1. Study Hölder regularity of the functions for all $\alpha > 0$

$$f_\alpha(x) = \begin{cases} 
  x^\alpha \sin(1/x), & 0 < x \leq 1, \\
  0, & x = 0.
\end{cases}$$

**Solution.** A simple calculation shows that when $\alpha \geq 2$ we have $f_\alpha \in C^1([0,1])$, in particular $f_\alpha$ is Lipschitz. Therefore we consider the case $\alpha \in (0,2)$.

Let $x_n = \frac{1}{(1+n^2)\pi}$, $y_n = \frac{1}{n \pi}$, so that

$$\frac{|f(x_n) - f(y_n)|}{|x_n - y_n|^\gamma} = 2^{\gamma - \alpha} n^{2\gamma - \alpha} \left( 1 + \frac{1}{2n} \right)^{\gamma - \alpha} \to \infty \quad \text{cualquier n} \to \infty \quad \text{si } \gamma > \alpha/2.$$ 

As a consequence, the Hölder exponent has to be at most $\alpha/2$. Let us check that it is exactly $\alpha/2$: consider the function

$$\phi(x) = \left( x^\alpha \sin \frac{1}{x} - a^\alpha \sin \frac{1}{a} \right)^{2/\alpha} = \phi(x) - \phi(a)$$

$$= \frac{2}{\alpha} \left( \xi^\alpha \sin \frac{1}{\xi} - a^\alpha \sin \frac{1}{a} \right)^{2-1} \left( \xi^{\alpha-1} \sin \frac{1}{\xi} - \xi^{-1} \sin \frac{1}{a} \right) (x-a),$$

where we have used the Mean Value Theorem with $0 \leq a < \xi < x \leq 1$. As a consequence,

$$\frac{\phi(x)}{x-a} = \frac{2}{\alpha} \left( \xi^{\alpha-}\sin \frac{1}{\xi} \sin \frac{1}{a} - \xi^{\alpha} \sin \frac{1}{\xi} \right) \leq C,$$

the result follows.

2. Let $\alpha \in (0,1)$ and consider the function

$$u(x) = (1 + x^2)^{-\alpha/2} (\log(2 + x^2))^{-1}, \quad x \in \mathbb{R}.$$

Show that $u \in W^{1,p}(\mathbb{R})$ for any $p \in [1/\alpha, \infty]$, and that $u \not\in L^q(\mathbb{R})$ when $q \in [1,1/\alpha)$.

**Solution.** We will use the following statements, whose proof is left as an (easy) exercise

$$\int_1^\infty \frac{dx}{x^\alpha} < \infty \iff \alpha > 1, \quad \int_2^\infty \frac{dx}{x \log^{\beta} x} < \infty \iff \beta > 1.$$ 

On one hand, the function $u$ is clearly bounded for any $\alpha \in (0,1)$, $u \in L^\infty(\mathbb{R})$.

On the other hand, if $p \in (1/\alpha, \infty)$,

$$\int_\mathbb{R} |u|^p = 2 \int_0^1 |u|^p + 2 \int_1^\infty |u|^p = C + 2 \int_1^\infty \frac{dx}{(1 + x^2)^{\alpha p/2} (\log(2 + x^2))} \leq C + \frac{2}{\log^p} \int_1^\infty \frac{dx}{x^{\alpha p}} < \infty.$$ 

The critical case, $p = 1/\alpha$,

$$\int_\mathbb{R} |u|^p = C + 2 \int_2^\infty \frac{dx}{(1 + x^2)^{1/2} (\log(2 + x^2))^{1/\alpha}} \leq C + \frac{2}{2^{1/\alpha}} \int_2^\infty \frac{dx}{x \log^{1/\alpha} x} < \infty.$$ 

Finally, the derivatives

$$u'(x) = -\alpha (1 + x^2)^{-\alpha/2 - 1} x (\log(2 + x^2))^{-1} - (1 + x^2)^{-\alpha/2} (\log(2 + x^2))^{-2} \frac{2x}{2 + x^2}.$$
has a decay at infinity which is faster than the function $u$, therefore it lies (at least) in the same $L^p(\mathbb{R})$ space as $u$. We can conclude that $u \in W^{1,p}(\mathbb{R})$, $p \in [1/(1/\alpha), \infty]$. If $q \in [1, 1/\alpha)$, taking $\varepsilon > 0$ so that $\alpha q + \varepsilon < 1$ (we can do it since $\alpha q < 1$),

$$
\int_{\mathbb{R}} |u|^q = C + 2 \int_{2}^{\infty} \frac{dx}{(1+x^2)^{\alpha q/2}\log(2+x^2)} \geq C + C \int_{2}^{\infty} \frac{dx}{x^{\alpha q + \varepsilon}} = \infty.
$$

3. Let $\Omega = \{x \in \mathbb{R}^2 : |x_1| < 1, |x_2| < 1\}$ and

$$
u(x) = \begin{cases}
1 - x_1 & \text{if } x_1 > 0, |x_2| < x_1, \\
1 + x_1 & \text{if } x_1 < 0, |x_2| < -x_1, \\
1 - x_2 & \text{if } x_2 > 0, |x_1| < x_2, \\
1 + x_2 & \text{if } x_2 < 0, |x_1| < -x_2.
\end{cases}
$$

Find the values of $p$, $1 \leq p \leq \infty$, such that $u \in W^{1,p}(\Omega)$.

**Solution #1.** It is trivial to check that $u \in L^\infty(\Omega)$, with $\|u\|_{L^\infty(\Omega)} = 1$. Since $\Omega$ is a bounded domain, then $u \in L^p(\Omega)$, for all $1 \leq p \leq \infty$.

Given $\phi \in C^\infty_c(\Omega)$ we have that

$$
\int_{\Omega} u \partial_{x_1} \phi = \sum_{j=1}^{4} \int_{T_j} u \partial_{x_1} \phi = -\sum_{j=1}^{4} \int_{T_j} \phi \partial_{x_1} u + \sum_{j=1}^{4} \int_{\partial T_j} u \phi e_1 \cdot \nu^j,
$$

where $\nu^j$ is the unit exterior normal to $T_j$ in $\partial T_j$. On one hand we get

$$
-\sum_{j=1}^{4} \int_{T_j} \phi \partial_{x_1} u = -\int_{\Omega} \phi (\chi_{T_1} + \chi_{T_2}).
$$

On the other hand, since $\phi$ is compactly supported in $\Omega$, and observing that $T_i$ and $T_j$ have a common side, it follows that $\nu^i = -\nu^j$ on that common side,

$$
\sum_{j=1}^{4} \int_{\partial T_j} u \phi e_1 \cdot \nu^j = 0.
$$

We conclude

$$
\int_{\Omega} u \partial_{x_1} \phi = -\int_{\Omega} \phi (\chi_{T_1} + \chi_{T_2}),
$$

that is

$$
\partial_{x_1} u = -\chi_{T_1} + \chi_{T_2}
$$

in the distributional sense. We notice that $\partial_{x_1} u \in L^\infty(\Omega)$. As a consequence, since the domain is bounded, $\partial_{x_1} u \in L^p(\Omega)$, $1 \leq p \leq \infty$. The same holds for $\partial_{x_2} u$ (simply by switching $x_1$ and $x_2$), we conclude that $u \in W^{1,p}(\Omega)$ for all $1 \leq p \leq \infty$.

**Solution #2.** It is easy to check that $\min\{f, g\} = -\{f - g\}_+ + f$. We know that $h \in W^{1,p}(\Omega)$, therefore $\{h\}_+ \in W^{1,p}(\Omega)$ (cf. Problem 11), and we can conclude that if $f, g \in W^{1,p}(\Omega)$, then $\min\{f, g\} \in W^{1,p}(\Omega)$.

The function $u$ satisfies

$$
u(x_1, x_2) = \min\{1 - x_1, 1 + x_1, 1 - x_2, 1 + x_2\}.
$$

Being the minimum of $W^{1,\infty}(\Omega)$ functions, it lies in the same space, hence in all $W^{1,p}(\Omega)$, with $1 \leq p \leq \infty$, since $\Omega$ is bounded.

**Remark.** The same holds true also for $\max\{f, g\} = \{f - g\}_+ + g$: indeed, if $f, g \in W^{1,p}(\Omega)$, then $\max\{f, g\} \in W^{1,p}(\Omega)$.
4. Let $N > 1$. Check that the unbounded function $u(x) = \log \log \left(1 + \frac{1}{|x|}\right)$ lies in $W^{1,n}(B_1(0))$.

**Solution.** Change to polar coordinates:

$$\int_{B_1(0)} |u|^n = C \int_0^1 r^{n-1} |\log \log(1 + \frac{1}{r})|^n dr < \infty \quad \text{si } n \geq 2,$$

the function under integral has a continuous extension on the whole interval $[0, 1]$, since its limit as $r \to 0^+$ is 0).

A simple calculation shows that

$$\partial_{x_i} u(x) = -\frac{x_i}{(|x|^3 + |x|^2) \log(1 + \frac{1}{|x|^2})} \quad \text{si } x \neq 0.$$ 

Change again to polar coordinates

$$\int_{B_1(0)} \left|\frac{x_i}{(|x|^3 + |x|^2) \log(1 + \frac{1}{|x|^2})}\right|^n dx \leq C \int_0^1 \frac{r^{n-1}}{(r^2 + r) \log(1 + \frac{1}{r})} dr \leq C - \int_0^{1/2} \frac{dr}{r \log^n r} < \infty,$$

if $N \geq 2$.

Let $T_k u(x) = \min\{u(x), k\}$. For each constant $k \geq 0$ this function is in $W^{1,n}(B_1(0))$, its weak derivatives are

$$\partial_{x_i} T_k u(x) = -\chi_{\{u<k\}} \frac{x_i}{(|x|^3 + |x|^2) \log(1 + \frac{1}{|x|^2})},$$

which are functions belonging to $L^n(B_1(0))$ uniformly in $k$, and also to $L^1(B_1(0))$. By Dominated convergence, the limit as $k \to \infty$ becomes

$$\int_{B_1(0)} T_k u \partial_{x_i} \phi = -\int_{B_1(0)} \phi \partial_{x_i} T_k u$$

from which we deduce

$$\partial_{x_i} u(x) = -\frac{x_i}{(|x|^3 + |x|^2) \log(1 + \frac{1}{|x|^2})} \quad \text{en } D'(B_1(0)),$$

which concludes the proof.

5. Let $\Omega \subseteq \mathbb{R}^N$ be open and connected and $u \in W^{1,p}(\Omega)$. Show that if $Du = 0$ a.e. in $\Omega$, then $u$ is constant a.e. in $\Omega$.

**Solution.** For any $\varepsilon > 0$ consider the regularization $u^\varepsilon = \eta_\varepsilon \ast u$, and we know that $u^\varepsilon : \Omega \to \mathbb{R} \in C^\infty(\Omega_\varepsilon)$. Its first order derivatives, $\partial^\alpha u^\varepsilon = \eta_\varepsilon \ast \partial^\alpha u$, $|\alpha| = 1$, are also zero on $\Omega_\varepsilon$. As a consequence $u$ is constant on each connected component of $\Omega_\varepsilon$.

Let $x, y \in \Omega$. Since $\Omega$ open and connected, there is a continuous path $\Gamma \subset \Omega$ joining $x$ and $y$. Let $\delta = \min_{z \in \Gamma} \text{dist}(z, \partial \Omega)$. For all $\varepsilon < \delta$ the whole path $\Gamma$ lies in $\Omega_\varepsilon$, hence $x$ and $y$ lie in the same connected component of $\Omega_\varepsilon$. Therefore, $u^\varepsilon(x) = u^\varepsilon(y)$.

Let $\tilde{u}(x) = \lim_{\varepsilon \to 0} u^\varepsilon(x)$. As a consequence of the above results, $\tilde{u}$ is constant in $\Omega$. We also know that $\tilde{u}(x) = u(x)$ a.e. in $\Omega$, and the proof is concluded.

6. (Fundamental Theorem of Calculus) Let $I \subset \mathbb{R}$ an interval (not necessarily bounded). Let $g \in L^1_{\text{loc}}(I)$. For any fixed $y_0 \in I$ we define

$$v(x) = \int_{y_0}^x g(t) dt, \quad x \in I.$$
Prove that \( v \in C(I) \) and that
\[
\int_I v \phi' = -\int_I g \phi \quad \text{for any } \phi \in C^1_c(I).
\]

**Solution.** We have that
\[
\int_I v \phi' = \int_I \left( \int_{y_0}^x g(t) dt \right) \phi'(x) = -\int_a^{y_0} \left( \int_x^{y_0} g(t) \phi'(x) dt \right) dx + \int_{y_0}^b \left( \int_x^{y_0} g(t) \phi'(x) dt \right) dx.
\]
By Fubini’s Theorem,
\[
\int_I v \phi' = -\int_a^{y_0} g(t) \left( \int_{a}^{t} \phi'(x) dx \right) dt + \int_{y_0}^b g(t) \left( \int_{t}^{b} \phi'(x) dx \right) dt
= -\int_I g(t) \phi(t) dt.
\]

7. Let \( I \subset \mathbb{R} \) an interval (not necessarily bounded). Let \( u \in W^{1,p}(I) \), \( 1 \leq p \leq \infty \). Prove that there exists a function \( \tilde{u} \in C(\overline{I}) \) such that \( u = \tilde{u} \) a.e. in \( I \), and that moreover we have
\[
\tilde{u}(x) - \tilde{u}(y) = \int_x^y u'(t) dt \quad \text{para todo } x, y \in \overline{I}.
\]

**Hint.** Use the two previous exercises

**Solution.** Fix \( y_0 \in I \) and let \( \bar{u}(x) = \int_{y_0}^x u'(t) dt \). Thanks to the previous exercise, we have
\[
\int_I \bar{u} \phi' = -\int_I u' \phi \quad \forall \phi \in C^1_c(I).
\]
As a consequence, \( \int_I (u - \bar{u}) \phi' = 0 \) for all \( \phi \in C^1_c(I) \). Thanks to Problem 5, \( u - \bar{u} = C \) a.e. in \( I \). The function \( \tilde{u} = \bar{u} + C \) has the required properties.

8. Let \( u, v \in H^1(\mathbb{R}) \). Show that
\[
\int_{\mathbb{R}} uv' = -\int_{\mathbb{R}} u'v.
\]

**Solution.** If \( u \in H^1(\mathbb{R}) \) and \( v \in C^\infty_c(\mathbb{R}) \), the identity is nothing but the definition of distributional derivative of \( u \). For the general case, \( v \in H^1(\mathbb{R}) \), let us take a sequence \( \{v_n\} \subset C^\infty_c(\mathbb{R}) \) so that \( v_n \to v \) en \( H^1(\mathbb{R}) \). We obtain the result just by taking the limits in
\[
\int_{\mathbb{R}} uv_n' = -\int_{\mathbb{R}} u'v_n.
\]

**Remark.** The very same proof works in ANY dimension \( N \geq 1 \).
9. (Leibnitz rule in Sobolev Spaces) Let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Show that $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and that

$$
\partial_x_i(uv) = v\partial_x_i u + u\partial_x_i v, \quad i = 1, \ldots, n.
$$

**Solution.** Let $\{u_n\}, \{v_k\} \subset C^\infty_c(\Omega)$ such that $u_n \to u$, $v_k \to v$ en $W^{1,p}(\Omega)$, $\|u_n\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$, $\|v_k\|_{L^\infty(\Omega)} \leq \|v\|_{L^\infty(\Omega)}$. We immediately get

$$
-\int_\Omega u_nv_k\partial_x_i\phi = \int_\Omega \partial_x_i(u_nv_k)\phi = \int_\Omega (v_k\partial_x_i u_n + u_n\partial_x_i v_k)\phi.
$$

Taking the limits, first in $n$ then in $k$ at the first and last terms of the above inequality, we obtain

$$
-\int_\Omega uv\partial_x_i\phi = \int_\Omega (v\partial_x_i u + u\partial_x_i v)\phi.
$$

This means that we satisfy Leibnitz rule for the derivative of a product, in the distributional sense. We then take the limit, recalling that the product of a bounded function with a function of $C^\infty_c(\Omega)$ lies $L^p$.

Once we have checked the identity in the distributional sense, we conclude by recalling that the product of a bounded function (in $L^\infty$) with a function of $L^p$ is still in $L^p$.

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10. (Chain Rule) Let $F : \mathbb{R} \to \mathbb{R}$ a $C^1$ function with bounded $F'$ and $F(0) = 0$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set. Let $u \in W^{1,p}(\Omega)$ for some $p$, $1 \leq p \leq \infty$. Show that $v = F(u)$ lies in $W^{1,p}(\Omega)$ and that $v_{x_i} = F'(u)u_{x_i}$, $i = 1, \ldots, n$.

**Solution.** Given $\phi \in C^\infty_c(\Omega)$, there is a sequence $\{u_n\} \subset C^\infty(\Omega)$ so that $u_n \to u$ in $W^{1,p}(\text{sop } \phi)$ and $u_n \to u$ a.e. in $\Omega$. We then have

$$
-\int_\Omega F(u_n)\partial_x_i \phi = \int_\Omega \phi F'(u_n)\partial_x_i u_n.
$$

Moreover,

$$
\left| \int_\Omega (F(u_n) - F(u))\partial_x_i \phi \, dx \right| \leq \|\partial_x_i \phi\|_{\infty} \sup_{\text{sop } \phi} |F'| \int_\Omega |u_n - u| \, dx \to 0 \quad \text{cuando } n \to \infty.
$$

We also have

$$
\left| \int_\Omega (F'(u_n)\partial_x_i u_n - F'(u)\partial_x_i u) \phi \, dx \right| \\
\leq \|\phi\|_{\infty} \sup_{\text{sop } \phi} |F'| \int_\Omega |\partial_x_i u_n - \partial_x_i u| \, dx + \int_\Omega |F'(u_n) - F'(u)||Du| \, dx \to 0 \quad \text{cuando } n \to \infty.
$$

We have used Dominated Convergence together with the pointwise convergence of $|F'(u_n) - F'(u)|$ to $0$, in order to prove the convergence of the second term in the right-hand side. Take the limit in (1),

to get

$$
-\int_\Omega F(u)\partial_x_i \phi = \int_\Omega \phi F'(u)\partial_x_i u,
$$

which is equivalent to $v_{x_i} = F'(u)u_{x_i}$. Under our assumptions on $F$ and $u$, we know that the right-hand side is in $L^p(\Omega)$, therefore also $v_{x_i} \in L^p(\Omega)$.

Finally,

$$
\int_\Omega |v|^p = \int_\Omega |F(u) - F(0)|^p \leq (\sup |F'|)^p \int_\Omega |u|^p < \infty,
$$

which gives the result.
11. Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain, and let $1 \leq p \leq \infty$.

(a) Prove that $u \in W^{1,p}(\Omega)$, implies $|u| \in W^{1,p}(\Omega)$.

(b) Prove that $u \in W^{1,p}(\Omega)$ implies $u^+, u^- \in W^{1,p}(\Omega)$, with

$$Du^+ = \begin{cases} Du & \text{a.e. in } \{u > 0\}, \\ 0 & \text{a.e. in } \{u \leq 0\}, \\ -Du & \text{a.e. in } \{u < 0\}, \end{cases}$$

$$Du^- = \begin{cases} 0 & \text{a.e. in } \{u \geq 0\}, \\ -Du & \text{a.e. in } \{u < 0\}. \end{cases}$$

Hint. $u^+ = \lim_{\varepsilon \to 0} F_\varepsilon(u)$, where

$$F_\varepsilon(z) = \begin{cases} (z^2 + \varepsilon^2)^{1/2} - \varepsilon & \text{if } z \geq 0, \\ 0 & \text{if } z < 0. \end{cases}$$

(c) Prove that if $u \in W^{1,p}(\Omega)$, then $Du = 0$ a.e. on the set $\{u = 0\}$.

**Solution.** It is sufficient to prove part (b). Parts (a) and (c) follow immediately, since $|u| = u^+ + u^-$ and $u = u^+ - u^-$. Let us show part (b). It is sufficient to prove it for $u^+$, since $u^- = (-u)^+$. Following the hint, we use the Chain Rule of Exercise 10, with $\phi \in C_c^\infty(\Omega)$

$$\int_\Omega F_\varepsilon(u) \partial_x \phi \, dx = - \int_{\{u > 0\}} \phi \frac{u \partial_x u}{(u^2 + \varepsilon^2)^{1/2}} \, dx.$$

Letting $\varepsilon \to 0$ and using Dominated Convergence, we get

$$\int_\Omega u^+ \partial_x \phi \, dx = - \int_{\{u > 0\}} \phi \partial_x u \, dx.$$

This concludes the proof.

12. Let $\Omega \subset \mathbb{R}^N$ an open set with $C^1$ boundary. Show by means of an example that $L^p(\Omega)$ functions, with $p \in [1, \infty)$, do not necessarily have a trace on $\partial \Omega$. More precisely, show that there can not exist a linear bounded operator $T : L^p(\Omega) \to L^p(\partial \Omega)$ such that $Tu = u|_{\partial \Omega}$ for all $u \in C(\bar{\Omega}) \cap L^p(\Omega)$.

**Solution.** Let us show a counterexample in dimension $N = 1$. We want to show that there there does not exists a constant $C > 0$ such that $\|Tu\|_{L^p(\partial \Omega)} \leq C\|u\|_{L^p(\Omega)}$ for all $u \in C(\bar{\Omega}) \cap L^p(\Omega)$. Assume by contradiction that this holds true. Choose a family of continuous functions on $\Omega = (0, 1)$ given by

$$f_n(x) = n^{\alpha+1} \left\{ \frac{1}{n^\alpha} - x \right\}_+.$$

We have that

$$\int_0^1 |f_n|^p \leq n^{p-\alpha} = 1 \quad \text{if } \alpha = p.$$

But we also have

$$\|Tf_n\|_{L^p(\partial \Omega)}^p = |f_n(0)|^p + |f_n(1)|^p = n^p,$$

which clearly contradicts the hypothesis.

Analogous counterexamples can be constructed in any dimension $N > 1$. 
13. (a) Show that there does not exists any constant \( C > 0 \) such that
\[
\int_{\mathbb{R}^N} u^2 \leq C \int_{\mathbb{R}^N} |\nabla u|^2 \quad \text{for all} \quad u \in H^1(\mathbb{R}^N).
\]

(b) (Hardy Inequality) For all \( N \geq 3 \) there exists \( C > 0 \) such that
\[
\int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \, dx \leq C \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \quad \text{for all} \quad u \in H^1(\mathbb{R}^N).
\]

Hint. \( |\nabla u + \lambda \frac{x}{|x|^2} u|^2 \geq 0 \) for all \( \lambda \in \mathbb{R} \).

Solution. (a) Let \( \zeta \in C^\infty(\mathbb{R}^N) \), be so that \( \zeta \geq 0 \), \( \zeta(x) = 1 \) if \( |x| \leq 1 \), \( \zeta(x) = 0 \) if \( |x| \geq 2 \). Define \( \zeta_k(x) = \zeta(x/k) \). If there would exist \( C > 0 \) for all functions of \( H^1(\mathbb{R}^N) \), we shall have
\[
\int_{\mathbb{R}^N} \zeta_k^2(x/k) \, dx \leq \frac{C}{k^2} \int_{\mathbb{R}^N} |\nabla \zeta_k|^2(x/k) \, dx \quad \text{for all} \quad k.
\]
Changing variables \( x = ky \),
\[
\int_{\mathbb{R}^N} \zeta_k^2 \leq \frac{C}{k^2} \int_{\mathbb{R}^N} |\nabla \zeta|^2 \quad \text{for all} \quad k.
\]
We can let \( k \to \infty \) to get a contradiction.

(b) Follow the hint and expand the square:
\[
0 \leq \int_{\mathbb{R}^N} \left| \nabla u + \frac{\lambda x}{|x|^2} u \right|^2 \, dx = \int_{\mathbb{R}^N} \left( |\nabla u|^2 + \frac{\lambda x \cdot \nabla (u^2)}{|x|^2} + \lambda^2 \frac{u^2}{|x|^2} \right) \, dx.
\]
Recalling that \( \nabla \cdot \left( \frac{x}{|x|^2} \right) = \frac{N-2}{|x|^2} \), we obtain
\[
0 \leq \int_{\mathbb{R}^N} |\nabla u|^2 - \lambda(N-2) \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \, dx + \lambda^2 \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \, dx.
\]
The (positive) minimum of the quadratic polynomial in \( \lambda \) is attained at \( \lambda = (N-2)/2 \). Substitute this value in the above inequality, gives the result with \( C = 4/(N-2)^2 \).

14. Let \( \alpha > 0 \). Show that there exists \( C = C(N, \alpha) > 0 \) so that
\[
\int_{B_1(0)} u^2 \leq C \int_{B_1(0)} |\nabla u|^2
\]
for all \( u \in H^1(B_1(0)) \) such that \( |\{ x \in B_1(0) : u(x) = 0 \} | \geq \alpha \).

Solution. Let \( B = B_1(0) \) and \( A = \{ x \in B : u(x) = 0 \} \). Using Poincaré inequality, we know that there exists \( C > 0 \) so that
\[
C ||\nabla u||_{L^2(B)} \geq \left| u - \frac{1}{|B|} \int_B u \right|_{L^2(B)} \geq \left| ||u||_{L^2(B)} - \frac{1}{|B|} \int_B u \right|_{L^2(B)}.
\]
By Hölder inequality,
\[
\left| \frac{1}{|B|} \int_B u \right|^2_{L^2(B)} = \frac{1}{|B|^2} \left( \int_{B \setminus A} u \right)^2 \leq \frac{|B \setminus A|}{|B|} ||u||^2_{L^2(B)}.
\]
As a consequence,
\[
C ||\nabla u||_{L^2(B)} \geq ||u||_{L^2(B)} \left( 1 - \left( \frac{|B| - \alpha}{|B|} \right)^{1/2} \right).
\]
The result follows, since \( 1 - \left( \frac{|B| - \alpha}{|B|} \right)^{1/2} > 0 \).
15. (Friedrichs’ Inequality) Let \( \Omega \subset \mathbb{R}^N \) be an open connected domain, with smooth boundary and let \( \Gamma \subset \partial \Omega \) a set with positive \((N-1)\)-dimensional measure. Show that there exists a constant \( C > 0 \) so that
\[
\|u\|_{H^1(\Omega)}^2 \leq C \left( \|u\|_{L^2(\Gamma)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 \right) \quad \forall u \in H^1(\Omega).
\]

**Solution.** We proceed by contradiction. Suppose that the inequality is false. Therefore, for all \( k \in \mathbb{N} \) there exists a function \( u_k \in H^1(\Omega) \) such that
\[
\|u_k\|_{H^1(\Omega)}^2 \geq k \left( \|u_k\|_{L^2(\Gamma)}^2 + \|\nabla u_k\|_{L^2(\Omega)}^2 \right).
\]
Let \( v_k = u_k/\|u_k\|_{H^1(\Omega)} \). As a consequence, \( \|v_k\|_{H^1(\Omega)} = 1 \) and \( \|v_k\|_{L^2(\Gamma)}^2 + \|\nabla v_k\|_{L^2(\Omega)}^2 < 1/k \). We deduce that \( v_k \to 0 \) in \( L^2(\Gamma) \) and that \( \partial_x v_k \to 0 \) in \( L^2(\Omega) \), \( i = 1, \ldots, N \). Next, since the sequence \( \{v_k\}_{k=1}^\infty \) is bounded in \( H^1(\Omega) \), using Rellich-Kondrachov Theorem, we can extract a subsequence, that we call \( \{v_k\} \) for simplicity, which is convergent in \( L^2(\Omega) \) to a limit function \( v \). Let us show that \( v_k \) converges to \( v \) in \( H^1(\Omega) \). Indeed,
\[
\|v_m - v_l\|_{H^1(\Omega)} \leq C \left( \|v_m - v_l\|_{L^2(\Omega)} + \|\nabla v_m\|_{L^2(\Omega)} + \|\nabla v_l\|_{L^2(\Omega)} \right).
\]
Since \( \{v_k\} \) converges in \( L^2(\Omega) \), it is a Cauchy sequence in that space, and since its gradient converges to 0, taking sufficiently big \( m \) and \( l \) we have that \( \|v_m - v_l\|_{H^1(\Omega)} \) can be as small as we want. As a consequence, \( v_k \to v \) in \( H^1(\Omega) \). This implies that \( \nabla v_k \to \nabla v \) in \( L^2(\Omega) \). But we already know that \( \nabla v_k \to (0, \ldots, 0) \) in \( L^2(\Omega) \). Since \( U \) is connected, hence \( v \) is constant in \( U \).

On the other hand, recall that \( \Gamma \) has positive \((N-1)\)-dimensional measure, hence, by trace inequality we get \( \|v_k - v\|_{L^2(\Gamma)} \leq C\|v_k - v\|_{H^1(\Omega)} \). As a consequence, \( v_k \to v \) in \( L^2(\Gamma) \). But we have shown that \( v_k \to 0 \) in \( L^2(\Gamma) \), which implies \( v = 0 \) in \( \Gamma \) in the trace sense. We deduce that \( v = 0 \) a.e. in \( U \), and that \( v_k \to 0 \) in \( H^1(\Omega) \). This gives a contradiction, since \( \|v_k\|_{H^1(\Omega)} = 1 \).

16. Integrate by parts to prove the following inequality
\[
\|Du\|_{L^2} \leq C\|u\|_{L^2}^{1/2} \|D^2u\|_{L^2}^{1/2} \quad \text{for all } u \in C_c^\infty(\Omega).
\]
Prove also that the inequality holds for \( u \in H^2(\Omega) \cap H_0^1(\Omega) \) if \( \Omega \) is a bounded domain with smooth boundary.

**Hint.** Take two sequences \( \{v_k\}_{k=1}^\infty \subset C_c^\infty(\Omega) \) converging to \( u \) in \( H^1(\Omega) \) and \( \{w_k\}_{k=1}^\infty \) converging to \( u \) in \( H^2(\Omega) \).

**Solution.** We follow the hint, and we integrate by parts and using Hölder inequality,
\[
\sum_{i=1}^n \int_\Omega \partial_x v_k \partial_x w_k = - \sum_{i=1}^n \int_\Omega v_k \partial^2_{xx} w_k \leq \sum_{i=1}^n \|v_k\|_{L^2(\Omega)} \|\partial^2_{xx} w_k\|_{L^2(\Omega)} \leq C\|v_k\|_{L^2(\Omega)} \|D^2w_k\|_{L^2(\Omega)}.
\]
Taking the limit as \( k \to \infty \) gives the result.

17. (Gagliardo-Nirenberg Inequality – First form, dimension \( N = 1 \)) Let \( \Omega = (0,1) \).

(a) Let \( 1 \leq q < \infty \) and \( 1 < r \leq \infty \). Show that
\[
\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{W^{1,r}(\Omega)}^{a} \|u\|_{L^q(\Omega)}^{1-a} \quad \text{para toda } u \in W^{1,r}(\Omega)
\]
for some constant \( C = C(q,r) > 0 \), where \( a \in (0,1) \) is given by
\[
a \left( \frac{1}{q} + 1 - \frac{1}{r} \right) = \frac{1}{q}.
\]
Hint. Begin with the case \( u(0) = 0 \) write \( G(u(x)) = \int_0^x G'(u(t))u'(t) \, dt \), where \( G(t) = |t|^\alpha - t \) and \( \alpha = 1/a \). When \( u(0) \neq 0 \), use the above inequality with \( \zeta u \), where \( \zeta \in C^1([0,1]) \), \( \zeta(0) = 0 \), \( \zeta(t) = 1 \) for all \( t \in [1/2,1] \).

(b) Let \( 1 \leq q < p < \infty \) \( 1 \leq r \leq \infty \). Show that
\[
\| u \|_{L^p(\Omega)} \leq C \| u \|_{W^{1,r}(\Omega)}^{b} \| u \|_{L^q(\Omega)}^{1-b} \quad \text{para toda } u \in W^{1,r}(\Omega)
\]
for some constant \( C = C(p,q,r) > 0 \), where \( b \in (0,1) \) is given by
\[
b \left( \frac{1}{q} + 1 - \frac{1}{r} \right) = \frac{1}{q} - \frac{1}{p}.
\]

Hint. Write \( \| u \|_{L^p(\Omega)} = \int_\Omega |u|^p \leq \| u \|_{L^q(\Omega)}^q \| u \|_{L^r(\Omega)}^r \) and use part (a) when \( r > 1 \).

(c) Under the same assumptions as in part (b), show that
\[
\| u \|_{L^p(\Omega)} \leq C \| u \|_{L^r(\Omega)}^{b} \| u \|_{L^q(\Omega)}^{1-b} \quad \text{for all } u \in W^{1,r}(\Omega) \text{ tal que } \int_\Omega u = 0.
\]

Solution. (a) Following the hint, using that \( G'(t) = \alpha |t|^{\alpha - 1} \), and Hölder inequality with conjugate exponents \( r \) and \( r' \), we get
\[
|u(x)|^a = |G(u(x))| \leq \int_0^1 |G'(u(t))| |u'(t)| \, dt \leq \alpha \| u' \|_{L^{r}(\Omega)} \| u \|_{L^{(\alpha-1)r'}(\Omega)}.
\]
The result for functions such that \( u(0) = 0 \) follows immediately, taking \( q = (\alpha - 1)r' \), and recalling that \( \alpha = 1/a \). The definition of \( q \) is equivalent to \( a \left( \frac{1}{q} + 1 - \frac{1}{r} \right) = \frac{1}{q} \). Let us notice that we actually get something better: instead of the norm \( W^{1,r} \) we get \( L^r \) norm of the derivative.

The general case follows again by the hint. Apply the previous case to
\[
|\zeta u(x)| \leq C \| \zeta u \|_{L^r(\Omega)}^{\alpha} \| \zeta u \|_{L^q(\Omega)}^{1-\alpha}.
\]
Recall that \( (\zeta u)' = \zeta' u + \zeta u' \), so that
\[
\| (\zeta u)' \|_{L^r(\Omega)} \leq C \left( \| \zeta u' \|_{L^r(\Omega)} + \| \zeta u \|_{L^r(\Omega)} \right) \leq C \| u' \|_{L^r(\Omega)} + C \| u \|_{L^r(\Omega)} \leq C \| u \|_{W^{1,r}(\Omega)}.
\]
We also have \( \| \zeta u \|_{L^q(\Omega)} \leq C \| u \|_{L^q(\Omega)} \), which leads to
\[
|u(x)| = |(\zeta u)(x)| \leq C \| u \|_{W^{1,r}(\Omega)}^{a} \| u \|_{L^q(\Omega)}^{1-a} \quad \text{si } x \in [1/2,1].
\]
To analyze the other half of the interval, let us consider the function \( \tilde{u}(x) = u(1 - x) \) and let us apply the result on \([1/2,1]\) to \( \tilde{u} \). We then get
\[
|u(x)| = |\tilde{u}(1-x)| \leq C \| \tilde{u} \|_{W^{1,r}(\Omega)}^{a} \| \tilde{u} \|_{L^q(\Omega)}^{1-a} \quad \text{si } x \in [0,1/2].
\]
The result follows once we notice that
\[
\| \tilde{u} \|_{W^{1,r}(\Omega)} = \| u \|_{W^{1,r}(\Omega)}, \quad \| \tilde{u} \|_{L^q(\Omega)} = \| u \|_{L^q(\Omega)}.
\]
(b) If \( r > 1 \), let us just follow the hint, taking \( b = a \left( 1 - \frac{2}{q} \right) \). If \( r = 1 \), we use again the hint, but instead of part (a) we now use the Sobolev inequality \( \| u \|_{L^\infty(\Omega)} \leq C \| u \|_{W^{1,1}(\Omega)} \) (NOTICE that we are in DIMENSION \( N \) = 1), and recall that in this case we have \( b = 1 - \frac{2}{q} \).

(c) Combine Poincaré-Wirtinger inequality with the result of part (b) as follows:
\[
\left\| u - \frac{1}{|\Omega|} \int_\Omega u \right\|_{L^{r}(\Omega)} \leq C \| u' \|_{L^{r}(\Omega)},
\]
which implies \( \| u \|_{L^{r}(\Omega)} \leq C \| u' \|_{L^{r}(\Omega)} \); hence we have \( \| u \|_{W^{1,r}(\Omega)} \leq C \| u' \|_{L^{r}(\Omega)} \), which combined with the result of part (b) proves the claim.