hspace{1cm} (K1)

- There exist constants $\gamma \in (0, 1]$ and $c_{0, \Omega}, c_{1, \Omega} > 0$ such that, for a.e. $x, y \in \Omega$:
  $${c_{0, \Omega}} \delta^\gamma(x) \delta^\gamma(y) \leq K(x, y) \leq \frac{c_{1, \Omega}}{|x-y|^{N-2s}} \left( \frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left( \frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1 \right),$$  
  \hspace{1cm} (K2)

where we adopt the notation $\delta(x) := \text{dist}(x, \partial\Omega)$.

Hypothesis (K2) introduces an exponent $\gamma$ which is a characteristic of the operator and will play a big role in the results. Notice that defining an inverse operator $L^{-1}$ implies that we are taking into account the Dirichlet boundary conditions. See more details in Section 2 of [14].

- The lower bound in (K2) is weaker than the known bounds on the Green function for many examples under consideration; indeed, the following stronger estimate holds in many cases:
  $$K(x, y) \asymp \frac{1}{|x-y|^{N-2s}} \left( \frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left( \frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1 \right).$$  
  \hspace{1cm} (K4)

**Remark.** We used the labels (A1), (A2), (K1), (K2), and (K4) to be consistent with the notation in [14].
Remark. Already in the classical local case $L = -\Delta$, the Green function $K$ satisfies (K4) only when $N \geq 3$, as the formulas slightly change when $N = 1, 2$. In the fractional case $s \in (0, 1)$ the same problem arises when $N = 1$ and $s \in [1/2, 1)$. Hence, treating also these cases would require a slightly different analysis based on different but related assumptions on $K$. Since our approach is very general, we expect it to work also in these remaining cases without any major difficulties. For this reason, to simplify the presentation, from now on we assume that

either $N \geq 2$, or $N = 1$ and $s \in (0, 1/2)$.

The role of the first eigenfunction of $L$. Under the assumption (K1) it is possible to show that the operator $L^{-1}$ has a first nonnegative eigenfunction $0 \leq \Phi_1 \in L^\infty(\Omega)$ satisfying $L\Phi_1 = \lambda_1 \Phi_1$ for some $\lambda_1 > 0$, cf. \cite{7}. As a consequence of (K2), the first eigenfunction satisfies

$$\Phi_1(x) \approx \delta^\gamma(x) = \text{dist}(x, \partial \Omega)^\gamma$$

for all $x \in \Omega$, (3.1)

hence it encodes the parameter $\gamma$ that takes care of describing the boundary behavior, as first noticed in \cite{13}. This holds under the assumption that the boundary of the domain $\Omega$ is smooth enough, for instance $C^{1,1}$.

Remark. Hypotheses (K1) and (K2) are basically the only requirement on the operator $L$ to obtain our main lower and upper estimates for the equation $\partial_t u = Lu^m$. To be more precise, it is shown in \cite{14} that (A1), (A2), and (K1) imply the validity of absolute upper bounds and some smoothing effects. It is also shown there that the stronger assumption (K2) implies weighted smoothing effects and weighted $L^1$ estimates. Here we strongly improve the results of \cite{14} by proving sharp upper and lower bounds boundary behavior as a consequence of (K2).

We note that our assumptions allow us to cover all the examples of operators described in Sections 2 and 3.3.

3.2 About the kernel of $L$

In this section we study the properties of the kernel of $L$. While in some cases $L$ is defined in terms of a kernel $K(x, y)$ as

$$L f(x) = P.V. \int_{\mathbb{R}^N} (f(x) - f(y)) K(x, y) \, dy$$

(see the RFL and CFL in Section 2), in other situations $L$ may not have a kernel (for instance, in the local case) or that may not be so obvious from its definition. Indeed, considering for instance the Spectral Fractional Laplacian (SFL), this operator is defined through the spectral formula

$$(-\Delta_\Omega)^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left( e^{t \Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}} = \sum_{j=1}^\infty \lambda_j^s \hat{g}_j \varphi_j(x)$$

(see Section 2). In the next lemma we show that, as a consequence of (3.3), the SFL admits a representation of the form (3.2), and $K(x, y)$ enjoys some precise interior and boundary estimates. This fact is needed in the study of the sharp boundary behavior, but has also an independent interest.
Lemma 3.1 (Spectral Kernels) Let \( s \in (0,1) \), and let \( \mathcal{L} \) be the \( s \)-th-spectral power of a linear elliptic second order operator \( \mathcal{A} \), and let \( \Phi_1 \) be the first positive eigenfunction of \( \mathcal{A} \). Let \( H(t,x,y) \) be the Heat Kernel of \( \mathcal{A} \), and assume that it satisfies the following bounds: there exist constants \( c_0, c_1, c_2 > 0 \) such that for all \( 0 < t \leq 1 \)

\[
c_0 \left[ \frac{\Phi_1(x)}{t^{\gamma/2}} \wedge 1 \right] \left[ \frac{\Phi_1(y)}{t^{\gamma/2}} \wedge 1 \right] e^{-c_1 \frac{|x-y|^2}{t}} \leq H(t,x,y) \leq c_0^{-1} \left[ \frac{\Phi_1(x)}{t^{\gamma/2}} \wedge 1 \right] \left[ \frac{\Phi_1(y)}{t^{\gamma/2}} \wedge 1 \right] \frac{e^{-c_1 \frac{|x-y|^2}{t}}}{t^{N/2}}
\]

and

\[
0 \leq H(t,x,y) \leq c_2 \Phi_1(x) \Phi_1(y) \quad \text{for all } t \geq 1.
\]

Then the operator \( \mathcal{L} \) can be expressed in the form (3.2) with a kernel \( K(x,y) \) satisfying

\[
c_3 \frac{1}{|x-y|^{N+2s}} \left[ \frac{\Phi_1(x)}{|x-y|^\gamma} \wedge 1 \right] \left[ \frac{\Phi_1(y)}{|x-y|^\gamma} \wedge 1 \right] \leq K(x,y)
\]

\[
\leq c_3^{-1} \frac{1}{|x-y|^{N+2s}} \left[ \frac{\Phi_1(x)}{|x-y|^\gamma} \wedge 1 \right] \left[ \frac{\Phi_1(y)}{|x-y|^\gamma} \wedge 1 \right],
\]

where \( c_3 > 0 \) depends only on \( c_0, c_1, c_2, N, s, \) and \( \gamma \). In particular, the kernel \( K \) is supported inside \( \Omega \times \Omega \).

Remark. Recall that \( \Phi_1(x) \asymp \delta^\gamma(x,\partial \Omega) \) under assumption (K2).

Proof. The complete proof of this result is given in [7]. Here we just explain the main idea. It follows from (3.2) that \( K(x,y) = -L \delta_y(x) \). Hence, using (3.3) we get

\[
K(x,y) = -L \delta_y(x) = -\frac{1}{\Gamma(-s)} \int_0^\infty (H(t,x,y) - \delta_y(x)) \frac{dt}{t^{1+s}}.
\]

Since \( \delta_y(x) = 0 \) when \( x \neq y \), it follows that

\[
K(x,y) = -\frac{1}{\Gamma(-s)} \int_0^\infty H(t,x,y) \frac{dt}{t^{1+s}}.
\]

Since \( \Gamma(-s) < 0 \), using the bounds (3.4) and (3.5) to estimate the right-hand side one obtains (3.6). \( \square \)

3.3 Examples

Here we briefly exhibit a number of examples to which our theory applies. These include a wide class of local and nonlocal operators. We just sketch the essential points, referring to [14] for a more detailed exposition.

The Restricted Fractional Laplacian (RFL). As already mentioned in Section 3.2 assumptions (A1), (A2), and (K2) are satisfied with \( \gamma = s \), cf. [35, 37]. Actually, in this case also (K4) holds.

Spectral Fractional Laplacian (SFL). As already mentioned in Section 2 assumptions (A1), (A2), and (K2) are satisfied with \( \gamma = 1 \). Assumption (K2) (and also (K4)) can be
obtained by the Heat kernel estimates valid for the case $s = 1$, cf. [23, 24, 25, 26, 27, 48], as explained in [13, 14].

Censored Fractional Laplacian (CFL) and operators with general kernels. As already mentioned in Section 2, assumptions (A1), (A2), and (K2) are satisfied with $\gamma = s - 1/2$. Moreover, it follows by [5, 19] that we can also consider operators of the form:

$$\mathcal{L}f(x) = \text{P.V.} \int_{\Omega} (f(x) - f(y)) \frac{a(x, y)}{|x - y|^{N+2s}} \, dy,$$

with $1/2 < s < 1$,

where $a(x, y)$ is a symmetric function of class $C^1$ bounded between two positive constants. Actually, the Green function $K(x, y)$ of $\mathcal{L}$ satisfies the stronger assumption (K4), cf. Corollary 1.2 of [19].

Fractional operators with general kernels. Consider integral operators of the form

$$\mathcal{L}f(x) = \text{P.V.} \int_{\mathbb{R}^N} (f(x) - f(y)) \frac{a(x, y)}{|x - y|^{N+2s}} \, dy,$$

where $a$ is a measurable symmetric function, bounded between two positive constants, and satisfying

$$|a(x, y) - a(x, x)| \chi_{|x-y|<1} \leq c|x - y|^\sigma,$$

with $0 < s < \sigma \leq 1$,

for some $c > 0$ (actually, one can allow even more general kernels, cf. [14, 36]). Then, for all $s \in (0, 1]$, the Green function $K(x, y)$ of $\mathcal{L}$ satisfies (K4) with $\gamma = s$, cf. Corollary 1.4 of [36].

Spectral powers of uniformly elliptic operators. Consider a linear operator $A$ in divergence form,

$$A = - \sum_{i,j=1}^{N} \partial_i (a_{ij} \partial_j),$$

with uniformly elliptic $C^1$ coefficients. The uniform ellipticity allows one to build a self-adjoint operator on $L^2(\Omega)$ with discrete spectrum $(\lambda_k, \phi_k)$. Using the spectral theorem, we can construct the spectral power of such operator as follows

$$\mathcal{L}f(x) := A^s f(x) := \sum_{k=1}^{\infty} \lambda_k^s \hat{f}_k \phi_k(x),$$

where $\hat{f}_k = \int_{\Omega} f(x) \phi_k(x) \, dx$

(we refer to the books [25, 26] for further details), and the Green function satisfies (K2) with $\gamma = 1$, cf. [26, Chapter 4.6]. Then, the first eigenfunction $\Phi_1$ is comparable to $\text{dist}(\cdot, \partial \Omega)$. Also, Lemma 3.1 applies (see for instance [26]) and allow us to get sharp upper and lower estimates for the kernel $K$ of $\mathcal{L}$, as in (3.6).

Other examples. As explained in Section 3 of [14], our theory may also be applied to: (i) Sums of two fractional operators; (ii) Sum of the Laplacian and a nonlocal operator kernels; (iii) Schrödinger equations for non-symmetric diffusions; (iv) Gradient perturbation of restricted fractional Laplacians. Finally, it is worth mentioning that our arguments readily extend to operators on manifolds for which the required bounds hold.
4 Main definitions and preliminary estimates

We denote by $L^p_{\Phi_1}(\Omega)$ the weighted $L^p$ space $L^p(\Omega, \Phi_1 \, dx)$, endowed with the norm
\[
\|f\|_{L^p_{\Phi_1}(\Omega)} = \left( \int_\Omega |f(x)|^p \Phi_1(x) \, dx \right)^{\frac{1}{p}}.
\]

Given $\gamma \in (0, 1]$ (this will be the Hölder regularity of the first eigenfunction of $\mathcal{L}$ near $\partial \Omega$), we recall the definition of the exponent $\sigma \in (0, 1]$ that will appear in the sharp boundary behavior:
\[
\sigma := 1 \land \frac{2sm}{\gamma(m-1)}.
\]

\[\text{(4.1)}\]

**Weak dual solutions: existence and uniqueness.** We recall the definition of weak dual solutions used in [14]. This is expressed in terms of the inverse operator $\mathcal{L}^{-1}$, and encodes the Dirichlet boundary condition. This is needed to build a theory of bounded nonnegative unique solutions to Equation (1.1) under the assumptions of the previous section. We just remark that in [14] we have adopted the setup with the weight $\delta_\gamma = \text{dist}(\cdot, \partial \Omega)^\gamma$, but the same arguments generalize immediately to the weight $\Phi_1$. Note that, under (K2), these two setups are equivalent.

**Definition 4.1** A function $u$ is a weak dual solution to the Dirichlet Problem for Equation (1.1) in $(0, \infty) \times \Omega$ if:
- $u \in C((0, \infty) : L^1_{\Phi_1}(\Omega))$, $u^m \in L^1((0, \infty) : L^1_{\Phi_1}(\Omega))$;
- The identity
  \[
  \int_0^\infty \int_\Omega \mathcal{L}^{-1} u \frac{\partial \psi}{\partial t} \, dx \, dt - \int_0^\infty \int_\Omega u^m \psi \, dx \, dt = 0 \tag{4.2}
  \]
  holds for every test function $\psi$ such that $\psi/\Phi_1 \in C^1_c((0, \infty) : L^\infty(\Omega))$.

We aim to solve equation (1.1) with homogeneous Dirichlet boundary (or exterior) conditions, plus given initial data. We will call this problem (CDP):

**Definition 4.2** A weak dual solution to the Cauchy-Dirichlet problem (CDP) is a weak dual solution to the Dirichlet Problem for (1.1) such that $u \in C([0, \infty) : L^1_{\Phi_1}(\Omega))$ and $u(0, x) = u_0 \in L^1_{\Phi_1}(\Omega)$.

This kind of solution has been first introduced in [13], cf. also [14]. Roughly speaking, we are considering the weak solution to the “dual equation” $\partial_t U = -u^m$, where $U = \mathcal{L}^{-1} u$, posed on the bounded domain $\Omega$ with homogeneous Dirichlet conditions. Such weak solution is obtained by approximation from below as the limit of the unique mild solution provided by the semigroup theory (cf. [14]), and it was used in [36] with space domain $\mathbb{R}^N$ in the study of Barenblatt solutions. We recall here the main existence and uniqueness result from [14].

**Theorem 4.3 (Existence and uniqueness of weak dual solutions)** Assume (K1) holds. Then, for every nonnegative $u_0 \in L^1_{\Phi_1}(\Omega)$, there exists a unique minimal weak dual solution to the (CDP).
Explicit solution. When trying to understand the behavior of positive solutions with general nonnegative data, it is natural to look for solutions obtained by separation of variables. These are given by

$$U_T(t, x) := (T + t)^{-\frac{1}{m-1}} S(x), \quad T \geq 0,$$

where $S$ solves the elliptic problem

$$\begin{cases} \mathcal{L}S^m = S & \text{in } (0, +\infty) \times \Omega, \\ S = 0 & \text{on the boundary}. \end{cases}$$

The properties of $S$ have been thoroughly studied in the companion paper [7], and we summarize them here for the reader’s convenience.

**Theorem 4.4 (Properties of asymptotic profiles)** Assume that $\mathcal{L}$ satisfies (A1), (A2), and (K2). Then there exists a unique positive solution $S$ to the Dirichlet Problem (4.4) with $m > 1$. Moreover, let $\sigma$ be as in (4.1), and assume that:
- either $\sigma = 1$;
- or $\sigma < 1$ and (K4) holds.

Then there exist positive constants $c_0$ and $c_1$ such that the following sharp absolute bounds hold true for a.e. $x \in \Omega$:

$$c_0 \Phi_1(x)^{\sigma/m} \leq S(x) \leq c_1 \Phi_1(x)^{\sigma/m}.$$  

**(4.5)**

**Remark.** As observed in the proof of Theorem 11.2, by applying Theorem 10.1 to the separate-variables solution $t^{-\frac{1}{m-1}} S(x)$ we deduce that (4.5) is still true when $\sigma < 1$ if, instead of assuming (K4), we suppose that $K(x, y) \leq c_1 |x - y|^{-(N+2s)}$ for a.e. $x, y \in \mathbb{R}^N$ and that $\Phi_1 \in C^\gamma(\Omega)$.

When $T = 0$, the solution $U_0$ in (4.3) is commonly named “Friendly Giant”, because it takes initial data $u_0 \equiv +\infty$ (in the sense of pointwise limit as $t \to 0$) but is bounded for all $t > 0$. This term was coined in the study of the standard porous medium equation.

## 5 Upper boundary estimates

In this section we shall assume that the operator $\mathcal{L}$ satisfies (A1) and (A2), and that its inverse $\mathcal{L}^{-1}$ satisfies (K2). Recall that, under these assumptions, $\Phi_1 \asymp \text{dist}(\cdot, \partial \Omega)^\gamma$.

**Theorem 5.1 (Absolute boundary estimates)** Let (A1), (A2), and (K2) hold. Let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in L^1_{\Phi_1}(\Omega)$, and let $\sigma$ be as in (4.1). Then, there exists a computable constant $k_1 > 0$, depending only on $N, s, m, \text{ and } \Omega$, such that

$$u(t, x) \leq k_1 \frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \quad \text{for all } t \geq 0 \text{ and all } x \in \Omega.$$  

**(5.1)**

This absolute bound proves a strong regularization which is independent of the initial datum, and improves the absolute bound in [14], as it exhibits a precise boundary behavior which turns out to be sharp in the nonspectral case (see Section 8).

The rest of the section is devoted to the proof of Theorems 5.1. To this end, we first recall a few results of [13] that we will be used in the rest of the paper, both in the proof of lower and upper estimates.
5.1 Pointwise and absolute upper estimates

Pointwise estimates. We begin by recalling the basic pointwise estimates which are crucial in the proof of all the upper and lower bounds of this paper.

Proposition 5.2 ([13, 14]) Let (A1), (A2), and (K2) hold. Let \( u \geq 0 \) be a weak dual solution to (CDP) corresponding to \( u_0 \in L^1_{\Phi_1}(\Omega) \). Then

\[
\int_{\Omega} u(t,x)K(x,x_0) \, dx \leq \int_{\Omega} u_0(x)K(x,x_0) \, dx \quad \text{for all } t > 0.
\] (5.2)

Moreover, for every \( 0 < t_0 \leq t_1 \leq t \) and almost every \( x_0 \in \Omega \), we have

\[
\left( \frac{t_0}{t_1} \right)^{\frac{m}{m-1}} (t_1 - t_0) u^m(t_0, x_0) \leq \int_{\Omega} \left[ u(t_0, x) - u(t_1, x) \right]K(x,x_0) \, dx \leq (m - 1) \frac{t_1^{\frac{m}{m-1}}}{t_0^{\frac{m}{m-1}}} u^m(t, x_0).
\] (5.3)

Absolute upper bounds. Using the estimates above, in Theorem 5.2 of [14] the authors proved that solutions corresponding to initial data \( u_0 \in L^1_{\Phi_1}(\Omega) \) satisfy

\[
\|u(t)\|_{L^\infty(\Omega)} \leq \frac{K_1}{t^{\frac{1}{m-1}}} \quad \text{for all } t > 0
\] (5.4)

where the constant \( K_1 \) is independent of the initial datum. For this reason, this is called “absolute bound”.

5.2 Upper bounds via Green function estimates

The proof of Theorem 5.1 (as well as some proofs of the lower bounds) requires the following general statement:

Lemma 5.3 Let (A1), (A2), and (K2) hold. Also, let \( u \geq 0 \) be a nonnegative bounded function, and let \( \sigma \) be as in (4.1). Assume that, for a.e. \( x_0 \in \Omega \),

\[
u(x_0)^m \leq \kappa_0 \int_{\Omega} \kappa(x) \Phi(x,x_0) \, dx.
\] (5.5)

Then, there exists a constant \( \kappa_\infty > 0 \), depending only on \( s, \gamma, m, N, \Omega \), such that the following bound holds true for a.e. \( x_0 \in \Omega \):

\[
\int_{\Omega} u(x) \Phi(x,x_0) \, dx \leq \kappa_\infty \kappa_0^{\frac{1}{m-1}} \Phi_1^\sigma(x_0).
\] (5.6)

The proof of the above lemma is long and technical, and it is given in full details in the companion paper about the elliptic problem [7] for a general nonlinearity \( F(u) \); for convenience of the reader, we have stated here the results in the case \( F(u) = u^m, m > 1 \).
5.3 Universal upper estimates: proof of Theorem 5.1

We already know that \( u(t) \in L^\infty(\Omega) \) for all \( t > 0 \) by (5.4). Also, choosing \( t_1 = 2t_0 \) in (5.3) we deduce that, for \( t \geq 0 \) and a.e. \( x_0 \in \Omega \),

\[
 u^m(t,x_0) \leq \frac{2^{m-1}}{t} \int_\Omega u(t,x)K(x,x_0) \, dx. \quad (5.7)
\]

The above inequality corresponds exactly to hypothesis (5.5) of Lemma 5.3 with the value \( \kappa_0 = \frac{2^m}{m!} t^{-1} \). As a consequence, inequality (5.6) holds, and we conclude that for a.e. \( x_0 \in \Omega \) and all \( t \geq t_0 \)

\[
 \int_\Omega u(t,x)K(x,x_0) \, dx \leq \kappa_\infty 2^{m-1} \Phi_1(x_0)^\sigma \frac{\Phi_1(x_0)^\sigma}{t^{m-1}}, \quad \text{for all } t > 0 \text{ and } x_0 \in \Omega. \quad (5.8)
\]

Hence, combining this bound with (5.7), we get

\[
 u^m(t,x_0) \leq K_1 \Phi_1(x_0)^\sigma \frac{t}{m-1}. \]

This proves the upper bounds (5.1) and concludes the proof. \( \square \)

6 Preliminary positivity estimates

In this section we shall always assume that the operator \( L \) satisfies (A1) and (A2), and that the kernel of \( L^{-1} \) satisfies (K2). In addition, we shall make some extra assumptions on the kernel of \( L \), depending on the situation. The results of these section will be used in the proof of the pointwise positivity results in the later sections.

6.1 Lower bounds for weighted norms

Here we prove some useful lower bounds for weighted norms. As we shall see, these follow from the \( L^1 \)-continuity for ordered solutions in the version proved in Proposition 8.1 of [14]:

**Proposition 6.1** Let \( u \geq v \) be two ordered weak dual solutions to (CDP) corresponding to the initial data \( 0 \leq u_0,v_0 \in L^1_{\Phi_1}(\Omega) \). Then, for all \( 0 \leq \tau_0 \leq \tau, t < \infty \) we have

\[
 \int_\Omega [u(\tau,x) - v(\tau,x)] \Phi_1(x) \, dx \leq \int_\Omega [u(t,x) - v(t,x)] \Phi_1(x) \, dx \\
+ K_1 \|u(\tau_0)\|_{L^1_{\Phi_1}(\Omega)} \|t - \tau\|^{2m/(m-1)\sigma} \int_\Omega [u(\tau_0,x) - v(\tau_0,x)] \Phi_1(x) \, dx, \quad (6.1)
\]

where \( \sigma := \frac{1}{2s+(N+\gamma)(m-1)} \) and \( K > 0 \) is a computable constant.

This result has interesting consequences.
Lemma 6.2 (Backward in time $L^1_{\Phi_1}$ lower bounds) Let $u$ be a solution to (CDP) corresponding to the initial datum $u_0 \in L^1_{\Phi_1}(\Omega)$. For all

$$0 \leq \tau_0 \leq t \leq \tau_0 + \frac{1}{(2\bar{K})^{1/(2s\vartheta_\gamma)}\|u(\tau_0)\|_{L^1_{\Phi_1}(\Omega)}^{m-1}}$$

(6.2)

we have

$$\frac{1}{2} \int_\Omega u(\tau_0,x)\Phi_1(x) \, dx \leq \int_\Omega u(t,x)\Phi_1(x) \, dx,$$

(6.3)

where $\bar{K}$ and $\vartheta_\gamma$ are as in Proposition 6.1.

Proof of Corollary 6.2 Choosing $v = 0$ and $\tau = \tau_0$ in (6.1), we get

$$\left[ 1 - K_0\|u(\tau_0)\|_{L^1_{\Phi_1}(\Omega)}^{2s(m-1)\vartheta_\gamma} \right] \int_\Omega u(\tau_0,x)\Phi_1(x) \, dx \leq \int_\Omega u(t,x)\Phi_1(x) \, dx.$$  

(6.4)

Then (6.3) follows from (6.2).

We also have a lower bound for $L^p_{\Phi_1}(\Omega)$ norms.

Lemma 6.3 Let $u$ be a solution to (CDP) corresponding to the initial datum $u_0 \in L^1_{\Phi_1}(\Omega)$. Then the following lower bound holds true for any $t \in [0, t_*)$ and $p \geq 1$:

$$c_2 \left( \int_\Omega u_0(x)\Phi_1(x) \, dx \right)^p \leq \int_\Omega u^p(t,x)\Phi_1(x) \, dx$$

(6.5)

Here $t_* = c_*\|u_0\|_{L^1_{\Phi_1}(\Omega)}^{-(m-1)}$, where $c_2, c_* > 0$ are positive constants that depend only on $N, s, m, p, \Omega$.

The proof of this Lemma is an easy adaptation of the proof of Lemma 2.2 of [6], so we skip it. Notice that $c_*$ has explicit form given in [13, 14, 6], while the form of $c_2$ is given in the proof of Lemma 2.2 of [6].

6.2 Approximate solutions

To prove our lower bounds, we will need a special class of approximate solutions $u_\delta$. We will list now the necessary details. In the case when $\mathcal{L}$ is the Restricted Fractional Laplacian (RFL) (see Section 3.3) these solutions have been used in the Appendix II of [6], where complete proofs can be found; the proof there holds also for the operators considered here. The interested reader can easily adapt the proofs in [6] to the current case.

Let us fix $\delta > 0$ and consider the problem:

$$\begin{cases}
\partial_t v_\delta = -\mathcal{L}[(v_\delta + \delta)^m - \delta^m] & \text{for any } (t, x) \in (0, \infty) \times \Omega \\
v_\delta(t, x) = 0 & \text{for any } (t, x) \in (0, \infty) \times (\mathbb{R}^N \setminus \Omega) \\
v_\delta(0, x) = u_0(x) & \text{for any } x \in \Omega.
\end{cases}$$

(6.6)

Next, we define

$$u_\delta := v_\delta + \delta.$$
We summarize here below the basic properties of \( u_\delta \).

Approximate solutions \( u_\delta \) exist, are unique, and bounded for all \((t, x) \in (0, \infty) \times \bar{\Omega}\) whenever \(0 \leq u_0 \in L^1_{\Phi_1}(\Omega)\). Also, they are uniformly positive: for any \( t \geq 0 \),

\[
u_\delta(t, x) \geq \delta > 0 \quad \text{for a.e. } x \in \Omega. \tag{6.7}\]

This implies that the equation for \( u_\delta \) is never degenerate in the interior, so solutions are smooth as the linear parabolic theory with the kernel \( K \) allows them to be (in particular, in the case of the fractional laplacian, they are \( C^\infty \) in space and \( C^1 \) in time). Also, by a comparison principle, for all \( \delta > \delta' > 0 \) and \( t \geq 0 \),

\[
u_\delta(t, x) \geq u_{\delta'}(t, x) \quad \text{for } x \in \Omega \tag{6.8}\]

and

\[
u_\delta(t, x) \geq u(t, x) \quad \text{for a.e. } x \in \Omega. \tag{6.9}\]

Furthermore, they converge in \( L^1_{\Phi_1}(\Omega) \) to \( u \) as \( \delta \to 0 \):

\[\|u_\delta(t) - u(t)\|_{L^1_{\Phi_1}(\Omega)} \leq \|u_\delta(0) - u_0\|_{L^1_{\Phi_1}(\Omega)} = \delta \|\Phi_1\|_{L^1(\Omega)}. \tag{6.10}\]

As a consequence of (6.8) and (6.10), we deduce that \( u_\delta \) converge pointwise to \( u \) at almost every point: more precisely, for all \( t \geq 0 \),

\[
u(t, x) = \lim_{\delta \to 0^+} u_\delta(t, x) \quad \text{for a.e. } x \in \Omega. \tag{6.11}\]

7 Infinite speed of propagation: universal lower bounds

We are going to quantitatively establish that all nonnegative weak dual solutions of our problems are in fact positive in \( \Omega \) for all \( t > 0 \).

**Theorem 7.1** Let \( \mathcal{L} \) satisfy (A1) and (A2), and assume that

\[
\mathcal{L} f(x) = P.V. \int_{\mathbb{R}^N} (f(x) - f(y)) K(x, y) \, dy, \quad \text{with } K(x, y) \geq c_0 \Phi_1(x) \Phi_1(y) \forall x, y \in \Omega. \tag{7.1}\]

Let \( u \geq 0 \) be a weak dual solution to the (CDP) corresponding to \( u_0 \in L^1_{\Phi_1}(\Omega) \). Then there exists a constant \( \kappa_0 > 0 \) such that the following inequality holds:

\[
u(t, x) \geq \kappa_0 \left(1 \wedge \frac{t}{t^*_s}\right) \left(\frac{\Phi_1(x)}{t^*_s}\right)^{\frac{m}{m-1}} \quad \text{for all } t > 0 \text{ and a.e. } x \in \Omega. \tag{7.2}\]

Here \( t^*_s = \kappa_s \|u_0\|_{L^1_{\Phi_1}(\Omega)}^{-(m-1)} \) is as in Lemma 6.3, and the constants \( \kappa_0 \) and \( \kappa_s \) depend only on \( N, s, \gamma, m, c_0, \) and \( \Omega \).

Note that, for \( t \geq t^*_s \), the dependence on the initial data disappears from the lower bound, as inequality reads

\[
u(t) \geq \kappa_0 \frac{\Phi_1}{t^*_s} \quad \forall t \geq t^*_s,\]
where $\kappa_0$ is an absolute constant. Assumption (7.1) on the kernel $K$ of $L$ holds for all examples mentioned in Section 3.3. Actually, in many cases the kernel of the nonlocal operator satisfies a stronger property, namely $\inf_{x,y \in \Omega} K(x,y) \geq \kappa \Omega > 0$. The latter case is somehow simpler to treat, and we can actually obtain sharp lower bounds for all times, see Theorem 7.3 below. Theorem 7.1 shows that, even in the “worst case scenario”, there is a quantitative lower bound to treat, and we can actually obtain sharp lower bounds for all times, see Theorem 8.1 below. It is remarkable that there are situations (for instance, when $\sigma < 1$) where this “minimal behavior” is sharp for short times, as shown by the following result:

**Theorem 7.2** Let $(A1)$, $(A2)$, and $(K2)$ hold. Let $u \geq 0$ be a weak dual solution to (CDP) corresponding to $u_0 \in L^1_\Phi(\Omega)$. Set $\bar{m} := m \land 2$ and assume that $2s < (\bar{m} - 1)\gamma$. Then, for every initial data $u_0 \leq A \Phi_1$ for some $A > 0$, we have

$$u(t) \leq \frac{\Phi_1}{[A^1 - \bar{m} - Ct]\bar{m}^{-1}} \text{ on } [0, T_A], \quad \text{where } T_A := \frac{1}{CA^{m-1}} ,$$

and the constant $\bar{C} > 0$, that depends only on $N, s, m, \lambda_1$, and $\Omega$.

**Remark 7.3** The assumption $2s < (\bar{m} - 1)\gamma$ is implied by $\sigma < 1$, since in that case $2s < \frac{m-1}{m} \gamma < (\bar{m} - 1)\gamma$. Still, there are cases when $\sigma = 1$ and the assumption $2s < (\bar{m} - 1)\gamma$ is satisfied.

**Proof of Theorem 7.1** The proof consists in showing that

$$u(t, x) \geq u(t, x) := k_0 t \Phi_1(x)$$

for all $t \in [0, t_*]$, where the parameter $k_0 > 0$ will be fixed later. Note that, once the inequality $u \geq u$ on $[0, t_*]$ is proved, we conclude as follows: since $t \mapsto t^{\frac{1}{m-1}} u(t, x)$ is nondecreasing in $t > 0$ for a.e. $x \in \Omega$ (cf. (2.3) in [14]) we have

$$u(t, x) \geq \left( \frac{t_*}{t} \right)^{\frac{1}{m-1}} u(t_*, x) \geq k_0 t_* \left( \frac{t_*}{t} \right)^{\frac{1}{m-1}} \Phi_1(x) \quad \text{for all } t \geq t_* .$$

Then, the result will follow $\kappa_0 = k_0 t_*^{\frac{m}{m-1}}$ (note that, as we shall see below, $k_0 t_*^{\frac{m}{m-1}}$ can be chosen independently of $u_0$). Hence, we are left with proving that $u \geq u$ on $[0, t_*]$. 

- **Step 1. Reduction to an approximate problem.** Let us fix $\delta > 0$ and consider the approximate solutions $u_\delta$ constructed in Section 6.2. We shall prove that $u_\delta \geq u$ on $[0, t_*]$, so that the result will follow by the arbitrariness of $\delta$.

- **Step 2.** We claim that $u(t, x) < u_\delta(t, x)$ for all $0 \leq t \leq t_*$ and $x \in \Omega$, for a suitable choice of $k_0 > 0$. Assume that the inequality $u < u_\delta$ is false in $[0, t_*] \times \Omega$, and let $(t_c, x_c)$ be the first contact point between $u$ and $u_\delta$. Since $u_\delta = \delta > 0 = u$ on the lateral boundary, $(t_c, x_c) \in (0, t_*) \times \Omega$. Now, since $(t_c, x_c) \in (0, t_*) \times \Omega$ is the first contact point, we necessarily have that

$$u_\delta(t_c, x_c) = u(t_c, x_c) \quad \text{and} \quad u_\delta(t, x) \geq u(t, x) \quad \forall t \in [0, t_c], \forall x \in \Omega . \quad (7.3)$$

Thus, as a consequence,

$$\partial_t u_\delta(t, x_c) \leq \partial_t u(t, x_c) = k_0 \Phi_1(x) . \quad (7.4)$$
Next, we observe the following Kato-type inequality holds: for any nonnegative function $f$,

$$\mathcal{L}(f^m) \leq mf^{m-1}\mathcal{L}f.$$  \hfill (7.5)

Indeed, by convexity, $f(x)^m - f(y)^m \leq m[f(x)]^{m-1}(f(x) - f(y))$, therefore

$$\mathcal{L}(f^m)(x) = \int_{\mathbb{R}^N} [f(x)^m - f(y)^m] K(x, y) \, dy$$

$$\leq m[f(x)]^{m-1}\int_{\mathbb{R}^N} [f(x) - f(y)] K(x, y) \, dy = m[f(x)]^{m-1}\mathcal{L}f(x).$$  \hfill (7.6)

As a consequence of (7.5), since $t_c \leq t_*$ and $\Phi_1$ is bounded,

$$\mathcal{L}(u^m)(t, x) \leq m\mathcal{L}(u) = m[k_0 t \Phi_1(x)]^{m-1} k_0 t \mathcal{L}(\Phi_1)(x)$$

$$= m\lambda_1[k_0 t \Phi_1(x)]^m \leq \kappa_1(t_* k_0)^m \Phi_1(x).$$  \hfill (7.7)

Then, using (7.4) and (7.7), we establish an upper bound for $-\mathcal{L}(u_\delta^m - u^m)(t_c, x_c)$ as follows:

$$-\mathcal{L}[u_\delta^m - u^m](t_c, x_c) = \partial_t u_\delta(t_c, x_c) + \mathcal{L}(u^m)(t_c, x_c) \leq k_0 \left[ 1 + \kappa_1 t_0^{m-1} \right] \Phi_1(x_c).$$  \hfill (7.8)

Next, we want to prove lower bounds for $-\mathcal{L}(u_\delta^m - \psi^m)(t_\delta, x_\delta)$, and this is the point where the nonlocality of the operator enters, since we make essential use of hypothesis (7.1). Using (7.3) and (7.1), we get

$$-\mathcal{L} [u_\delta^m - u^m](t_c, x_c)$$

$$= -\int_{\mathbb{R}^N} \left[ (u_\delta^m(t_c, x_c) - u_\delta^m(t_c, y)) - (u^m(t_c, x_c) - u^m(t_c, y)) \right] K(x, y) \, dy$$

$$= \int_{\Omega} [u_\delta^m(t_c, y) - u^m(t_c, y)] K(x, y) \, dy \geq c_0 \Phi_1(x_c) \int_{\Omega} [u_\delta^m(t_c, y) - u^m(t_c, y)] \Phi_1(y) \, dy,$$

from which it follows (since $u^m = [k_0 t \Phi_1(x)]^m \leq \kappa_2(t_* k_0)^m$)

$$-\mathcal{L} [u_\delta^m - u^m](t_c, x_c)$$

$$\geq c_0 \Phi_1(x_c) \int_{\Omega} u_\delta^m(t_c, y) \Phi_1(y) \, dy - c_0 \Phi_1(x_c) \int_{\Omega} u^m(t_c, y) \Phi_1(y) \, dy.$$  \hfill (7.9)

Combining the upper and lower bounds (7.8) and (7.9) we obtain

$$c_0 \Phi_1(x_c) \int_{\Omega} u_\delta^m(t_c, y) \Phi_1(y) \, dy \leq k_0 \left[ 1 + (\kappa_1 + \kappa_3) t_0^{m-1} \right] \Phi_1(x_c).$$  \hfill (7.10)

Hence, recalling (6.5), we get

$$c_2 \left( \int_{\Omega} u_0(x) \Phi_1(x) \, dx \right)^{m/2} \leq \int_{\Omega} u_\delta^m(t_c, y) \Phi_1(y) \, dy \leq k_0 \left[ 1 + (\kappa_1 + \kappa_3) t_0^{m-1} \right].$$

Since $t_* = \kappa_* \|u_0\|_{L^m_{\Phi_1}(\Omega)}$, this yields

$$c_2 \kappa_*^{m-1} t_*^{m-1} \leq k_0 \left[ 1 + (\kappa_1 + \kappa_3) t_0^{m-1} \right].$$

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which gives the desired contradiction provided we choose \( k_0 \) so that 
\[
K_0 := k_0 t^{m - 1} \quad \text{is universally small.}
\]

**Proof of Theorem 7.2.** In view of our assumption on the initial datum, namely \( u_0 \leq A \Phi_1 \), by comparison it is enough to prove that the function
\[
\overline{u}(t, x) = F(t)\Phi_1(x), \quad F(t) = \frac{1}{[A^{1 - \bar{m}} - \bar{C}t]^{\bar{m} - 1}}
\]
is a supersolution (i.e., \( \partial_t \overline{u} \geq -\mathcal{L}\overline{u}^m \)) in \((0, T_A) \times \Omega\) provided we choose \( \bar{C} \) sufficiently large.

To this aim, we use the following elementary inequality, whose proof is left to the interested reader: for any \( \overline{b} > 0 \) there exists \( B = B(\overline{b}) > 0 \) such that
\[
a^m - b^m \leq m b^{m-1}(a - b) + B |a - b|^\bar{m}, \quad \text{for all } 0 \leq a, b \leq \overline{b}, \quad (7.11)
\]
We apply inequality \((7.11)\) to \( a = \overline{u}(t, y) \) and \( b = \overline{u}(t, x) \) to obtain
\[
\overline{u}^m(t, y) - \overline{u}^m(t, x) \leq m F(t)^m \Phi_1(x)^{m-1} [\Phi_1(y) - \Phi_1(x)] + B F(t)^{\bar{m}} |\Phi_1(y) - \Phi_1(x)|^{\bar{m}} \leq m F(t)^m \Phi_1(x)^{m-1} [\Phi_1(y) - \Phi_1(x)] + B F(t)^{\bar{m}} c_{\gamma} |x - y|^{\bar{m} \gamma}, \quad (7.12)
\]
where in the last step we have used that \( |\Phi_1(y) - \Phi_1(x)| \leq c_{\gamma} |x - y|^{\gamma} \). Since
\[
\int_{\mathbb{R}^N} [\Phi_1(y) - \Phi_1(x)] K(x, y) \, dy = -\mathcal{L} \Phi_1(x) = -\lambda_1 \Phi_1(x) \leq 0,
\]
it follows that
\[
-\mathcal{L}[\overline{u}^m](x) = \int_{\mathbb{R}^N} [\overline{u}^m(t, y) - \overline{u}^m(t, x)] K(x, y) \, dy \leq B F(t)^{\bar{m}} c_{\gamma} \int_{\mathbb{R}^N} |x - y|^{\bar{m} \gamma} K(x, y) \, dy. \quad (7.13)
\]
Next, (K2) yields
\[
K(x, y) \leq \frac{c_1 \Phi_1(x)}{|x - y|^{N + 2s + \gamma}} \quad \text{and} \quad \text{supp}(K) \subseteq \overline{\Omega} \times \overline{\Omega}, \quad (7.14)
\]
which gives
\[
\int_{\mathbb{R}^N} |x - y|^{\bar{m} \gamma} K(x, y) \, dy \leq c_1 \Phi_1(x) \int_{\Omega} \frac{|x - y|^{\bar{m} \gamma}}{|x - y|^{N + 2s + \gamma}} \, dy \leq c_2 \Phi_1(x) .
\]
Notice that in the last step we have used that the integral is finite since \( 2s < (\bar{m} - 1) \gamma \). In conclusion,
\[
-\mathcal{L}\overline{u}^m \leq c_3 F(t)^{\bar{m}} \Phi_1(x) = F'(t)\Phi_1(x) = \partial_t \overline{u}
\]
where we used that \( F'(t) = c_3 F(t)^{\bar{m}} \) provided \( \bar{C} = c_3 (\bar{m} - 1) \). This proves that \( \overline{u} \) is a supersolution in \((0, T) \times \Omega\), concluding the proof. \( \square \)
8 Sharp lower bounds I: the non-spectral case

We now consider the case of operators \( \mathcal{L} \) whose kernel is strictly positive in \( \Omega \times \Omega \), and we obtain sharp estimates for all times. As far as examples are concerned, this property is enjoyed both by the RFL and by CFL, but not by the SFL (or, more in general, spectral powers of elliptic operators), see Sections 2 and 3. In the case of the RFL, this result was obtained in Theorem 1 of [6]. Note that, in the present situation, the boundary behavior is the same both for small and large times.

**Theorem 8.1 (Sharp lower boundary estimates for all times)** Let \( \mathcal{L} \) satisfy (A1) and (A2), and assume moreover that

\[
\mathcal{L} f(x) = \int_{\mathbb{R}^N} (f(x) - f(y)) K(x,y) \, dy, \quad \text{with} \quad \inf_{x,y \in \Omega} K(x,y) \geq \kappa_\Omega > 0. \tag{8.1}
\]

Furthermore, suppose that \( \mathcal{L} \) has a first eigenfunction \( \Phi_1 \approx \text{dist}(x, \partial \Omega)^\gamma \). Let \( \sigma \) be as in (4.1) and assume that:

- either \( \sigma = 1 \);
- or \( \sigma < 1 \), \( K(x,y) \leq c_1 |x - y|^{-(N+2s)} \) for a.e. \( x,y \in \mathbb{R}^N \), and \( \Phi_1 \in C^\gamma(\overline{\Omega}) \).

Let \( u \geq 0 \) be a weak dual solution to the (CDP) corresponding to \( u_0 \in L^1_{\Phi_1}(\Omega) \). Then there exists a constant \( \kappa_1 > 0 \) such that the following inequality holds:

\[
u(t,x) \geq \kappa_1 \left(1 \wedge \frac{t}{t_\ast} \right) \left(\frac{m}{m-1} \right)^{\frac{m}{m-1}} \Phi_1(x)^{\sigma/m} \quad \text{for all} \ t > 0 \ \text{and a.e.} \ x \in \Omega. \tag{8.2}\]

where \( t_\ast = \kappa \|u_0\|_{L^1_{\Phi_1}(\Omega)}^{(m-1)} \). The constants \( \kappa \) and \( \kappa_1 \) depend only on \( N,s,\gamma,m,\kappa_\Omega,c_1,\Omega, \) and \( \|\Phi_1\|_{C^\gamma(\Omega)} \).

**Remarks.** (i) As in the case of the Theorem 7.1 for large times the dependence on the initial data disappears from the lower bound and we have absolute lower bounds.

(ii) The boundary behavior is sharp in view of the upper bound from Theorem 5.1

**Proof of Theorem 8.1.** The proof proceeds along the lines of the proof of Theorem 7.1 so we will just briefly mention the common parts. We want to show that

\[
u(t,x) := \kappa_0 t \Phi_1(x)^{\sigma/m}, \tag{8.3}\]

is a lower barrier for our problem on \([0,t_\ast] \times \Omega \) provided \( \kappa_0 \) is small enough. More precisely, as in the proof of Theorem 7.1, we aim to prove that \( \nu < u_\delta \) on \([0,t_\ast] \), as the lower bound for \( t \geq t_\ast \) then follows by monotonicity.

Assume by contradiction that the inequality \( \nu(t,x) < u_\delta(t,x) \) is false inside \([0,t_\ast] \times \overline{\Omega} \). Since \( \nu < u_\delta \) on the parabolic boundary, letting \((t_c,x_c)\) be the first contact point, we necessarily have that \((t_c,x_c) \in (0,t_\ast] \times \partial \Omega \). The desired contradiction will be obtained by combining the upper and lower bounds (that we prove below) for the quantity \(-\mathcal{L}[u_\delta^n - u^m](t_c,x_c)\), and then choosing \( \kappa_0 > 0 \) suitably small. In this direction, it is convenient in what follows to assume that

\[
\kappa_0 \leq 1 \wedge \frac{t_\ast}{m-1}, \quad \text{so that} \quad \kappa_0^{m-1} t_\ast \leq 1. \tag{8.4}\]
Upper bound. We first establish the following upper bound: there exists a constant $\overline{A} > 0$ such that
\[
-\mathcal{L}[u_{\delta}^m - u^m](t_c, x_c) \leq \partial_t u_\delta(t_c, x_c) + \mathcal{L} u^m(t_c, x_c) \leq \overline{A} \kappa_0.
\] (8.5)

To prove this, we estimate $\partial_t u_\delta(t_c, x_c)$ and $\mathcal{L} u^m(t_c, x_c)$ separately. First we notice that, since $(t_\delta, x_\delta)$ is the first contact point, we have
\[
u_\delta(t_\delta, x_\delta) = u(t_\delta, x_\delta) \quad \text{and} \quad u_\delta(t, x) \geq u(t, x) \quad \forall t \in [0, t_\delta], \ \forall x \in \Omega.
\] (8.6)

Hence, since $t_\delta \leq t_*$,
\[
\partial_t u_\delta(t_\delta, x_\delta) \leq \partial_t u(t_\delta, x_\delta) = \kappa_0 \Phi_1(x)^{\sigma/m} \leq \kappa_0 \|\Phi_1\|_{L^\infty(\Omega)}^{\sigma/m} = A_1 \kappa_0,
\] (8.7)
where we defined $A_1 := \|\Phi_1\|_{L^\infty(\Omega)}^{\sigma/m}$. Next we estimate $\mathcal{L} u^m(t_c, x_c)$, using the Kato-type inequality [7.5], namely $\mathcal{L}[u^m] \leq m u^{m-1} \mathcal{L} u$. This implies
\[
\mathcal{L}[u^m](t) \leq m u^{m-1}(t) \mathcal{L} u(t, x) = m(\kappa_0 t)^m \Phi_1(x)^{\sigma(m-1)/m} \mathcal{L} \Phi_1^\sigma(x)
\leq m(\kappa_0 t_*)^m \|\Phi_1\|_{L^\infty(\Omega)}^{\sigma(m-1)/m} \|\mathcal{L} \Phi_1^\sigma\|_{L^\infty(\Omega)} := A_2 \kappa_0.
\] (8.8)
Since $\kappa_0^{m-1} t_*^m \leq 1$ (see (8.4)), in order to prove that $A_2$ is finite it is enough to bound $\|\mathcal{L} \Phi_1^\sigma\|_{L^\infty(\Omega)}$. When $\sigma = 1$ we simply have $\mathcal{L} \Phi_1 = -\lambda_1 \Phi_1$, hence $A_2 \leq m \lambda_1 \|\Phi_1\|_{L^\infty(\Omega)}^{2-1/m}$. When $\sigma < 1$, we use the assumption $\Phi_1 \in C^\gamma(\Omega)$ to estimate
\[
|\Phi_1^\sigma(x) - \Phi_1^\sigma(y)| \leq |\Phi_1(x) - \Phi_1(y)|^\sigma \leq C|x - y|^\gamma \quad \forall x, y \in \Omega.
\] (8.9)

Hence, since $\gamma \sigma = 2sm/(m - 1) > 2s$ and $K(x, y) \leq c_1|x - y|^{-(N+2s)}$, we see that
\[
|\mathcal{L} \Phi_1^\sigma(x)| = \left|\int_{\mathbb{R}^N} [\Phi_1^\sigma(x) - \Phi_1^\sigma(y)] K(x, y) dy\right|
\leq \int_{\Omega} |x - y|^\gamma K(x, y) dy + C \|\Phi_1\|_{L^\infty(\Omega)}^\sigma \int_{\mathbb{R}^N \setminus B_1} |y|^{-(N+2s)} dy \leq C_{1, \sigma},
\]
hence $A_2$ is again finite. Combining (8.7) and (8.8), we obtain (8.5) with $\overline{A} := A_1 + A_2$.

Lower bound. We want to prove that there exists $\underline{A} > 0$ such that
\[
-\mathcal{L}[u_{\delta}^m - u^m](t_c, x_c) \geq \frac{\underline{A}}{\|\Phi_1\|_{L^\infty(\Omega)}} \int_{\Omega} u_{\delta}^m(t_c, y) \Phi_1(y) dy - \underline{A} \kappa_0.
\] (8.10)

This follows by (8.1) and (8.6) as follows:
\[
-\mathcal{L}[u_{\delta}^m - u^m](t_c, x_c)
= - \int_{\mathbb{R}^N} \left[(u_{\delta}^m(t_c, x_c) - u_{\delta}^m(t_c, y)) - (u^m(t_c, x_c) - u^m(t_c, y))\right] K(x, y) dy
= \int_{\Omega} [u_{\delta}^m(t_c, y) - u^m(t_c, y)] K(x, y) dy
\geq \kappa_0 \int_{\Omega} [u_{\delta}^m(t_c, y) - u^m(t_c, y)] dy \geq \frac{\kappa_0}{\|\Phi_1\|_{L^\infty(\Omega)}} \int_{\Omega} u_{\delta}^m(t_c, y) \Phi_1(y) dy - \underline{A} \kappa_0,
\] (8.11)
where in the last step we used that \( u_m(t, y) = [\kappa_0 t \Phi_1(y)]^m \leq \kappa_2 (\kappa_0 t^*_m)^m \) and \( \kappa_0^{m-1} t^*_m \leq 1 \) (see (8.4)).

End of the proof. The contradiction can be now obtained by joining the upper and lower bounds (8.5) and (8.10). More precisely, we have proved that

\[
\int_{\Omega} u^m_0(t_c, y) \Phi_1(y) \, dy \leq \frac{\|\Phi_1\|_{L^\infty(\Omega)}}{\kappa_0} (A + A) \kappa_0 := \pi \kappa_0,
\]

that combined with the lower bound (6.5) yields

\[
c_2 \left( \int_{\Omega} u_0(x) \Phi_1(x) \, dx \right)^m \leq \int_{\Omega} u^m_0(t_c, y) \Phi_1(y) \, dy \leq \pi \kappa_0.
\]

Setting \( \kappa_0 := (1 \wedge \frac{c_2}{\pi}) t^*_m/m-1 \) we obtain the desired contradiction.

9 Sharp lower bounds II: the remaining cases

We have already seen the example of the separate-variables solutions (4.3) that have a very definite behavior at the boundary \( \partial \Omega \). The analysis of general solutions leads to completely different situations for \( \sigma = 1 \) and \( \sigma < 1 \).

9.1 Sharp lower bounds for large times: the case \( \sigma = 1 \)

When \( \sigma = 1 \) we can establish a quantitative lower bound near the boundary that matches the separate-variables behavior for large times.

**Theorem 9.1 (Absolute boundary estimates for large times)**

Let (A1), (A2), and (K2) hold, and let \( \sigma = 1 \). Let \( u \geq 0 \) be a weak dual solution to the (CDP) corresponding to \( u_0 \in L^1_{\Phi_1}(\Omega) \). There exists a constant \( \kappa_2 > 0 \) such that

\[
u(t, x) \geq \kappa_2 \frac{\Phi_1(x)^{1/m}}{t^{1/m-1}} \quad \text{for all } t \geq t^* \text{ and a.e. } x \in \Omega.
\]

(9.1)

Here, \( t^* = \kappa_2 \| u_0 \|_{L^1_{\Phi_1}(\Omega)}^{-(m-1)} \), and the constants \( \kappa_2 \) depend only on \( N, s, \gamma, m, \) and \( \Omega \).

**Remarks.** (i) At first sight, this theorem may seem weaker than the positivity results of the two previous sections. However, this result has wider applicability since it holds under the only assumption (K2) on the kernel \( K \). In particular it is valid in the local case \( s = 1 \), where the finite speed of propagation makes it impossible to have global lower bounds for small times.

(ii) When \( L = -\Delta \) the result has been proven in [2] and [44] by quite different methods. On the other hand, our method is very general and immediately applies to the case when \( L \) is an elliptic operator with \( C^1 \) coefficients, see Section [3.3]

(iii) This result fixes a small error in Theorem 7.1 of [13] where the power \( \sigma \) was not present.
Proof of Theorem 9.1. We first recall the upper pointwise estimates (5.3): for all \(0 \leq t_0 \leq t_1 \leq t\) and a.e. \(x_0 \in \Omega\), we have that
\[
\int_\Omega u(t_0, x)K(x, x_0) \, dx - \int_\Omega u(t_1, x)K(x, x_0) \, dx \leq (m - 1) \frac{t_1}{t_0} \| u^m(t, x) \|_{L^1(\Omega)}.
\]
(9.2)

The proof follows by estimating the two integrals on the left-hand side separately.

We begin by using the upper bounds (5.8) to get
\[
\int_\Omega u(t_1, x)K(x, x_0) \, dx \leq \frac{\Phi_1(x_0)}{t_1^{m-1}} \quad \text{for all } (t_1, x) \in (0, +\infty) \times \Omega.
\]
(9.3)

Then we note that, as a consequence of (K2) and Lemma 6.2
\[
\int_\Omega u(t_0, x)K(x, x_0) \, dx \geq \kappa_0 \Phi_1(x_0) \int_\Omega u(t_0, x) \Phi_1(x) \, dx \geq \frac{\kappa_0}{2} \Phi_1(x_0) \int_\Omega u_0(x) \Phi_1(x) \, dx
\]
(9.4)

provided \(t_0 \leq \frac{\tau_0}{\| u_0 \|_{L_{\infty}^1(\Omega)}^{1/m-1}}\). Combining (9.2), (9.3), and (9.4), for all \(t \geq t_1 \geq t_0 \geq 0\) we obtain
\[
u^m(t, x) \geq \frac{t_0^{1/m-1}}{m-1} \left( \frac{\kappa_0}{2} \| u_0 \|_{L_{\infty}^1(\Omega)} - \kappa t_1^{1/m-1} \right) \frac{\Phi_1(x_0)}{t_1^{m-1}}.
\]

Choosing
\[
t_0 := \frac{\tau_0}{\| u_0 \|_{L_{\infty}^1(\Omega)}^{1/m-1}} \leq t_1 := t_* = \frac{\kappa_*}{\| u_0 \|_{L_{\infty}^1(\Omega)}^{1/m-1}} \quad \text{with} \quad \kappa_* \geq \tau_0 \vee \left( \frac{\kappa_0}{4\kappa} \right)^{m-1}
\]
so that \(\frac{\kappa_0}{2} \| u_0 \|_{L_{\infty}^1(\Omega)} - \kappa t_1^{1/m-1} \geq \frac{\kappa_0}{4} \| u_0 \|_{L_{\infty}^1(\Omega)}\), the result follows. \(\Box\)

9.2 Positivity for large times: the case \(\sigma < 1\)

As shown in Theorem 7.1 the lower bound \(u(t) \geq \Phi_1\) is always valid. We now discuss the possibility of improving this bound.

Let \(S\) solve the elliptic problem (1.4). It follows by comparison whenever \(u_0 \geq \epsilon_0 S\) with \(\epsilon_0 > 0\) then \(u(t) \geq \frac{S}{(t_0 + t)^{1/(m-1)}}\), where \(T_0 = \epsilon_0^{1-m}\). Since \(S \asymp \Phi_1^{\sigma/m}\) under (K4), there are initial data for which the lower behavior is dictated by \(\Phi_1(x)^{\sigma/m} t^{1/(m-1)}\). More in general, as we shall see in Theorem 11.1, given any initial datum \(u_0 \in L_{\infty}^1(\Omega)\) the function \(v(t, x) := \frac{1}{m-1} u(t)\) always converges to \(S\) in \(L^\infty(\Omega)\) as \(t \to \infty\), independently of the value of \(\sigma\). Hence, one may conjecture that there should exist a waiting time \(t_* > 0\) after which the lower behavior is dictated by \(\Phi_1(x)^{\sigma/m} t^{-1/(m-1)}\), in analogy with what happens for the classical porous medium equation. As we shall see, this is actually false when \(\sigma < 1\). Since for large times \(v(t, x)\) must look like \(S(x)\) in uniform norm away from the boundary (by the interior regularity that we will prove later), the contrasting situation for large times could be described as ‘dolphin’s head’ with the ‘snout’ flatter than the ‘forehead’. As \(t \to \infty\) the forehead progressively fills the whole domain.

The next result shows that, in general, we cannot hope to prove that \(u(t) > \Phi_1^{1/m}\). In particular, when \(\sigma < 1\), this shows that the behavior \(u(t) \asymp \Phi_1^{\sigma/m}\) cannot hold.
Proposition 9.2 Let \((A1), (A2),\) and \((K2)\) hold, and \(u \geq 0\) be a weak dual solution to the \((CDP)\) corresponding to a nonnegative initial datum \(u_0 \in L^{1}_\Phi(\Omega)\). Assume that \(u_0(x) \leq C_0 \Phi_1(x)\) a.e. in \(\Omega\) for some \(C_0 > 0\). Then there exists a constant \(\kappa,\) depending only \(N, s, \gamma, m,\) and \(\Omega,\) such that

\[
    u(t, x)^m \leq C_0 \kappa \frac{\Phi_1(x)}{t} \quad \text{for all } t > 0 \text{ and a.e. } x \in \Omega.
\]

In particular, if \(\sigma < 1,\) then

\[
    \lim_{x \to \partial \Omega} \Phi_1(x)^{\sigma/m} = 0 \quad \text{for any } t > 0.
\]

The proposition above could make one wonder whether the sharp general lower bound could be given by \(\Phi_1^{1/m},\) as in the case \(\sigma = 1.\) Recall that, under rather minimal assumptions on the kernel \(K\) associated to \(L,\) we have a universal lower bound for \(u(t)\) in terms of \(\Phi_1\) (see Theorem 7.1). Here we shall see that, under \((K4),\) the bound \(u(t) \gtrsim \Phi_1^{1/m}\) is false for \(\sigma < 1,\)

Proposition 9.3 Let \((A1), (A2),\) and \((K4)\) hold, and let \(u \geq 0\) be a weak dual solution to the \((CDP)\) corresponding to a nonnegative initial datum \(u_0 \leq C_0 \Phi_1\) for some \(C_0 > 0.\) Assume that there exist constants \(\kappa, T, \alpha > 0\) such that

\[
    u(T, x) \geq \kappa \Phi_1^\alpha(x) \quad \text{for a.e. } x \in \Omega.
\]

Then \(\alpha \geq 1 - \frac{2s}{\gamma}.\) In particular \(\alpha > \frac{1}{m}\) if \(\sigma < 1.\)

Proof of Proposition 9.2 Since \(u_0 \leq C_0 \Phi_1\) and \(L \Phi = \lambda_1 \Phi_1,\) we have

\[
    \int_\Omega u_0(x) \mathbb{K}(x, x_0) \, dx \leq C_0 \int_\Omega \Phi_1(x) \mathbb{K}(x, x_0) \, dx = C_0 \mathcal{L}^{-1} \Phi_1(x_0) = \frac{C_0}{\lambda_1} \Phi_1(x_0).
\]

Since \(t \mapsto \int_\Omega u(t, y) \mathbb{K}(x, y) \, dy\) is decreasing (see (5.2)), it follows that

\[
    \int_\Omega u(t, y) \mathbb{K}(x_0, y) \, dy \leq \frac{C_0}{\lambda_1} \Phi_1(x_0) \quad \text{for all } t \geq 0.\quad (9.5)
\]

Combining this estimate with (5.7) concludes the proof. □

Proof of Proposition 9.3 Given \(x_0 \in \Omega,\) set \(R_0 := \text{dist}(x_0, \partial \Omega).\) Since \(\mathbb{K}(x, x_0) \gtrsim |x - x_0|^{-(N-2s)}\) inside \(B_{R_0/2}(x_0)\) (by \((K4),\)) using our assumption on \(u(T)\) we get

\[
    \int_\Omega \mathbb{K}(x, x_0) u(T, x) \, dx \gtrsim \int_{B_{R_0/2}(x_0)} \frac{\Phi_1(x)^{\alpha}}{|x - x_0|^{N-2s}} \gtrsim \Phi_1(x_0)^{\alpha} R_0^{2s}.
\]

Recalling that \(\Phi_1(x_0) \asymp R_0^{\gamma},\) this yields

\[
    \Phi_1(x_0)^{\alpha + \frac{2s}{\gamma}} \lesssim \int_\Omega \mathbb{K}(x, x_0) u(T, x) \, dx.
\]

Combining the above inequality with (9.5) gives

\[
    \Phi_1(x_0)^{\alpha + \frac{2s}{\gamma}} \lesssim \Phi_1(x_0) \quad \forall x_0 \in \Omega, \quad \text{which implies } \alpha \geq 1 - \frac{2s}{\gamma}.
\]

Noticing that \(1 - \frac{2s}{\gamma} > \frac{1}{m}\) if and only if \(\sigma < 1,\) this concludes the proof. □
10 Harnack inequalities

In this section we combine the upper and lower bounds from the previous sections to generate various forms of Harnack inequalities both of global and local type. Such inequalities are important for regularity issues (see Section 12), and they play a fundamental role in formulating the sharp asymptotic behavior (see Section 11).

In this section we assume that the operator $L$ satisfies (A1) and (A2), and that its inverse $L^{-1}$ satisfies (K2). Recall that

$$\Phi_1 \asymp \text{dist}(\cdot, \partial \Omega)^\gamma, \quad \sigma = 1 \land \frac{2sm}{\gamma(m-1)}, \quad t_* = \kappa_* \|u_0\|_{L^1_\Phi_1(\Omega)}^{-(m-1)}.$$

Further assumptions will be made in each statement, depending on the desired result we want.

Theorem 10.1 (Global Harnack Principle I. The non-spectral case.) Let (A1), (A2), (K2), and (8.1) hold. Also, when $\sigma < 1$, assume that $K(x, y) \leq c_1|x - y|^{-(N + 2s)}$ for a.e. $x, y \in \mathbb{R}^N$ and that $\Phi_1 \in C^\gamma(\Omega)$. Let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in L^1_\Phi_1(\Omega)$. Then, there exist constants $\kappa, \hat{\kappa} > 0$, so that the following inequality holds:

$$\kappa \left(1 \lor \frac{t}{t_*}\right)^{-\frac{m}{m-1}} \frac{\Phi_1(x)^{\sigma/m}}{t^{1/m-1}} \leq u(t, x) \leq \hat{\kappa} \frac{\Phi_1(x)^{\sigma/m}}{t^{1/m-1}} \quad \text{for all } t > 0 \text{ and all } x \in \Omega. \quad (10.1)$$

The constants $\kappa, \hat{\kappa}$ depend only on $N, s, \gamma, m, c_1, \kappa_\Omega, \Omega$, and $\|\Phi_1\|_{C^\gamma(\Omega)}$.

Proof. We combine the upper bound (5.1) with the lower bound (8.1). The expression of $t_*$ is explicitly given in Theorem 8.1. \qed

As a consequence, we can obtain more standard local Harnack inequalities, of elliptic and forward type, as follows.

Theorem 10.2 (Local Harnack Inequalities of Elliptic/Backward Type) Under the assumptions of Theorem 10.1, there exists a constant $\hat{H} > 0$, depending only on $N, s, \gamma, m, c_1, \kappa_\Omega, \Omega$, such that for all the following holds for all balls $B_R(x_0)$ such that $B_{2R}(x_0) \subset \Omega$:

$$\sup_{x \in B_R(x_0)} u(t, x) \leq \hat{H} \left(1 \lor \frac{t}{t_*}\right)^{-\frac{m}{m-1}} \inf_{x \in B_R(x_0)} u(t, x) \quad \text{for all } t > 0. \quad (10.2)$$

Moreover, for all $t > 0$ and all $h > 0$ we have the following:

$$\sup_{x \in B_R(x_0)} u(t, x) \leq \hat{H} \left[\left(1 + \frac{h}{t}\right) \left(1 \lor \frac{t}{t_*}\right)^{-\frac{m}{m-1}} \right] \inf_{x \in B_R(x_0)} u(t + h, x). \quad (10.3)$$

Proof. Recalling (10.1), the bound (10.2) follows easily from the following Harnack inequality for the first eigenfunction, see for instance [7]:

$$\sup_{x \in B_R(x_0)} \Phi_1(x) \leq H_{N, s, \gamma, \Omega} \inf_{x \in B_R(x_0)} \Phi_1(x).$$
Since \( u(t, x) \leq (1 + h/t)^{\frac{1}{m-1}} u(t + h, x) \) (by the time monotonicity of \( t \mapsto t^{\frac{1}{m-1}} u(t, x) \)), (10.3) follows. \( \square \)

**Remark.** Already in the local case \( s = 1 \), these Harnack inequalities are stronger than the known Harnack inequalities for the porous medium equation, cf. [1, 22, 29, 30, 31], which are of forward type and are often stated in terms of the so-called intrinsic geometry. Note that elliptic and backward Harnack-type inequalities usually occur in the fast diffusion range \( m < 1 \) [8, 9, 11, 12], or for linear equations in bounded domains [33, 42].

### 10.1 Degenerate kernels

When the kernel \( K \) vanishes on \( \partial \Omega \), there are two combinations of upper/lower bounds that provide Harnack inequalities, one for small times and one for large times. As we have already seen, there is a strong difference between the case \( \sigma = 1 \) and \( \sigma < 1 \).

**Theorem 10.3** (Global Harnack Principle II.) Let (A1), (A2), and (K2) hold, and let \( u \geq 0 \) be a weak dual solution to the (CDP) corresponding to \( u_0 \in L^1_{\Phi_1}(\Omega) \). Assume that:
- either \( \sigma = 1 \);
- or \( \sigma < 1 \), \( u_0 \geq \kappa_0 \Phi_1^{\sigma/m} \) for some \( \kappa_0 > 0 \), and (K4) holds.

Then there exist constants \( \kappa, \overline{\kappa} > 0 \) such that the following inequality holds:

\[
\kappa \frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \leq u(t, x) \leq \overline{\kappa} \frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \quad \text{for all } t \geq t_* \text{ and all } x \in \Omega. \tag{10.4}
\]

The constants \( \kappa, \overline{\kappa} \) depend only on \( N, s, \gamma, m, \kappa_0, \overline{\kappa}_0, \kappa_\Omega \), and \( \Omega \).

**Proof.** In the case \( \sigma = 1 \), we combine the upper bound (5.1) with the lower bound (9.1). The expression of \( t_* \) is explicitly given in Theorem 9.1. When \( \sigma < 1 \), the upper bound is still given (5.1), while the lower bound follows by comparison with the solution \( S(x)(\kappa_0^{1-m} + t)^{-\frac{1}{m-1}} \), recalling that \( S \asymp \Phi_1^{\sigma/m} \) (see Theorem 4.4). \( \square \)

**Remark.** The same elliptic/backward local Harnack inequalities of Theorem 10.2 hold true for \( t \geq t_* \).

Note that, for small times, we cannot find matching powers for a global Harnack inequality (except for some special initial data), and such result is actually false for \( s = 1 \) (in view of the finite speed of propagation). Hence, in the remaining cases, we have only the following general result.

**Theorem 10.4** (Global Harnack Principle III.) Let (A1), (A2), (K2), and (7.1) hold. Let \( u \geq 0 \) be a weak dual solution to the (CDP) corresponding to \( u_0 \in L^1_{\Phi_1}(\Omega) \). Then, there exist constants \( \kappa, \overline{\kappa} > 0 \), so that the following inequality holds:

\[
\kappa \left( 1 \wedge \frac{t}{t_*} \right)^{\frac{m}{m-1}} \Phi_1(x)^{\sigma/m} \leq u(t, x) \leq \overline{\kappa} \frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \quad \text{for all } t > 0 \text{ and all } x \in \Omega. \tag{10.5}
\]

**Proof.** We combine the upper bound (5.1) with the lower bound (7.2). The expression of \( t_* \) is explicitly given in Theorem 7.1 \( \square \)
11 Asymptotic behavior

An important application of the Global Harnack inequalities of the previous section concerns the sharp asymptotic behavior of solutions. More precisely, we first show that for large times all solutions behave like the separate-variables solution \( U(t, x) = S(x) t^{-\frac{1}{m-1}} \) introduced at the end of Section 4. Then, whenever the (GHP) holds, we can improve this result to an estimate in relative error.

**Theorem 11.1 (Asymptotic behavior)** Assume that \( L \) satisfies (A1), (A2), and (K2), and let \( S \) be as in Theorem 4.4. Let \( u \) be any weak dual solution to the (CDP). Then, unless \( u \equiv 0 \),

\[
\left\| t^{\frac{1}{m-1}} u(t, \cdot) - S \right\|_{L^{\infty}(\Omega)} \xrightarrow{t \to \infty} 0.
\]  

(11.1)

**Proof.** The proof uses rescaling and time monotonicity arguments, and it is a simple adaptation of the proof of Theorem 2.3 of [10]. In those arguments, the interior \( C^{\alpha} \) continuity is needed to improve the \( L^1(\Omega) \) convergence to \( L^\infty(\Omega) \), but the interior Hölder continuity is guaranteed by Theorem 12.1(i) below.

We now exploit the (GHP) to get a stronger result.

**Theorem 11.2 (Sharp asymptotic behavior)** Under the assumptions of Theorem 11.1, assume that \( u \not\equiv 0 \). Furthermore, suppose that either the assumptions of Theorem 10.1 or of Theorem 10.3 hold. Set \( U(t, x) := t^{-\frac{1}{m-1}} S(x) \). Then there exists \( c_0 > 0 \) such that, for all \( t \geq t_0 := c_0 \| u_0 \|_{L^{1}_b(\Omega)} \), we have

\[
\left\| \frac{u(t, \cdot)}{U(t, \cdot)} - 1 \right\|_{L^{\infty}(\Omega)} \leq \frac{2}{m-1} \frac{t_0}{t_0 + t}.
\]  

(11.2)

We remark that the constant \( c_0 > 0 \) only depends on \( N, s, \gamma, m, \kappa_0, \kappa_\Omega \), and \( \Omega \).

**Remark.** This asymptotic result is sharp, as it can be checked by considering \( u(t, x) = U(t + 1, x) \). For the classical case, that is \( L = \Delta \), we recover the classical results of [2, 44] with a different proof.

**Proof.** Notice that, as a consequence of the (GHP) in the previous section,

\[
u(t) \asymp \Phi_{1}^{\sigma/m} t^{\frac{1}{m-1}} \quad \text{for all } t \geq t_*.
\]

In particular, applying this result to \( U(t, x) = S(x)(1 + t)^{-\frac{1}{m-1}} \), it follows that \( S \asymp \Phi_{1}^{\sigma/m} \). Hence, we can rewrite the bound above saying that there exist \( \kappa, \overline{\kappa} > 0 \) such that

\[
\kappa \frac{S(x)}{t^{\frac{1}{m-1}}} \leq u(t, x) \leq \overline{\kappa} \frac{S(x)}{t^{\frac{1}{m-1}}} \quad \text{for all } t \geq t_* \text{ and a.e. } x \in \Omega.
\]  

(11.3)

Since \( t_* = \kappa_\| u_0 \|_{L^{1}_b(\Omega)} \), the first inequality implies that

\[
\frac{S}{(t_* + t_0)^{\frac{1}{m-1}}} \leq \kappa \frac{S}{t_*^{\frac{1}{m-1}}} \leq u(t_*)
\]
for some $t_0 = c_0 ||u_0||^{-\frac{1}{m-1}}_{1,\Omega} \geq t_*$. Hence, by comparison principle,

$$\frac{S}{(t + t_0)^{\frac{1}{m-1}}} \leq u(t) \text{ for all } t \geq t_*.$$  

On the other hand, it follows by (11.3) that $u(t, x) \leq U_T(t, x) := S(x) (t - T)^{\frac{1}{m-1}}$ for all $t \geq T$ provided $T$ is large enough. If we now start to reduce $T$, the comparison principle combined with the upper bound (5.1) shows that $u$ can never touch $U_T$ from below in $(T, \infty) \times \Omega$. Hence we can reduce $T$ until $T = 0$, proving that $u \leq U_0$ (for an alternative proof, see Lemma 5.4 in [10]). Since $t_0 \geq t_*$, this shows that

$$\frac{S(x)}{(t + t_0)^{\frac{1}{m-1}}} \leq u(t, x) \leq \frac{S(x)}{t^{\frac{1}{m-1}}} \text{ for all } t \geq t_0 \text{ and a.e. } x \in \Omega,$$

therefore

$$\left| 1 - \frac{u(t, x)}{U(t, x)} \right| \leq 1 - \left( 1 - \frac{t_0}{t_0 + t} \right)^{\frac{1}{m-1}} \leq \frac{2}{m-1} \frac{t_0}{t_0 + t} \text{ for all } t \geq t_0 \text{ and a.e. } x \in \Omega,$$

as desired. \(\square\)

12 Regularity results

In order to obtain the regularity results, we need to assume that the operator $L$ satisfies (A1) and (A2), and that its inverse $L^{-1}$ satisfies (K2). We also require the validity of a Global Harnack Principle (GHP), namely Theorems 10.1, 10.3, or 10.4, depending on the situation under study. For convenience we refer to all these assumptions as (R).

As a consequence, for any ball $B_{2r}(x_0) \subset \subset \Omega$ and $0 < t_0 < T_1$, there exist $\delta, M > 0$ such that

$$0 < \delta \leq u(t, x) \text{ for a.e. } (t, x) \in (T_0, T_1) \times B_{2r}(x_0),$$

$$0 \leq u(t, x) \leq M \text{ for a.e. } (t, x) \in (T_0, T_1) \times \Omega.$$  

The constants in the regularity estimates will depend on the solution only through the upper and lower bounds on $u$. For some higher regularity results, we will eventually need some extra assumptions on the kernels.

**Theorem 12.1 (Interior Regularity)** Assume (R) and that

$$Lf(x) = P.V. \int_{\mathbb{R}^N} (f(x) - f(y)) K(x, y) \, dy,$$

with

$$K(x, y) \asymp |x - y|^{-(N + 2s)} \text{ in } B_{2r}(x_0) \subset \Omega, \quad K(x, y) \lesssim |x - y|^{-(N + 2s)} \text{ in } \mathbb{R}^N \setminus B_{2r}(x_0).$$

Let $u$ be a nonnegative bounded weak dual solution to problem (CDP).
(i) Then $u$ is Hölder continuous in the interior. More precisely, there exists $\alpha > 0$ such that, for all $0 < T_0 < T_2 < T_1$,

$$\|u\|_{C_t^{\alpha/2s,\alpha}((T_2,T_1) \times B_r(x_0))} \leq C. \quad (12.1)$$

(ii) Assume in addition $|K(x,y) - K(x',y)| \leq c|x - x'|^{\beta} |y|^{-(N+2s)}$ for some $\beta \in (0, 1 \wedge 2s)$ such that $\beta + 2s$ is not an integer. Then $u$ is a classical solution in the interior. More precisely, for all $0 < T_0 < T_2 < T_1$,

$$\|u\|_{C_t^{1+\beta/2s,2s+\beta}((T_2,T_1) \times B_r(x_0))} \leq C. \quad (12.2)$$

When $s > \gamma/2$, we can also prove Hölder regularity up to the boundary:

**Theorem 12.2 (Hölder continuity up to the boundary)** Under assumptions of Theorem 12.1(ii), assume in addition that $2s > \gamma$. Then $u$ is Hölder continuous up to the boundary. More precisely, for all $0 < T_0 < T_2 < T_1$ there exists a constant $C > 0$ such that

$$\|u\|_{C_t^{\gamma/m,\gamma}((T_2,T_1) \times \Omega)} \leq C \quad \text{with} \quad \vartheta := 2s - \gamma \left(1 - \frac{1}{m}\right). \quad (12.3)$$

**Remark.** Since $u(t,x) \asymp \Phi_1(x)^{1/m} \asymp \text{dist}(x,\partial\Omega)^{\gamma/m}$ (note that $2s > \gamma$ implies that $\sigma = 1$), the spacial Hölder exponent is sharp, while the Hölder exponent in time is the natural one by scaling.

### 12.1 Proof of interior regularity

The strategy to prove Theorem 12.1 follows the lines of [6] but with some modifications. The basic idea is that, because $u$ is bounded away from zero and infinity, the equation is non-degenerate and we can use parabolic regularity for nonlocal equations to obtain the results. More precisely, interior Hölder regularity will follow by applying $C_t^{\alpha/2s,\alpha}$ estimates of [34] for a “localized” linear problem. Once Hölder regularity is established, under an Hölder continuity assumption on the kernel we can use the Schauder estimates proved in [32] to conclude.

#### 12.1.1 Localization of the problem

Up to a rescaling, we can assume $r = 2$, $T_0 = 0$, $T_1 = 1$. Also, by a standard covering argument, it is enough to prove the results with $T_2 = 1/2$.

Take a cutoff function $\rho \in C_c^\infty(B_1)$ such that $\rho \equiv 1$ on $B_3$, $\eta \in C_c^\infty(B_2)$ a cutoff function such that $\eta \equiv 1$ on $B_1$, and define $v = \rho u$. By construction $u = v$ on $(0,1) \times B_1$. Since $\rho \equiv 1$ on $B_3$, we can write the equation for $v$ on the small cylinder $(0,1) \times B_1$ as

$$\partial_t v(t,x) = -L[v^m](t,x) + g(t,x) = -L_a v(t,x) + f(t,x) + g(t,x)$$

where

$$L_a[v](t,x) := \int_{\mathbb{R}^N} (v(t,x) - v(t,y))a(t,x,y)K(x,y) \, dy,$$
\[ a(t, x, y) := \frac{v^m(t, x) - v^m(t, y)}{v(t, x) - v(t, y)} \eta(x - y) + [1 - \eta(x - y)] \]
\[ = m\eta(x - y) \int_0^1 [(1 - \lambda v(t, x) + \lambda v(t, y))]^{m-1} d\lambda + [1 - \eta(x - y)], \]
\[ f(t, x) := \int_{\mathbb{R}^N \setminus B_1(x)} \left( v^m(t, x) - v^m(t, y) - v(t, x) + v(t, y) \right) [1 - \eta(x - y)] K(x, y) dy, \]
and
\[ g(t, x) := -\mathcal{L} [(1 - \rho^m) u^m](t, x) = \int_{\mathbb{R}^N \setminus B_1} (1 - \rho^m(y)) u^m(t, y) K(x, y) dy \]
(recall that \((1 - \rho^m) u^m \equiv 0\) on \((0, 1) \times B_3\)).

12.1.2 Hölder continuity in the interior

Set \(b := f + g\), with \(f\) and \(g\) as above. It is easy to check that, since \(K(x, y) \lesssim |x - y|^{-(N+2s)}\), \(b \in L^\infty((0, 1) \times B_1)\). Also, since \(0 < \delta \leq u \leq M\) inside \((0, 1) \times B_1\), there exists \(\Lambda > 1\) such that \(\Lambda^{-1} \leq a(t, x, y) \leq \Lambda\) for \(a.e.\) \((t, x, y) \in (0, 1) \times B_1 \times B_1\) with \(|x - y| \leq 1\). This guarantees that the linear operator \(L_a\) is uniformly elliptic, so we can apply the results in [34] to ensure that
\[ \|u\|_{C^{\alpha/2s, \alpha}_t((1/2, 1) \times B_1/2)} \leq C \left( \|b\|_{L^\infty((0, 1) \times B_1)} + \|u\|_{L^\infty((0, 1) \times \mathbb{R}^N)} \right) \]
for some universal exponent \(\alpha > 0\). This proves Theorem [12.1](i).

12.1.3 Classical solutions in the interior

Now that we know that \(u \in C^{\alpha/2s, \alpha}_t((1/2, 1) \times B_1/2)\), we repeat the localization argument above with cutoff functions \(\rho\) and \(\eta\) supported inside \((1/2, 1) \times B_1/2\) to ensure that \(v := \rho u\) is Hölder continuous in \((1/2, 1) \times \mathbb{R}^N\). Then, to obtain higher regularity we argue as follows.

Set \(\beta_1 := \min\{\alpha, \beta\}\). Thanks to the assumption on \(K\) and Theorem [12.1](i), it is easy to check that \(K_a(t, x, y) := a(t, x, y) K(x, y)\) satisfies
\[ |K_a(t, x, y) - K_a(t', x', y)| \leq C \left( |x - x'|^{\beta_1} + |t - t'|^{\beta_1/2s} \right) |y|^{-(N+2s)} \]
inside \((1/2, 1) \times B_1/2\). Also, \(f, g \in C^{\beta_1/2s, \beta_1}_t((1/2, 1) \times B_1/2)\). This allows us to apply the Schauder estimates from [32] (see also [18]) to obtain that
\[ \|v\|_{C^{1+\beta_1/2s, 2s+\beta_1}_t((3/4, 1) \times B_{1/4})} \leq C \left( \|b\|_{C^{\beta_1/2s, \beta_1}_t((1/2, 1) \times B_{1/2})} + \|v\|_{C^{\beta_1/2s, \beta_1}_t((1/2, 1) \times \mathbb{R}^N)} \right). \]
In particular, \(u \in C^{1+\beta_1/2s, 2s+\beta_1}_t((3/4, 1) \times B_{1/8})\). In case \(\beta_1 = \beta\) we stop here. Otherwise we set \(\alpha_1 := 2s + \beta\) and we repeat the argument above with \(\beta_2 := \min\{\alpha_1, \beta\}\) in place of \(\beta_1\). In this way, we obtain that \(u \in C^{1+\beta_1/2s, 2s+\beta_1}_t((1 - 2^{-4}, 1) \times B_{2^{-5}})\). Iterating this procedure finitely many times, we finally obtain that
\[ u \in C^{1+\beta/2s, 2s+\beta}_t((1 - 2^{-k}, 1) \times B_{2^{-k-1}}) \]
for some universal \(k\). Finally, a covering argument completes the proof of Theorem [12.1](ii).
12.2 Proof of boundary regularity

The proof of Theorem [12.2] follows by scaling and interior estimates. Notice that the assumption $2s > γ$ implies that $σ = 1$, hence $u(t)$ has matching upper and lower bounds.

Given $x_0 ∈ Ω$, set $r = \text{dist}(x_0, ∂Ω)/2$ and define

$$u_r(t, x) := r^{-γ/m} u \left( t_0 + r^γ t, x_0 + rx \right), \quad \text{with} \quad γ := 2s - γ \left( 1 - \frac{1}{m} \right).$$

Note that, because $2s > γ$, we have $γ > 0$. With this definition, we see that $u_r$ satisfies the equation $∂_t u_r + L_r u_r^m = 0$ in $Ω_r := (Ω - x_0)/r$, where

$$L_r f(x) = P.V. \int_{\mathbb{R}^N} (f(x) - f(y)) K_r(x, y) dy, \quad K_r(x, y) := r^{N+2s} K(x_0 + rx, x_0 + ry).$$

Note that, since $σ = 1$, it follows by the (GHP) that $u(t) ∼ \text{dist}(x, ∂Ω)^γ/m$. Hence,

$$0 < δ ≤ u_r(t, x) ≤ M, \quad \text{for all} \quad t ∈ [r^{-δ}T_0, r^{-δ}T_1] \text{ and } x ∈ B_1,$n

with constants $δ, M > 0$ that are independent of $r$ and $x_0$. In addition, using again that $u(t) ∼ \text{dist}(x, ∂Ω)^γ/m$, we see that

$$u_r(t, x) ≤ C(1 + |x|^γ/m) \quad \text{for all} \quad t ∈ [r^{-δ}T_0, r^{-δ}T_1] \text{ and } x ∈ \mathbb{R}^N.$$

Noticing that $u_r^m(t, x) ≤ C(1 + |x|^γ)$ and that $γ < 2s$ by assumption, we see that the tails of $u_r$ will not create any problem. Indeed, for any $x ∈ B_1$,

$$\int_{\mathbb{R}^N \setminus B_2} u_r^m(t, y)K_r(x, y)^{-(N+2s)} dy ≤ C \int_{\mathbb{R}^N \setminus B_2} |y|^γ |y|^{-(N+2s)} dy ≤ C_0,$$

where $C_0$ is independent of $r$. This means that we can localize the problem using cutoff functions as done in Section [12.1] and the integrals defining the functions $f$ and $g$ will converge uniformly with respect to $x_0$ and $r$. Hence, we can apply Theorem [12.1(ii)] to get

$$\|u_r\|_{C^{1+β/2s, 2s+β}([r^{-δ}T+1/2, r^{-δ}T+1] × B_{1/2})} ≤ C \quad \text{(12.4)}$$

for all $T ∈ [T_0, T_1 - r^{-δ}]$. Since $γ/m < 2s + β$ (because $γ < 2s$), it follows that

$$\|u_r\|_{L^∞([r^{-δ}T+1/2, r^{-δ}T+1], C^{γ/m}(B_{1/2}))} ≤ \|u_r\|_{C^{1+β/2s, 2s+β}([r^{-δ}T+1/2, r^{-δ}T+1] × B_{1/2})} ≤ C.$$

Noticing that

$$\sup_{t ∈ [r^{-δ}T+1/2, r^{-δ}T+1]} [u_r]_{C^{γ/m}(B_{1/2})} = \sup_{t ∈ [T+δT/2, r^{-δ}T+1]} [u]_{C^{γ/m}(B_r(x_0))},$$

and that $T ∈ [T_0, T_1 - r^{-δ}]$ and $x_0$ are arbitrary, arguing as in [41] we deduce that, given $T_2 ∈ (T_0, T_1)$,

$$\sup_{t ∈ [T_2, T_1]} [u]_{C^{γ/m}(Ω)} ≤ C. \quad \text{(12.5)}$$

This proves the global Hölder regularity in space. To show the regularity in time, we start again from (12.4) to get

$$\|\partial_t u_r\|_{L^∞([r^{-δ}T+1/2, r^{-δ}T+1] × B_{1/2})} ≤ C.$$
By scaling, this implies that
\[ \| \partial_t u \|_{L^\infty([T+r^{\alpha}/2,T+r^{\alpha}]) \times B_r(x_0)} \leq C r^{\frac{2}{m}-\vartheta}, \]
and by the arbitrariness of \( T \) and \( x_0 \) we obtain (recall that \( r = \text{dist}(x_0, \partial \Omega)/2 \))
\[ |\partial_t u(t, x)| \leq C \text{dist}(x, \partial \Omega)^{\frac{2}{m}-\vartheta} \quad \forall t \in [T_2, T_1], x \in \Omega. \tag{12.6} \]

Note that \( \frac{2}{m}-\vartheta = \gamma - 2s < 0 \) by our assumption.

Now, given \( t_0, t_1 \in [T_2, T_1] \) and \( x \in \Omega \), we argue as follows: if \( |t_0 - t_1| \leq \text{dist}(x, \partial \Omega)\vartheta \) then we use (12.6) to get (recall that \( \frac{2}{m}-\vartheta < 0 \))
\[ |u(t_1, x) - u(t_0, x)| \leq C \text{dist}(x, \partial \Omega)^{\frac{2}{m}-\vartheta} |t_0 - t_1| \leq C |t_0 - t_1|^{\frac{2}{m\vartheta}}. \]

On the other hand, if \( |t_0 - t_1| \geq \text{dist}(x, \partial \Omega)\vartheta \), then we use (12.5) and the fact that \( u \) vanishes on \( \partial \Omega \) to obtain
\[ |u(t_1, x) - u(t_0, x)| \leq |u(t_1, x)| + |u(t_0, x)| \leq C \text{dist}(x, \partial \Omega)^{\frac{2}{m}} \leq C |t_0 - t_1|^{\frac{2}{m\vartheta}}. \]
This proves that \( u \) is \( \frac{2}{m\vartheta} \)-Hölder continuous in time, and completes the proof of Theorem 12.2.

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References


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