Nonlinear and Nonlocal Degenerate Diffusions on Bounded Domains

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References:


Outline of the talk

- The abstract setup of the problem
- Some important examples
- Existence and uniqueness
- First pointwise estimates
- Upper and Lower Estimates
- Harnack Inequalities
- Regularity Estimates
Introduction

Homogeneous Dirichlet Problem for Fractional Nonlinear Degenerate Diffusion Equations

(HDP) \[
\begin{align*}
    u_t + \mathcal{L} F(u) &= 0, & \text{in } (0, +\infty) \times \Omega \\
    u(0, x) &= u_0(x), & \text{in } \Omega \\
    u(t, x) &= 0, & \text{on the lateral boundary}.
\end{align*}
\]

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $L^1(\Omega)$.
  A wide class of linear operators fall in this class:
  \textit{all fractional Laplacians on domains}.
- The most studied nonlinearity is $F(u) = |u|^{m-1}u$, with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of “degenerate” nonlinearities $F$ are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$. 

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- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator \( \mathcal{L} \).
The linear operator $\mathcal{L} : \text{dom}(A) \subseteq L^1(\Omega) \to L^1(\Omega)$ is assumed to be densely defined and *sub-Markovian*, more precisely satisfying $(A1)$ and $(A2)$ below:

**(A1)** $\mathcal{L}$ is $m$-accretive on $L^1(\Omega)$,

**(A2)** If $0 \leq f \leq 1$ then $0 \leq e^{-t\mathcal{L}}f \leq 1$, or equivalently,

**(A2')** If $\beta$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$, $u \in \text{dom}(\mathcal{L})$, $\mathcal{L}u \in L^p(\Omega)$, $1 \leq p \leq \infty$, $v \in L^{p/(p-1)}(\Omega)$, $v(x) \in \beta(u(x))$ a.e., then

$$\int_{\Omega} v(x) \mathcal{L}u(x) \, dx \geq 0$$

**Remark.** These assumptions are needed for existence (and uniqueness) of semigroup (mild) solutions for the nonlinear equation $u_t = \mathcal{L}F(u)$, through a variant of the celebrated Crandall-Liggett theorem, as done by Benilan, Crandall and Pierre:


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- M. Crandall, M. Pierre, \textit{Regularizing Effects for} \( u_t = A\varphi(u) \) \textit{in} \( L^1 \), J. Funct. Anal. \textbf{45}, (1982), 194–212
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Assumption on the nonlinearity $F$

Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous and non-decreasing function, with $F(0) = 0$. Moreover, it satisfies the condition:

$$(N1) \quad F \in C^1(\mathbb{R} \setminus \{0\}) \text{ and } F/F' \in \text{Lip}(\mathbb{R})$$

and there exists $\mu_0, \mu_1 > 0$ s.t.

$$\frac{1}{m_1} = 1 - \mu_1 \leq \left(\frac{F}{F'}\right)' \leq 1 - \mu_0 = \frac{1}{m_0}$$

where $F/F'$ is understood to vanish if $F(r) = F'(r) = 0$ or $r = 0$.

The main example will be

$$F(u) = |u|^{m-1}u,$$

with $m > 1$, and $\mu_0 = \mu_1 = \frac{m-1}{m} < 1$.

which corresponds to the nonlocal porous medium equation studied in [BV1].

A simple variant is the combination of two powers:

- $m_0$ gives the behaviour at zero, when $u \sim 0$
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Let $\mathcal{L}$ satisfy (A1) and (A2) and let $F$ as satisfy (N1). Then for all $0 \leq u_0 \in L^1(\Omega)$, there exists a unique mild solution $u$ to equation

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and the function

$$(1) \quad t \mapsto t^{\frac{1}{\mu_0}} F(u(t,x)) \quad \text{is nondecreasing in } t > 0 \text{ for a.e. } x \in \Omega.$$ 

Moreover, the semigroup is contractive on $L^1(\Omega)$ and $u \in C([0, \infty) : L^1(\Omega))$.

We notice that (1) is a weak formulation of the monotonicity inequality:

$$\partial_t u \geq - \frac{1}{\mu_0} \frac{F(u)}{t F'(u)},$$

which implies

$$\partial_t u \geq - \frac{1 - \mu_0}{\mu_0} \frac{u}{t}$$

or equivalently, that the function

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Assumptions on the inverse of $\mathcal{L}$

We will assume that the operator $\mathcal{L}$ has an inverse $\mathcal{L}^{-1} : L^1(\Omega) \to L^1(\Omega)$ with a kernel $K$ such that

$$\mathcal{L}^{-1}f(x) = \int_{\Omega} K(x, y) f(y) \, dy,$$

and that satisfies (one of) the following estimates for some $\gamma, s \in (0, 1]$ and $c_{i,\Omega} > 0$

\begin{align*}
(\text{K1}) & \quad 0 \leq K(x, y) \leq \frac{c_{1,\Omega}}{|x - y|^{N-2s}} \\
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where

$$\delta_\gamma(x) := \text{dist}(x, \partial\Omega)^\gamma.$$

Indeed, (K1) implies that $\mathcal{L}$ has a first eigenfunction $0 \leq \Phi_1 \in L^\infty(\Omega)$. Moreover, (K2) implies that $\Phi_1 \asymp \text{dist}(\cdot, \partial\Omega)^\gamma = \delta_\gamma$ and we can rewrite (K2) as

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Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$:

1. **By means of Fourier Transform,**

   $$(((-\Delta_{\mathbb{R}^N})^sf)(\xi)) = |\xi|^{2s}\hat{f}(\xi).$$

   This formula can be used for positive and negative values of $s$.

2. **By means of an Hypersingular Kernel:**

   if $0 < s < 1$, we can use the representation

   $$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x) - g(z)}{|x - z|^{N+2s}} \, dz,$$

   where $c_{N,s} > 0$ is a normalization constant.

3. **Spectral definition,** in terms of the heat semigroup associated to the standard Laplacian operator:

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The Spectral Fractional Laplacian operator (SFL)

\[(−Δ_Ω)^s g(x) = \sum_{j=1}^{∞} λ_j^s \hat{g}_j \phi_j(x) = \frac{1}{Γ(−s)} \int_0^{∞} (e^{tΔ_Ω} g(x) − g(x)) \frac{dt}{t^{1+s}}.\]

- \(Δ_Ω\) is the classical Dirichlet Laplacian on the domain \(Ω\)
- EIGENVALUES: \(0 < λ_1 ≤ λ_2 ≤ \ldots ≤ λ_j ≤ λ_{j+1} ≤ \ldots\) and \(λ_j \asymp j^{2/N}\).
- EIGENFUNCTIONS: \(ϕ_j\) are as smooth as the boundary of \(Ω\) allows, namely when \(∂Ω\) is \(C^k\), then \(ϕ_j \in C^∞(Ω) \cap C^k(\overline{Ω})\) for all \(k \in \mathbb{N}\).

\[\hat{g}_j = \int_Ω g(x)ϕ_j(x) \, dx, \quad \text{with} \quad \|ϕ_j\|_{L^2(Ω)} = 1.\]

Lateral boundary conditions for the SFL

\[u(t,x) = 0, \quad \text{in } (0, ∞) \times ∂Ω.\]

The Green function of SFL satisfies a stronger assumption than (K2) or (K3), i.e.

\[(K4) \quad K(x,y) \asymp \frac{1}{|x−y|^{N−2s}} \left( \frac{δ_γ(x)}{|x−y|^γ} ∧ 1 \right) \left( \frac{δ_γ(y)}{|x−y|^γ} ∧ 1 \right), \quad \text{with } γ = 1\]
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- \( \Delta_\Omega \) is the classical Dirichlet Laplacian on the domain \( \Omega \)
- **Eigenvlues:** \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \lambda_{j+1} \leq \ldots \) and \( \lambda_j \approx j^{2/N} \).
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Outline of the talk

Part 1. First Pointwise Estimates
Part 2. Upper Bounds
Part 3. Lower bounds
Part 4. Harnack Inequalities
Part 5. Regularity for RFL
Part 6. Asymptotic behaviour

Examples of operators $\mathcal{L}$

Definition via the hypersingular kernel in $\mathbb{R}^N$, “restricted” to functions that are zero outside $\Omega$.

**The Restricted Fractional Laplacian operator (RFL)**

\[
(-\Delta_{|\Omega})^s g(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x) - g(z)}{|x - z|^{N + 2s}} \, dz, \quad \text{with supp}(g) \subseteq \overline{\Omega}.
\]

where $s \in (0, 1)$ and $c_{N,s} > 0$ is a normalization constant.

- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $L^2(\Omega)$ with a discrete spectrum:
- **EIGENVALUES**: $0 < \bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \ldots \leq \bar{\lambda}_j \leq \bar{\lambda}_{j+1} \leq \ldots$ and $\bar{\lambda}_j \asymp j^{2s/N}$.
  
  Eigenvalues of the RFL are smaller than the ones of SFL: $\bar{\lambda}_j \leq \lambda_j^s$ for all $j \in \mathbb{N}$.
- **EIGENFUNCTIONS**: $\phi_j$ are the normalized eigenfunctions, are only Hölder continuous up to the boundary, namely $\phi_j \in C^s(\overline{\Omega})$.

Lateral boundary conditions for the RFL

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(K4) \[
K(x, y) \asymp \frac{1}{|x - y|^{N - 2s}} \left( \frac{\delta_\gamma(x)}{|x - y|^\gamma} \wedge 1 \right) \left( \frac{\delta_\gamma(y)}{|x - y|^\gamma} \wedge 1 \right), \quad \text{with } \gamma = s
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$$( - \Delta |_{\Omega} )^s g(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x) - g(z)}{|x-z|^{N+2s}} \, dz , \quad \text{with supp}(g) \subseteq \overline{\Omega}.$$  

where $s \in (0, 1)$ and $c_{N,s} > 0$ is a normalization constant.

- $( - \Delta |_{\Omega} )^s$ is a self-adjoint operator on $L^2(\Omega)$ with a discrete spectrum:
- **EIGENVALUES**: $0 < \bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \ldots \leq \bar{\lambda}_j \leq \bar{\lambda}_{j+1} \leq \ldots$ and $\bar{\lambda}_j \asymp j^{2s/N}$.
  
  Eigenvalues of the RFL are smaller than the ones of SFL: $\bar{\lambda}_j \leq \lambda_j^s$ for all $j \in \mathbb{N}$.

- **EIGENFUNCTIONS**: $\overline{\phi}_j$ are the normalized eigenfunctions, are only Hölder continuous up to the boundary, namely $\overline{\phi}_j \in C^s(\overline{\Omega})$.

**Lateral boundary conditions for the RFL**

$$u(t, x) = 0 , \quad \text{in } (0, \infty) \times (\mathbb{R}^N \setminus \Omega).$$

The Green function of RFL satisfies a stronger assumption than (K2) or (K3), i.e.

$$(K4) \quad \mathbb{K}(x, y) \asymp \frac{1}{|x-y|^{N-2s}} \left( \frac{\delta_\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left( \frac{\delta_\gamma(y)}{|x-y|^\gamma} \wedge 1 \right) , \quad \text{with } \gamma = s$$

Censored Fractional Laplacians (CFL)

\[ \mathcal{L}f(x) = \text{P.V.} \int_{\Omega} (f(x) - f(y)) \frac{a(x, y)}{|x - y|^{N+2s}} \, dy, \quad \text{with} \quad \frac{1}{2} < s < 1, \]

where \( a(x, y) \) is a measurable, symmetric function bounded between two positive constants, satisfying some further assumptions; for instance \( a \in C^1(\overline{\Omega} \times \overline{\Omega}) \).

The Green function \( \mathbb{K}(x, y) \) satisfies \((K4)\), proven by Chen, Kim and Song (2010)

\[ \mathbb{K}(x, y) \asymp \frac{1}{|x - y|^{N-2s}} \left( \frac{\delta_\gamma(x)}{|x - y|^{\gamma}} \wedge 1 \right) \left( \frac{\delta_\gamma(y)}{|x - y|^{\gamma}} \wedge 1 \right), \quad \text{with} \quad \gamma = s - \frac{1}{2}. \]

Remarks.

- This is a third model of Dirichlet fractional Laplacian when \( a(x, y) = \text{const} \). This is not equivalent to SFL nor to RFL.
- Roughly speaking, \( s \in (0, 1/2] \) corresponds to Neumann boundary conditions.

References.

Examples of operators $\mathcal{L}$

Introduced in 2003 by Bogdan, Burdzy and Chen.

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### References.

**Spectral powers of uniformly elliptic operators.** Consider a linear operator $A$ in divergence form, with uniformly elliptic bounded measurable coefficients:

$$A = \sum_{i,j=1}^{N} \partial_i(a_{ij}\partial_j) , \quad s\text{-power of } A \text{ is: } \mathcal{L}f(x) := A^sf(x) := \sum_{k=1}^{\infty} \lambda_k f_k \phi_k(x)$$

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$$(K3) \quad c_{0,\Omega} \phi_1(x) \phi_1(y) \leq \mathcal{K}(x,y) \leq \frac{c_{1,\Omega}}{|x-y|^{N-2s}} \left( \frac{\phi_1(x)}{|x-y|} \wedge 1 \right) \left( \frac{\phi_1(y)}{|x-y|} \wedge 1 \right)$$

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**Fractional operators with “rough” kernels.** Integral operators of Levy-type

$$\mathcal{L}f(x) = \text{P.V.} \int_{\mathbb{R}^N} (f(x+y) - f(y)) \frac{K(x,y)}{|x-y|^{N+2s}} \, dy .$$

where $K$ is measurable, symmetric, bounded between two positive constants, and

$$|K(x,y) - K(x,x)| \chi_{|x-y|<1} \leq c|x-y|\sigma , \quad \text{with } 0 < s < \sigma \leq 1 ,$$

for some positive $c > 0$. We can allow even more general kernels.

The Green function satisfies a stronger assumption than (K2) or (K3), i.e.

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Sums of two Restricted Fractional Laplacians. Operators of the form

\[ \mathcal{L} = (\Delta|_{\Omega})^s + (\Delta|_{\Omega})^\sigma, \quad \text{with } 0 < \sigma < s \leq 1, \]

where \((\Delta|_{\Omega})^s\) is the RFL. Satisfy \((K4)\) with \(\gamma = s\).

Sum of the Laplacian and operators with general kernels. In the case

\[ \mathcal{L} = a\Delta + A_s, \quad \text{with } 0 < s < 1 \quad \text{and} \quad a \geq 0, \]

where

\[ A_s f(x) = \text{P.V.} \int_{\mathbb{R}^N} (f(x+y) - f(y) - \nabla f(x) \cdot y \chi_{|y| \leq 1}) \chi_{|y| \leq 1} d\nu(y), \]

the measure \(\nu\) on \(\mathbb{R}^N \setminus \{0\}\) is invariant under rotations around origin and satisfies

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together with other assumptions.

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\[ \mathcal{L} = c - \left(c^{1/s} - \Delta\right)^s, \quad \text{with } c > 0, \quad \text{and } 0 < s \leq 1. \]

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Many other interesting examples. Schrödinger equations for non-symmetric diffusions, Gradient perturbation of RFL...

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The "dual" formulation of the problem

Recall the homogeneous Dirichlet problem:

\[
\begin{aligned}
\partial_t u &= -\mathcal{L} F(u), \quad \text{in } (0, +\infty) \times \Omega \\
u(0, x) &= u_0(x), \quad \text{in } \Omega \\
u(t, x) &= 0, \quad \text{on the lateral boundary}.
\end{aligned}
\]

(CDP)

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

\[
\partial_t U = -F(u),
\]

where

\[
U(t, x) := \mathcal{L}^{-1}[u(t, \cdot)](x) = \int_\Omega K(x, y)u(t, y)\,dy.
\]

This formulation encodes the lateral boundary conditions in the inverse operator $\mathcal{L}^{-1}$.

Remark. This formulation has been used before by Pierre, Vázquez [...] to prove (in the $\mathbb{R}^N$ case) uniqueness of the "fundamental solution", i.e. the solution corresponding to $u_0 = \delta_{x_0}$, known as the Barenblatt solution.
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Recall that $\Phi_1 \asymp \delta_{\gamma}$ and

$$\|f\|_{L^1_{\Phi_1}(\Omega)} = \int_\Omega f(x) \Phi_1(x) \, dx,$$

and $L^1_{\Phi_1}(\Omega) := \{f : \Omega \rightarrow \mathbb{R} \mid \|f\|_{L^1_{\Phi_1}(\Omega)} < \infty\}$.

**Weak Dual Solutions**

A function $u$ is a weak dual solution to the Dirichlet Problem for $\partial_t u + \mathcal{L}F(u) = 0$ in $Q_T = (0, T) \times \Omega$ if:

- $u \in C((0, T) : L^1_{\Phi_1}(\Omega))$, $F(u) \in L^1((0, T) : L^1_{\Phi_1}(\Omega))$;
- The following identity holds for every $\psi/\Phi_1 \in C^1_c((0, T) : L^\infty(\Omega))$:

$$\int_0^T \int_\Omega L^{-1}_F(u) \frac{\partial \psi}{\partial t} \, dx \, dt - \int_0^T \int_\Omega F(u) \psi \, dx \, dt = 0.$$

**Weak Dual Solutions for the Cauchy Dirichlet Problem (CDP)**

A weak dual solution to the Cauchy-Dirichlet problem (CDP) is a weak dual solution to Equation $\partial_t u + \mathcal{L}F(u) = 0$ such that moreover

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Existence and uniqueness of weak dual solutions

**Theorem. Existence of weak dual solutions** (M.B. and J. L. Vázquez)

For every nonnegative $u_0 \in L^1_{\Phi_1}(\Omega)$ there exists a minimal weak dual solution to the $(CDP)$. Such a solution is obtained as the monotone limit of the semigroup (mild) solutions that exist and are unique. The minimal weak dual solution is continuous in the weighted space $u \in C([0, \infty) : L^1_{\Phi_1}(\Omega))$. Mild solutions (constructed by Crandall and Pierre) are weak dual solutions and if $u_0 \in L^p(\Omega)$ then $u(t) \in L^p(\Omega)$ for all $t > 0$.

**Theorem. Uniqueness of weak dual solutions** (M.B. and J. L. Vázquez)

The solution constructed in the above Theorem by approximation of the initial data from below is unique. We call it the *minimal solution*. In this class of solutions the standard comparison result holds, and also the weighted $L^1$ estimates.
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First Pointwise Estimates

**Theorem.** (M.B. and J. L. Vázquez)

Let $u \geq 0$ be a weak dual solution to Problem (CDP) with $u_0 \in L^p(\Omega)$, $p > N/2s$. Then,

$$
\int_{\Omega} u(t_1, x) K(x, x_0) \, dx \leq \int_{\Omega} u(t_0, x) K(x, x_0) \, dx,
$$

for all $t_1 \geq t_0 \geq 0$.

Moreover, for almost every $0 \leq t_0 \leq t_1$ and almost every $x_0 \in \Omega$, we have

$$
\left( \frac{t_0}{t_1} \right)^{\frac{1}{\mu_0}} (t_1 - t_0) F(u(t_0, x_0)) \leq \int_{\Omega} \left[ u(t_0, x) - u(t_1, x) \right] K(x, x_0) \, dx
$$

$$
\leq (m_0 - 1) \frac{t_1}{\frac{1}{\mu_0} - \frac{1}{\mu_0}} \frac{1}{t_0^{\frac{1}{\mu_0}}} F(u(t_1, x_0)).
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**Remark.** As a consequence of the above inequality and Hölder inequality, we have that $u(t) \in L^\infty(\Omega)$ when $u_0 \in L^p(\Omega)$, with $p > N/(2s)$. 
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Proof of the First Pointwise Estimates

**Sketch of the proof of the First Pointwise Estimates**

We would like to take as test function

\[
\psi(t, x) = \psi_1(t)\psi_2(x) = \chi_{[t_0, t_1]}(t) K(x_0, x),
\]

(This is not an admissible test in the Definition of Weak Dual solutions)

Plugging such test function in the definition of weak dual solution gives the formula

\[
\int_\Omega u(t_0, x) K(x_0, x) \, dx - \int_\Omega u(t_1, x) K(x_0, x) \, dx = \int_{t_0}^{t_1} F(u(\tau, x_0)) \, d\tau.
\]

This formula can be proven rigorously though careful approximation.

Next, we use the monotonicity estimates,

\[
t \mapsto t^{\frac{1}{\mu_0}} F(u(t, x)) \quad \text{is nondecreasing in } t > 0 \text{ for a.e. } x \in \Omega.
\]

to get for all \(0 \leq t_0 \leq t_1\), recalling that \(\frac{1}{\mu_0} = \frac{m_0}{m_0 - 1}\)

\[
\left(\frac{t_0}{t_1}\right)^{\frac{1}{\mu_0}} (t_1 - t_0) F(u(t_0, x_0)) \leq \int_{t_0}^{t_1} F(u(\tau, x_0)) \, d\tau \leq \frac{m_0 - 1}{t_0^{\frac{1}{\mu_0}}} F(u(t_1, x_0)). \]

\[\Box\]
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This formula can be proven rigorously though careful approximation.

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$$\left( \frac{t_0}{t_1} \right)^{1/\mu_0} (t_1 - t_0) F(u(t_0, x_0)) \leq \int_{t_0}^{t_1} F(u(\tau, x_0)) \, d\tau \leq \frac{m_0 - 1}{t_0^{1/\mu_0}} F(u(t_1, x_0)). \Box$$
Upper Bounds

For the rest of the talk we deal with the special case:

\[ F(u) = u^m := |u|^{m-1}u \]
The power case. Absolute bounds and boundary behaviour

**Theorem. (Absolute upper bounds and boundary behaviour)** (M.B. & J. L. Vázquez)

Let \( u \) be a weak dual solution, then there exists constants \( K_1, K_2 > 0 \) depending only on \( N, s, m, \Omega \) (but not on \( u_0 \)!!), such that

(K1) assumption implies:

\[
\|u(t)\|_{L^\infty(\Omega)} \leq \frac{K_1}{t^{m-1}}, \quad \text{for all } t > 0.
\]

Moreover, (K2) assumption implies, for \( 0 < \gamma \leq 2sm/(m - 1) \)

\[
u(t, x) \leq K_2 \frac{\Phi_1(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} , \quad \text{for all } t > 0 \text{ and } x \in \Omega.
\]

When \( \gamma > 2sm/(m - 1) \) the power of \( \Phi_1 \) becomes

\[
\frac{\sigma}{m} := \frac{2s}{(m - 1)\gamma} < \frac{1}{m}
\]

Remarks.

- This is a very strong regularization *independent* of the initial datum \( u_0 \).
- Sharp boundary estimates: we will show lower bounds with matching powers. The power decay of \( u^m \) is \( \sigma = 1 \wedge 2sm/[(m - 1)\gamma] \)
  In examples, only for SFL-type, \( \gamma = 1 \), and \( s \) small, \( 0 < s < 1/2 - 1/(2m) \)
- Time decay is sharp, but only for large times, say \( t \geq 1 \). For small times when \( 0 < t < 1 \) a better time decay is obtained in the form of smoothing effects
Theorem. (Absolute upper bounds and boundary behaviour) (M.B. & J. L. Vázquez)

Let $u$ be a weak dual solution, then there exists constants $K_1, K_2 > 0$ depending only on $N, s, m, \Omega$ (but not on $u_0$ !!), such that

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The power case. Absolute bounds and boundary behaviour

Sketch of the proof of Absolute Bounds

• **STEP 1. First upper estimates.** Recall the pointwise estimate:
\[
\left(\frac{t_0}{t_1}\right)^{m-1} (t_1 - t_0) u^m(t_0, x_0) \leq \int_{\Omega} u(t_0, x) G_\Omega(x, x_0) \, dx - \int_{\Omega} u(t_1, x) G_\Omega(x, x_0) \, dx.
\]
for any \(u \in S_p\), all \(0 \leq t_0 \leq t_1\) and all \(x_0 \in \Omega\). Choose \(t_1 = 2t_0\) to get
\[
(\star) \quad u^m(t_0, x_0) \leq \frac{2^{m-1}}{t_0} \int_{\Omega} u(t_0, x) G_\Omega(x, x_0) \, dx.
\]

Recall that \(u \in S_p\) with \(p > N/(2s)\), means \(u(t) \in L^p(\Omega)\) for all \(t > 0\), so that:
\[
u^m(t_0, x_0) \leq \frac{c_0}{t_0} \int_{\Omega} u(t_0, x) G_\Omega(x, x_0) \, dx \leq \frac{c_0}{t_0} \|u(t_0)\|_{L^p(\Omega)} \|G_\Omega(\cdot, x_0)\|_{L^q(\Omega)} < +\infty
\]

since \(G_\Omega(\cdot, x_0) \in L^q(\Omega)\) for all \(0 < q < N/(N - 2s)\), so that \(u(t_0) \in L^\infty(\Omega)\) for all \(t_0 > 0\).

• **STEP 2.** Let us estimate the r.h.s. of (\(\star\)) as follows:
\[
u^m(t_0, x_0) \leq \frac{c_0}{t_0} \int_{\Omega} u(t_0, x) G_\Omega(x, x_0) \, dx \leq \|u(t_0)\|_{L^\infty(\Omega)} \frac{c_0}{t_0} \int_{\Omega} G_\Omega(x, x_0) \, dx.
\]
Taking the supremum over \(x_0 \in \Omega\) of both sides, we get:
\[
\|u(t_0)\|_{L^\infty(\Omega)} \leq \frac{c_0}{t_0} \sup_{x_0 \in \Omega} \int_{\Omega} G_\Omega(x, x_0) \, dx \leq \frac{K_{I_1}^{m-1}}{t_0}
\]
Sketch of the proof of Absolute Bounds

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since \( G_{\Omega}(\cdot, x_0) \in L^q(\Omega) \) for all \( 0 < q < N/(N - 2s) \), so that \( u(t_0) \in L^\infty(\Omega) \) for all \( t_0 > 0 \).

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\|u(t_0)\|_{L^\infty(\Omega)}^{m-1} \leq \frac{c_0}{t_0} \sup_{x_0 \in \Omega} \int_{\Omega} G_{\Omega}(x, x_0) \, dx \leq \frac{K_1^{m-1}}{t_0}
\]
Outline of the talk

Part 1
First Pointwise Estimates

Part 2. Upper Bounds
Part 3. Lower bounds
Part 4. Harnack Inequalities
Part 5. Regularity

Asymptotic behaviour

Smoothing Effects

Define the exponents:
\[ \vartheta_{1,\gamma} = \frac{1}{2s + (N + \gamma)(m - 1)} \quad \text{and} \quad \vartheta_1 = \vartheta_{1,0} = \frac{1}{2s + N(m - 1)} \]

**Theorem. (Smoothing effects)** (M.B. & J. L. Vázquez)

There exist universal constants \( K_3, K_4 > 0 \) such that:

**L^1-L^\infty** SMOOTHING EFFECT: (K1) assumption implies for all \( t > 0 \):
\[
\|u(t)\|_{L^\infty(\Omega)} \leq \frac{K_3}{t^{N\vartheta_1}} \|u(t)\|_{L^1(\Omega)}^{2s\vartheta_1} \leq \frac{K_3}{t^{N\vartheta_1}} \|u_0\|_{L^1(\Omega)}^{2s\vartheta_1}
\]

**L^1_{\Phi_1}-L^\infty** SMOOTHING EFFECT: (K2) assumption implies for all \( t > 0 \):
\[
\|u(t)\|_{L^\infty(\Omega)} \leq \frac{K_4}{t^{(N+\gamma)\vartheta_{1,\gamma}}} \|u(t)\|_{L^1_{\Phi_1}(\Omega)}^{2s\vartheta_{1,\gamma}} \leq \frac{K_4}{t^{(N+\gamma)\vartheta_{1,\gamma}}} \|u_0\|_{L^1_{\Phi_1}(\Omega)}^{2s\vartheta_{1,\gamma}}.
\]

- A novelty is that we get instantaneous smoothing effects.
- Also the weighted smoothing effect is new (as far as we know).
- The time decay is better for small times \( 0 < t < 1 \) than the one given by absolute bounds:
\[
(N + \gamma)\vartheta_{1,\gamma} = \frac{N + \gamma}{2 + (N + \gamma)(m - 1)} < \frac{1}{m - 1}.
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\| u(t) \|_{L^\infty(\Omega)} \leq \frac{K_3}{t^{\vartheta_1}} \| u(t) \|_{L^1(\Omega)}^{2s \vartheta_1} \leq \frac{K_3}{t^{\vartheta_1}} \| u_0 \|_{L^1(\Omega)}^{2s \vartheta_1}
\]

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\[
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\]
Theorem. (Backward Smoothing effects) (M.B. & J. L. Vázquez)

There exists a universal constant $K_4 > 0$ such that for all $t, h > 0$

$$\|u(t)\|_{L^\infty(\Omega)} \leq \frac{K_4}{t^{(d+\gamma)\vartheta_1,\gamma}} \left(1 \vee \frac{h}{t}\right)^{\frac{2s\vartheta_1,\gamma}{m-1}} \|u(t+h)\|_{L^1_{\Phi_1}(\Omega)}.$$  

Proof. By the monotonicity estimates, the function $u(x, t)t^{1/(m-1)}$ is non-decreasing in time for fixed $x$, therefore using the smoothing effect, we get for all $t_1 \geq t$:

$$\|u(t)\|_{L^\infty(\Omega)} \leq \frac{K_4}{t^{(N+1)\vartheta_1,\gamma}} \left(\int_\Omega u(t, x) \Phi_1(x) \, dx\right)^{2s\vartheta_1,\gamma}$$

$$\leq \frac{K_4}{t^{(N+1)\vartheta_1,\gamma}} \left(\frac{t_1^{\frac{1}{m-1}}}{t^{\frac{1}{m-1}}} \int_\Omega u(t_1, x) \Phi_1(x) \, dx\right)^{2s\vartheta_1,\gamma}$$

where $K_4$ is as in the smoothing effects. Finally, let $t_1 = t + h$. □
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Upper Bounds for general $F$
Theorem. (Absolute upper estimate) (M.B. & J. L. Vázquez)

Let $u$ be a nonnegative weak dual solution corresponding to $u_0 \in L^1_{\delta \gamma}(\Omega)$. Then, there exists universal constants $K_0, K_1, K_2 > 0$ such that the following estimates hold true for all $t > 0$:

$$F\left(\|u(t)\|_{L^\infty(\Omega)}\right) \leq F^*\left(\frac{K_1}{t}\right).$$

Moreover, there exists a time $\tau_1(u_0)$ with $0 \leq \tau_1(u_0) \leq K_0$ such that

$$\|u(t)\|_{L^\infty(\Omega)} \leq 1 \quad \text{for all } t \geq \tau_1,$$

so that

$$\|u(t)\|_{L^\infty(\Omega)} \leq \frac{K_2}{t^{m_i-1}} \quad \text{with} \quad \begin{cases} i = 0 & \text{if } t \leq K_0 \\ i = 1 & \text{if } t \geq K_0 \end{cases}$$

The Legendre transform of $F$ is defined as a function $F^* : \mathbb{R} \to \mathbb{R}$ with

$$F^*(z) = \sup_{r \in \mathbb{R}} (zr - F(r)) = z(F')^{-1}(z) - F\left((F')^{-1}(z)\right) = F'(r) r + F(r),$$

with the choice $r = (F')^{-1}(z)$. 
Let $\gamma, s \in [0, 1]$ be the exponents appearing in assumption $(K2)$. Define

$$\vartheta_{i, \gamma} = \frac{1}{2s + (N + \gamma)(m_i - 1)}$$

with

$$m_i = \frac{1}{1 - \mu_i} > 1$$

Theorem. (Weighted $L^1 - L^\infty$ smoothing effect) (M.B. & J. L. Vázquez)

As a consequence of $(K2)$ hypothesis, there exists a constant $K_6 > 0$ s.t.

$$F\left(\|u(t)\|_{L^\infty(\Omega)}\right) \leq K_6 \frac{\|u(t_0)\|_{L^1_{\delta\gamma}(\Omega)}^{2sm_i \vartheta_{i, \gamma}}}{t^{m_i(N + \gamma) \vartheta_{i, \gamma}}} , \quad \text{for all } 0 \leq t_0 \leq t ,$$

with $i = 1$ if $t \geq \|u(t_0)\|_{L^1_{\delta\gamma}(\Omega)}^{N + \gamma}$ and $i = 0$ if $t \leq \|u(t_0)\|_{L^1_{\delta\gamma}(\Omega)}^{N + \gamma}$.

- A novelty is that we get instantaneous smoothing effects, new even when $s = 1$.
- The weighted smoothing effect is new even for $s = 1$.
- Corollary. Under the weaker assumption $(K1)$ instead of $(K2)$, the above result holds true with $\gamma = 0$ and replacing $\| \cdot \|_{L^1_{\delta\gamma}(\Omega)}$ with $\| \cdot \|_{L^1(\Omega)}$.
- The time decay is better for small times $0 < t < 1$ than the one given by absolute bounds:

$$(N + \gamma)\vartheta_{i, \gamma} = \frac{N + \gamma}{2s + (N + \gamma)(m_i - 1)} < \frac{1}{m_i - 1}.$$
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for all $0 \leq t_0 \leq t$,

with $i = 1$ if $t \geq \|u(t_0)\|_{L^1_{\delta \gamma}(\Omega)}^{\frac{2s}{N+\gamma}}$ and $i = 0$ if $t \leq \|u(t_0)\|_{L^1_{\delta \gamma}(\Omega)}^{\frac{2s}{N+\gamma}}$.

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for all $0 \leq t_0 \leq t$,

with $i = 1$ if $t \geq \|u(t_0)\|_{L^{1, \vartheta_{i, \gamma}}(\Omega)}^{2s} \frac{1}{N+\gamma}$ and $i = 0$ if $t \leq \|u(t_0)\|_{L^{1, \vartheta_{i, \gamma}}(\Omega)}^{2s} \frac{1}{N+\gamma}$.

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$$\frac{(N + \gamma) \vartheta_{i, \gamma}}{2s + (N + \gamma)(m_i - 1)} < \frac{1}{m_i - 1}.$$
Lower bounds and speed of propagation
Theorem. (Lower absolute and boundary estimates) (M.B. & J. L. Vázquez)

Let let $m > 1$ and let $u \geq 0$ be a weak dual solution to the (CDP), corresponding to the initial datum $0 \leq u_0 \in L^1_{\Phi_1}(\Omega)$. Then, there exist constants $l_0(\Omega), l_1(\Omega) > 0$, so that, setting

$$t^\ast = \frac{l_0(\Omega)}{\left(\int_\Omega u_0 \Phi_1 \, dx\right)^{m-1}},$$

we have that for all $t \geq t^\ast$ and all $x_0 \in \Omega$, the following inequality holds when $0 < \gamma \leq 2sm/(m - 1)$

$$u(t, x_0) \geq l_1(\Omega) \frac{\Phi_1(x_0)^{1/m}}{t^{m-1}}.$$

When $\gamma > 2sm/(m - 1)$ the power of $\Phi_1$ changes to $2s/[(m - 1)\gamma] < 1/m$

The constants $l_0(\Omega), l_1(\Omega) > 0$, depend on $N, m, s$ and on $\Omega$, but not on $u$ (or any norm of $u$); they have an explicit form. Recall that $\Phi_1 \asymp \delta_\gamma$. 
Remarks.

- This boundary behaviour is sharp because we have upper bounds with matching powers of $\Phi_1$.
- $t_*$ is an estimate the time that it takes “to fill the hole”: if $u_0$ is concentrated close to the border (leaves an hole in the middle of $\Omega$), then $\int_{\Omega} u_0 \Phi_1 \, dx$ is small, therefore $t_*$ becomes very large, therefore it takes a lot of time to fill the hole.

When $s = 1$ it is known that the PME has finite speed of propagation. **Question:** Is the speed of propagation finite when $s < 1$?

- These estimates can also be rewritten “à la” Aronson-Caffarelli:

  either $t \leq t_* = \frac{l_0}{(\int_{\Omega} u_0 \Phi_1 \, dx)^{m-1}}$, or $u(t, x_0) \geq l_1 \frac{\Phi_1(x_0)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \quad \forall t \geq t_*$,

  which gives, for all $t \geq 0$ and all $x_0 \in \Omega$:

  $$u(t, x_0) \geq \frac{l_1 \Phi_1(x_0)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \left[ 1 - \left( \frac{t_*}{t} \right)^{\frac{1}{m-1}} \right].$$

- **Open problem:** find precise lower bounds for small times, $0 < t < t_*$.  
- **Solved for RFL, with $s < 1$:** precise lower bounds for small times proven for Restricted-type Fractional Laplacians (on any domain), by MB, A. Figalli and X. Ros-Oton.
Remarks.

- This boundary behaviour is sharp because we have upper bounds with matching powers of $\Phi_1$.
- $t_*$ is an estimate the time that it takes “to fill the hole”: if $u_0$ is concentrated close to the border (leaves an hole in the middle of $\Omega$), then $\int_{\Omega} u_0 \Phi_1 \, dx$ is small, therefore $t_*$ becomes very large, therefore it takes a lot of time to fill the hole.

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- These estimates can also be rewritten “á la” Aronson-Caffarelli:

  
  
  \begin{align*}
  \text{either} \quad t & \leq t_* = \frac{l_0}{(\int_{\Omega} u_0 \Phi_1 \, dx)^{m-1}}, \quad \text{or} \quad u(t, x_0) \geq l_1 \frac{\Phi_1(x_0)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \quad \forall t \geq t_*,
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- **Open problem:** find precise lower bounds for small times, $0 < t < t_*$. 

- **Solved for RFL, with $s < 1$:** precise lower bounds for small times proven for Restricted-type Fractional Laplacians (on any domain), by MB, A. Figalli and X. Ros-Oton.
Remarks.

- This boundary behaviour is sharp because we have upper bounds with matching powers of $\Phi_1$.
- $t_*$ is an estimate the time that it takes “to fill the hole”: if $u_0$ is concentrated close to the border (leaves an hole in the middle of $\Omega$), then $\int_{\Omega} u_0 \Phi_1 \, dx$ is small, therefore $t_*$ becomes very large, therefore it takes a lot of time to fill the hole.

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Harnack Inequalities
Joining our upper and lower bounds we obtain

**Theorem. (Global Harnack Principle) (M.B. & J. L. Vázquez)**

There exist universal constants $H_0, H_1, l_0 > 0$ such that setting

$$t_* = l_0 \left( \int_{\Omega} u_0 \Phi_1 \, dx \right)^{-(m-1)},$$

we have that for all $t \geq t_*$ and all $x \in \Omega$, when $0 < \gamma \leq 2sm/(m-1)$

$$H_0 \frac{\Phi_1(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \leq u(t, x) \leq H_1 \frac{\Phi_1(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}}$$

When $\gamma > 2sm/(m-1)$ the power of $\Phi_1$ changes to $2s/[(m-1)\gamma] < 1/m$

Recall that $\Phi_1 \approx \text{dist}(\cdot, \partial \Omega)^\gamma$, is the first eigenfunction of $\mathcal{L}$.

**Remarks.**

- This inequality implies local Harnack inequalities of elliptic type
- As a corollary we get the sharp asymptotic behaviour
- For $s = 1$ similar results by Aronson and Peletier [JDE, 1981], Vázquez [Monatsh. Math. 2004]
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Solutions $u$ to the parabolic problem inherit the Harnack inequality for $\Phi_1$:

$$\sup_{x \in B_R(x_0)} \Phi_1(x) \leq \mathcal{H} \inf_{x \in B_R(x_0)} \Phi_1(x) \quad \forall B_R(x_0) \in \Omega.$$

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There exist universal constants $H_0, H_1, l_0 > 0$ such that setting $t_\ast = l_0 \|u_0\|^{-\frac{(m-1)}{L_{\Phi_1}^1(\Omega)}}$, we have that for all $t \geq t_\ast$ and all $B_R(x_0) \in \Omega$:

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**Corollary. (Local Harnack Inequalities of Backward Type)**

Under the running assumptions, for all $t \geq t_\ast$ and all $B_R(x_0) \in \Omega$, we have:

$$\sup_{x \in B_R(x_0)} u(t, x) \leq 2 \frac{H_1 \mathcal{H}^{\frac{1}{m}}}{H_0} \inf_{x \in B_R(x_0)} u(t + h, x) \quad \text{for all } 0 \leq h \leq t_\ast.$$

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Harnack Inequalities

and

Higher Regularity for RFL

For the rest of the talk we deal with the special case:

\[ \mathcal{L}(u)(x) = (-\Delta|_{\Omega})^s u(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy \]
For the RFL we solve the problem of sharp lower bounds for small times. Recall that here $\gamma = s$ and $\Phi_1 \simeq \delta_\gamma = \text{dist}(\cdot, \partial \Omega)^s$.

**Theorem. (Global quantitative positivity) (M.B., A. Figalli, X. Ros-Oton)**

Let $m > 1$, $0 < s < 1$, and $N > 2s$. Let $\Omega$ be a bounded domain of class $C^{1,1}$, and let $u$ be a weak dual solution to the (CDP) corresponding to $0 \leq u_0 \in L^1_{\Phi_1}(\Omega)$. Then the following bound holds true:

$$u(t, x) \geq \kappa \|u_0\|_{L^1_{\Phi_1}(\Omega)}^m t^{\Phi_1(x)} \left( \frac{1}{m} \right)^{\frac{1}{m}},$$

for all $0 \leq t \leq t_*$ and all $x \in \overline{\Omega}$,

where $t_* = l_0 \|u_0\|_{L^1_{\Phi_1}(\Omega)}^{-(m-1)}$ and $l_0, \kappa > 0$ depend only on $N, s, m, \Omega$.

As a consequence, solutions to the (CDP) corresponding to nonnegative and nontrivial initial data, have **infinite speed of propagation**.

- No free boundaries when $s < 1$, contrary to the “local” case $s = 1$, cf. Barenblatt, Aronson, Caffarelli, Vázquez, Wolansky [...]
- Different from the so-called Caffarelli-Vázquez model (on $\mathbb{R}^N$) that has finite speed of propagation [ARMA 2011, DCDS 2011] and also Stan, del Teso Vázquez [CRAS 2014, NLTMA 2015, JDE 2015], cf. also Coxeter lecture by Caffarelli yesterday :)
Theorem. (Global Harnack Principle for all times) (M.B., A. Figalli, X. Ros-Oton)

Let $m > 1$, $0 < s < 1$, and $N > 2s$. Let $\Omega$ be a bounded domain of class $C^{1,1}$, and let $u$ be a weak dual solution to the (CDP) corresponding to $0 \leq u_0 \in L^1_{\Phi_1}(\Omega)$. Let $t_*$ be as above. Then for all $t > 0$ and all $x \in \overline{\Omega}$

$$\kappa \left(1 \wedge \frac{t}{t_*}\right)^{-\frac{1}{m-1}} \Phi_1(x)^{-\frac{1}{m}} \leq u(t,x) \leq \overline{\kappa} \Phi_1(x)^{-\frac{1}{m}} \left(1 \wedge \frac{t}{t_*}\right)^{-\frac{1}{m-1}},$$

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Theorem. (Local Harnack inequalities for all times) (M.B., A. Figalli, X. Ros-Oton)

Under the above assumptions, for all balls $B_R(x_0) \subset \subset \Omega$, we have

$$\sup_{x \in B_R(x_0)} u(t,x) \leq \frac{\mathcal{H}}{(1 \wedge \frac{t}{t_*})^{\frac{m}{m-1}}} \inf_{x \in B_R(x_0)} u(t,x), \quad \text{for all } t > 0,$$

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Harnack inequalities for all times
Hölder Regularity up to the boundary.
The following regularity results hold true under the running assumptions:

(R) Let $m > 1$, $0 < s < 1$, and $N > 2s$. Let $\Omega$ be a bounded domain of class $C^{1,1}$, and let $u$ be a solution to the (CDP) corresponding to a nonnegative initial datum $u_0 \in L^1_{\Phi}(\Omega)$.

**Theorem. (Hölder regularity up to the boundary)** (M.B., A. Figalli, X. Ros-Oton)

Under the running assumptions (R), then, for each $0 < t_0 < T$ we have

$$\|u\|_{C^{\frac{s}{m}}_{x,t}, \frac{1}{2m}\sqrt{\Omega \times [t_0,T]}} \leq C,$$

where $C$ depends only on $N, s, m, \Omega, t_0$, and $\|u_0\|_{L^1_{\Phi}(\Omega)}$.

**Remarks.**

- Notice that the $C^{s/m}_x$ regularity up to the boundary is optimal, since we have that $u(t, x) \geq c(u_0, t) \text{dist}(x, \partial \Omega)^{s/m}$, with $c(u_0, t) > 0$ for all $t > 0$, and therefore $u(t, \cdot) \notin C^{\frac{s}{m} + \epsilon}_x(\overline{\Omega})$ for any $\epsilon > 0$.
- Previous result on $C^\alpha$ regularity by Athanasopoulos and Caffarelli [Adv. Math, 2010].
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Higher Regularity. Under the running assumptions (R), we prove interior $C^\infty$ regularity in the $x$-variable and interior $C^{1,\alpha}$ regularity in the $t$-variable.

**Theorem. (Higher interior regularity in space)** (M.B., A. Figalli, X. Ros-Oton)

Under the running assumptions (R), then $u \in C^\infty_x((0, \infty) \times \Omega)$. More precisely, let $k \geq 1$ be any positive integer, and $d(x) = \text{dist}(x, \partial\Omega)$, then, for any $t \geq t_0 > 0$ we have

$$\left| D_x^k u(t, x) \right| \leq C \left[ d(x) \right]^{\frac{s}{m} - k},$$

where $C$ depends only on $N, s, m, k, \Omega, t_0$, and $\|u_0\|_{L^1_{\Phi_1}(\Omega)}$.

**Theorem. (C$^{1,\alpha}$ interior regularity in time)** (M.B., A. Figalli, X. Ros-Oton)

Under the running assumptions (R), then $u \in C^{1,\alpha}_t((0, \infty) \times \Omega)$ for some $\alpha > 0$ that depends only on $s$ and $m$. Moreover, for any compact set $K \subset \subset \Omega$, and any $0 < t_0 < T$, we have

$$\|u\|_{C^{1,\alpha}_t([t_0, T] \times K)} \leq C,$$

where $C$ depends only on $N, s, m, \Omega, t_0, \|u_0\|_{L^1_{\Phi_1}(\Omega)}$, and $K$. 
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Under the running assumptions \((R)\), then \(u \in C^\infty_x((0, \infty) \times \Omega)\). More precisely, let \(k \geq 1\) be any positive integer, and \(d(x) = \text{dist}(x, \partial \Omega)\), then, for any \(t \geq t_0 > 0\) we have

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\left| D_x^k u(t, x) \right| \leq C \left[ d(x) \right]^{s \frac{k}{m} - k},
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Remarks.

- A possible value for the exponent $\alpha$ in the previous theorem on time regularity is $\alpha = \min \{ \frac{1}{2m}, 1 - s \}$.
- Notice that the above regularity results imply that solutions to (CDP) are classical for any nonnegative initial datum $u_0 \in L^1_{\Phi_1}(\Omega)$.
- Higher regularity in time is a difficult open problem. It is connected to higher order boundary regularity in $t$. To our knowledge also open for the local case $s = 1$.
- Even for the Fractional Heat Equation (FHE) $u_t + (-\Delta|_\Omega)^s u = 0$ on $(0, 1) \times B_1$ we have that $u \in C^\infty$ in $x$, namely
  \[
  \|u\|_{C^{k,\alpha}_x((\frac{1}{2}, 1) \times B_{1/2})} \leq C \|u\|_{L^\infty((0,1) \times \mathbb{R}^N)}, \quad \text{for all } k \geq 0.
  \]
  Analogous estimates in time do not hold for $k \geq 1$ and $\alpha \in (0, 1)$.
  Indeed, one can construct a solution to the (FHE) which is bounded in all of $\mathbb{R}^N$, but which is not $C^1$ in $t$ in $(\frac{1}{2}, 1) \times B_{1/2}$.
  [H. Chang-Lara, G. Davila, JDE (2014)]
- Our techniques allow to prove regularity also in unbounded domains, and also to treat operator with more general kernels.
- Also the “classical/local” case $s = 1$ works after the waiting time $t_*$:
  \[
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$$\|u\|_{C^{k,\alpha}_x((1/2,1)\times B_{1/2})} \leq C\|u\|_{L^\infty((0,1)\times \mathbb{R}^N)}, \quad \text{for all } k \geq 0.$$ 

Analogous estimates in time do not hold for $k \geq 1$ and $\alpha \in (0, 1)$. Indeed, one can construct a solution to the (FHE) which is bounded in all of $\mathbb{R}^N$, but which is not $C^1$ in $t$ in $(1/2, 1) \times B_{1/2}$.
[H. Chang-Lara, G. Davila, JDE (2014)]
- Our techniques allow to prove regularity also in unbounded domains, and also to treat operator with more general kernels.
- Also the “classical/local” case $s = 1$ works after the waiting time $t_*$: $u \in C^\frac{1}{m}, \frac{1}{2m} \left( \Omega \times [t_*, T] \right)$, $C^\infty_x((0, \infty) \times \Omega)$ and $C^1_{t;\alpha}([t_0, T] \times K)$. 
Remarks.

- A possible value for the exponent $\alpha$ in the previous theorem on time regularity is $\alpha = \min \left\{ \frac{1}{2m}, 1 - s \right\}$.

- Notice that the above regularity results imply that solutions to (CDP) are classical for any nonnegative initial datum $u_0 \in L^{1}_{\Phi_1}(\Omega)$.

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The End

Thank You!!!

Merci Beaucoup!!!

Muchas Gracias!!!
Asymptotic behaviour of nonnegative solutions

- Convergence to the stationary profile
- Convergence with optimal rate
In the rest of the talk we consider the nonlinearity $F(u) = |u|^{m-1}u$ with $m > 1$.

**Theorem. (Asymptotic behaviour)  (M.B., Y. Sire, J. L. Vázquez)**

There exists a unique nonnegative selfsimilar solution of the above Dirichlet Problem

$$U(\tau, x) = \frac{S(x)}{\tau^{\frac{1}{m-1}}} ,$$

for some bounded function $S: \Omega \to \mathbb{R}$. Let $u$ be any nonnegative weak dual solution to the (CDP), then we have (unless $u \equiv 0$)

$$\lim_{\tau \to \infty} \tau^{\frac{1}{m-1}} \|u(\tau, \cdot) - U(\tau, \cdot)\|_{L^\infty(\Omega)} = 0 .$$

The previous theorem admits the following corollary.

**Theorem. (Elliptic problem)  (M.B., Y. Sire, J. L. Vázquez)**

Let $m > 1$. There exists a unique weak dual solution to the elliptic problem

$$\left\{ \begin{array}{l}
\mathcal{L}(S^m) = \frac{S}{m-1} \quad \text{in } \Omega, \\
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Notice that the previous theorem is obtained in the present paper through a parabolic technique.
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Theorem. (Sharp asymptotic with rates) (M.B., Y. Sire, J. L. Vázquez)

Let $u$ be any nonnegative weak dual solution to the (CDP), then we have (unless $u \equiv 0$) that there exist $t_0 > 0$ of the form

$$t_0 = \bar{k} \left[ \frac{\int_{\Omega} \Phi_1 \, dx}{\int_{\Omega} u_0 \Phi_1 \, dx} \right]^{m-1}$$

such that for all $t \geq t_0$ we have

$$\left\| \frac{u(t, \cdot)}{U(t, \cdot)} - 1 \right\|_{L^\infty(\Omega)} \leq \frac{2}{m-1} \frac{t_0}{t_0 + t}.$$ 

The constant $\bar{k} > 0$ only depends on $m, N, s,$ and $|\Omega|.$

Remarks.

- We provide two different proofs of the above result.
- One proof is based on the construction of the so-called Friendly-Giant solution, namely the solution with initial data $u_0 = +\infty,$ and is based on the Global Harnack Principle of Part 4.
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