1. Let \( M = \{ u \in C([0, 1]) : \int_0^{1/2} u - \int_{1/2}^1 u = 1 \} \) and \( I : M \to \mathbb{R} \) given by \( I(u) = \|u\|_\infty \).

   (i) Show that \( \inf_{u \in M} I(u) = 1. \)

   (ii) Show that there is no function \( u \in M \) such that \( I(u) = 1. \)

   The problem is that this space is not reflexive.

2. Let \( M \) the convex closed subset of \( H^1([0, 1]) \) given by \( M = \{ u \in H^1([0, 1]) : u(0) = 1, u(1) = 0 \} \). Consider the functional \( I : M \to \mathbb{R} \) defined by \( I(u) = \int_0^1 x |u'(x)|^2 dx. \)

   (i) Show that \( \inf_{u \in M} I(u) = 0. \)

   (ii) Show that there does not exist any \( u \in M \) such that \( I(u) = 0. \)

   In this case the problem is that the functional is not coercive.

3. Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with smooth boundary, let \( \beta : \mathbb{R} \to \mathbb{R} \) be a smooth function such that there exist \( a \) and \( b \) such that

   \[ 0 < a \leq \beta'(z) \leq b \quad \text{for all } z \in \mathbb{R}, \]

   and \( f \in L^2(\Omega) \).

   (i) Define a concept of weak solution for the nonlinear problem

   \[ -\Delta u = f \quad \text{en } \Omega, \quad \partial u/\partial n + \beta(u) = 0 \quad \text{in } \partial \Omega. \]

   (ii) Prove that there exists a weak solution (and is unique).

4. Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with smooth boundary. Given \( u \in H^1(\Omega) \), we define the surface of the graphic of \( u \) by

   \[ F(u) = \int_\Omega \sqrt{1 + |\nabla u|^2} \, dx. \]

   (i) Prove that the functional \( F \) is \( C^1 \) in \( H^1(\Omega) \).

   (ii) Let \( g \in H^1(\Omega) \), and \( \mathcal{A} = \{ u = g + v : v \in H^1_0(\Omega) \} \). Prove that a critical point of \( F \) in \( \mathcal{A} \) is a weak solution to the equation of minimal surfaces:

   \[ \nabla \cdot \left( \frac{\nabla u}{\left(1 + |\nabla u|^2\right)^{1/2}} \right) = 0 \quad \text{en } \Omega, \quad u = g \quad \text{en } \partial \Omega. \]

   The expression on the left of this equality is \( N \)-times the mean curvature of the graphic of \( u \). Hence a minimal surface has zero mean curvature.

   (iii) Check whether or not the direct method of calculus of variations can be used to deduce existence of a minimizer of \( F \) in \( \mathcal{A} \).

   (iv) Let \( J(w) = \int_\Omega w \, dx \). Assume that \( u \) is a smooth minimizer of \( F \) in \( \mathcal{A} \cap \{ w : J(w) = 1 \} \). Show that the graphic of \( u \) is a minimal surface with constant mean curvature.
5. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Consider the eigenvalue problem for the bi-harmonic operator with Dirichlet boundary conditions,

$$\Delta^2 u = \lambda u \quad \text{en} \quad \Omega, \quad u = \partial u / \partial n = 0 \quad \text{in} \quad \partial \Omega.$$ 

Show that there exists a non-trivial weak solution $(\lambda, u)$ to the problem when $\lambda > 0$.

6. Let $f \in L^2(\Omega)$. Show that there exists a unique minimizer $u$ of

$$J(w) = \int_{\Omega} \left( \frac{1}{2} |\nabla w|^2 - f w \right)$$

in $\mathcal{A} = \{w \in H^1_0(\Omega) : |\nabla w| \leq 1 \text{ a.e.} \}$. Show that

$$\int_{\Omega} \nabla u \cdot \nabla (w - u) \geq \int_{\Omega} f(w - u) \quad \text{for all} \ w \in \mathcal{A}.$$