MONOIDAL TRANSFORMS AND INVARIANTS OF SINGULARITIES IN POSITIVE CHARACTERISTIC

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Abstract. The problem of resolution of singularities in positive characteristic can be reformulated as follows: Fix a hypersurface $X$, embedded in a smooth scheme, with points of multiplicity at most $n$. Let an $n$-sequence of transformations of $X$ be a finite composition of monoidal transformations with centers included in the $n$-fold points of $X$, and of its successive strict transforms. The open problem (in positive characteristic) is to prove that there is an $n$-sequence such that the final strict transform of $X$ has no points of multiplicity $n$ (no $n$-fold points).

In characteristic zero, such an $n$-sequence is defined in two steps: the first consisting in the transformation of $X$ to a hypersurface with $n$-fold points in the so called monomial case. The second step consists in the elimination of these $n$-fold points (in the monomial case), which is achieved by a simple combinatorial procedure for choices of centers.

The invariants treated in this work allow us to define a notion of strong monomial case which parallels that of monomial case in characteristic zero: If a hypersurface is within the strong monomial case we prove that a resolution can be achieved in a combinatorial manner.

Recently a canonical $n$-sequence of $X$ has been defined, which together with invariants introduced is this work, yield resolution in small dimension. We include here a simple proof of embedded resolution of 2-dimensional schemes simply as an application of the invariants introduced in this work.

Contents

1. Introduction. 1

Part I. Inductive functions and the tight monomial algebra. 6

2. Differential algebras, elimination and local presentations. 6

3. Monomial algebras and the behavior of elimination under monoidal transformations. 11

4. Invariants defined in terms of $p$-presentations. 13

5. Well adapted $p$-presentations. 16

6. Transformations of $p$-presentations. 19

7. On the two Main Theorems 22

Part II. Strong monomial case. 22

Part III. Embedded Resolution of 2-dimensional schemes. 28

Part IV. Proofs of Theorems

Appendix A. Proof of Main Theorem 1. 37

Appendix B. The tight monomial algebra and Proof of Main Theorem 2. 39

References 40

1. Introduction.

1.1. The objective of this paper is to study invariants of singularities in positive characteristic. It is motivated by the open problem of elimination of points of highest multiplicity of a singular
hypersurface $X$ by means of monoidal transformations. To be precise, let $V$ be a smooth scheme of dimension $d$ over a perfect field $k$ of characteristic $p > 0$ and let $X$ be hypersurface in $V$ with highest multiplicity $n$. The problem is to define a sequence

$$\begin{align*}
(1.1.1) \quad & X \xrightarrow{\pi_C} X_1 \xrightarrow{\pi_{C_2}} \cdots \xrightarrow{\pi_{C_N}} X_N \\
& V \xleftarrow{\pi_{C_1}} V_1 \xleftarrow{\pi_{C_2}} \cdots \xleftarrow{\pi_{C_N}} V_N
\end{align*}$$

where each $V_{i-1} \xrightarrow{\pi_{C_i}} V_i$ is defined with center $C_{i-1}$ included in the $n$-fold points of $X_{i-1}$, and each $X_i \subset V_i$ denotes the strict transform of $X_{i-1}$ by $\pi_{C_i}$, so that $X_N$ has no point of multiplicity $n$. We require, in addition, that the exceptional locus of $V \leftarrow V_N$ be a union of $N$ hypersurfaces with normal crossings at $V_N$. This sequence is said to be a simplification of $n$-fold points of $X$.

In characteristic zero, a simplification of $n$-fold points can always be defined. This is usually done in two steps. The first step consists in a sequence of, say $r$, monoidal transformations, so that the set of points of highest multiplicity $n$ of $X_r$ is within the so called monomial case. The second step consists in the elimination of the $n$-fold points of the hypersurface $X_r$, which is assumed to be in the monomial case. This latter step can be achieved by a combinatorial choice of centers.

Both steps rely on Hironaka’s main inductive invariant, $\text{ord}^{(d-1)}(x) \in \mathbb{Q}$, defined for $x$ in the highest multiplicity locus of the hypersurface. In fact, these invariants lead to the definition of a sequence in such a way $X_1$ is in the monomial case; and also to the very definition of the monomial considered in the monomial case. The role of Hironaka’s main inductive function in both steps mentioned above, always in characteristic zero, will be recalled in 1.3.

In this work we study the extension of Hironaka’s inductive function (on the rational numbers) to positive characteristic. This will lead us to a notion of strong monomial case for a hypersurface in positive characteristic. It parallels that of monomial case in characteristic zero, i.e., if $X_r$ is in the strong monomial case, then elimination of $n$-fold points is achieved in a combinatorial manner.

A canonical sequence of transformations of $X$ was defined in [9], for hypersurfaces in positive characteristic. This sequence transforms $X$ to an embedded hypersurface, say $X_r$, which is closely related to the monomial case, but still weaker than the strong monomial case we introduce here. The simplification of $n$-fold problem (and hence the resolution problem) would be solved if one could fill the gap between the weak monomial case in [9] and our strong monomial case. This can be achieved in low dimension, and we fill the gap and prove resolution of 2-dimensional schemes simply as a consequence of the invariants introduced here. A detailed proof of this fact can be found in Part III.

1.2. Assume for simplicity that $V$ is affine and $X = V(f)$ is a hypersurface with highest multiplicity $n$. We will first attach to the previous data the algebra $\mathcal{O}_V[fW^n]$, where $n$ as above. The notion of transformation of hypersurfaces (with the center included in the subset of $n$-fold points) has a natural reformulation in the language of algebras. Moreover, the task of defining a sequence (1.1.1) that eliminates $n$-fold points of $X$, as above, by means of monoidal transformations, can be also expressed in terms of algebras and transformations of algebras.

This reformulation of the problem in terms of algebras is well justified. We have attached to $X = V(f) \subset V$ and to $n$, the algebra $\mathcal{O}_V[fW^n]$. For the purpose of finding a simplification of $n$-fold points one can also replace $\mathcal{O}_V[fW^n]$ by the algebra

$$\text{(1.2.1)} \quad \mathcal{O}_V[fW^n, \Delta^\alpha(f)W^n-|\alpha|]$$

where $\Delta^\alpha$ denotes some differential operators of order $|\alpha| < n$ on the smooth scheme $V$.

This second algebra, called here differential algebra (see 2.5), is closely related to the theory of maximal contact in characteristic 0. This theory allows us to reformulate the problem of simplification of $n$-fold points with an algebra defined in a smooth hypersurface $\overline{V}$, and hence in one dimension less. This hypersurface $\overline{V}$ is usually called hypersurface of maximal contact. This form of induction is formulated in the language of algebras, and gives rise to the so called coefficient algebra.

1.3. It is natural to consider sequences of transformations of the form

$$\begin{align*}
(1.3.1) \quad & V \xrightarrow{\pi_{C_1}} V_1 \xleftarrow{\pi_{C_2}} \cdots \xleftarrow{\pi_{C_N}} V_N
\end{align*}$$

where the exceptional locus of the sequence, say $E_r = \{H_1, \ldots, H_s\}$, has normal crossings at $V_r$. A monomial algebra in $V_r$ is an algebra of the form $\mathcal{O}_V[I(H_1)^{\alpha_1} \cdots I(H_s)^{\alpha_s}W^n]$ for some $s, \alpha_i \in \mathbb{Z}_{\geq 0}$. 


In this case of characteristic 0, the simplification of n-fold points can be achieved in two steps, both of them expressed in terms of algebras, once a hypersurface of maximal contact, say $V_r$, is fixed:

(***STEP 1***) in which a sequence of monoidal transformations is defined over the hypersurface of maximal contact, say

$$\pi_1: V \leftarrow \pi_2 \leftarrow \cdots \leftarrow \pi_r: V_r,$$

so that the coefficient algebra is transformed into a monomial algebra, say $O_{V_r}[I(H_1)^{\alpha_1} \ldots I(H_r)^{\alpha_r}W^s]$. This sequence can be defined so as to induce a sequence (1.3.1) and in this case, the n-fold points of $X_r$ (the strict transform of $X$) are said to be in the monomial case.

(***STEP 2***) in which a simplification of the n-fold points of $X_r$ (monomial case) is defined, say

$$\pi_{r+1}: V_{r+1} \leftarrow \cdots \leftarrow \pi_N: V_N.$$

This step is achieved in an easy combinatorial manner. This process of choice of centers is defined only in terms of the exponents $\alpha_i$ of the monomial algebra.

All these arguments (always in characteristic 0), rely strongly on the Hironaka’s inductive functions $\text{ord}_{d-i}^{(d-1)}$ (see (2.3.1)), defined in terms of the coefficient algebra. In fact, Hironaka’s functions allow us to attach to an arbitrary sequence (1.3.1) a monomial algebra $O_{V_r}[I(H_1)^{\alpha_1} \ldots I(H_r)^{\alpha_r}W^s]$.

To be precise, this is done by setting $\frac{\alpha_i}{s} + 1 = \text{ord}_{d-i}^{(d-1)}(y_i)$ (i = 1, ..., r); here the right hand side is the evaluation of the inductive function at $y_i$, the generic point of the center $C_{i-1}$.

1.4. Main objectives of this work. In this work we consider schemes over perfect fields of positive characteristic. The two main objectives are:

1. to define an analogue to Hironaka’s inductive functions, called here $v - \text{ord}_{d-i}^{(d-1)}$ (Main Theorem 1 in 7.2), with values in $\mathbb{Q}$. These functions enable us to attach a monomial algebra $O_{V_r}[I(H_1)^{\alpha_1} \ldots I(H_r)^{\alpha_r}W^s]$ to a sequence of transformations (1.3.1), setting as before $\frac{\alpha_i}{s} + 1 = v - \text{ord}_{d-i}^{(d-1)}(y_i)$ (see Main Theorem 2 in 7.5).

2. To characterize, by numerical invariants, a case called here strong monomial case (Definition 8.4), in which a combinatorial resolution of the monomial algebra defines, as in Step 2, a simplification of n-fold points (Theorem 8.13). This property will rely strongly on Main Theorem 2.

1.5. Differences with characteristic zero. In characteristic zero, Hironaka’s inductive function $\text{ord}_{d-i}^{(d-1)}$ is upper semi-continuous. This property follows from a form of coherence, and the proof of this property requires some form of patching of local data, and all together it is quite involved. In positive characteristic the function $v - \text{ord}_{d-i}^{(d-1)}$ is not upper semi-continuous and therefore we do not go through this kind of difficulty. So there is no coherence or patching to be proved in the positive characteristic case. Despite this fact, this function is essential in the study of singularities and we show that it leads to (1) and (2) in 1.4.

In characteristic zero the value of the function $\text{ord}_{d-i}^{(d-1)}$, at a given point, is computed by fixing a hypersurface of maximal contact. As there is no maximal contact in positive characteristic, we replace reduction to hypersurfaces of maximal contact by transversal projections: $V^{(d)} \rightarrow V^{(d-1)}$ defined in étale topology (Definition 2.10). In this setting, algebras over the smooth scheme $V^{(d-1)}$ are defined; they are called elimination algebra (2.11). In characteristic zero elimination algebras parallel the role of the coefficients algebras.

We use here transversal projections and elimination algebras to calculate the value of the function $v - \text{ord}_{d-i}^{(d-1)}$ at a given point, which is a rational number. To fix ideas let $x$ be an n-fold point of $X = V(f) \subset V^{(d)}$. Weierstrass preparation Theorem ensures that one can choose a regular system of parameters $\{z, x_1, \ldots, x_{r-1}\}$ so that at the completion $\hat{O}_{V(x),x} = k'[\{z, x_1, \ldots, x_{r-1}\}]$ (r = d if x is closed) we can take

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_n \text{ with } a_i \in k'[\{x_1, \ldots, x_{r-1}\}].$$
A rational number $\geq 1$ is defined as

\[
\max_{1 \leq i \leq n} \left\{ \frac{\nu_x(a_i)}{i} \right\} \in \mathbb{Q}, \quad (\nu_x(a_i) \text{ is the order at } k'[\![x_1, \ldots, x_{r-1}]\!]).
\]

This is the maximal slope, for the different choices of $z$, but always fixing the inclusion of rings $k'[\![x_1, \ldots, x_{r-1}]\!] \subset k'[\![z, x_1, \ldots, x_{r-1}]\!] = \mathcal{O}_{V(d), x}$. Fixing an inclusion of rings is formulated here by fixing a morphism of smooth schemes $V(d) \rightarrow V(d-1)$ (called here projection). In order to parallel the presentation in (1.5.1) (Weierstrass preparation Theorem) we need to consider étale topology. Projections for which (1.5.1) holds (where $n$ is the multiplicity of $X$ at the point), will be said to be transversal at $x$.

Our setting will be slightly more general. Once a transversal projection $V(d) \xrightarrow{\beta} V(d-1)$ is fixed, we will consider an expression

\[
f(z) = z^n + a_1z^{n-1} + \cdots + a_n \in \mathcal{O}_{V(d-1)}[z]
\]

where $a_i$ are global functions on $V(d-1)$ and where $z$ is a global function on $V(d)$ so that $\{dz\}$ is a basis of $\Omega^1_{\beta}$, the sheaf of relative $\beta$-differentials. In this case, the smooth hypersurface $\{z = 0\}$ is a section of $V(d) \xrightarrow{\beta} V(d-1)$. We will abuse notation and say that the function $z$ is a transversal section of $\beta$.

We show here that the rational number in (1.5.2) is independent of the chosen transversal projection, and hence intrinsic of the singularity (Main Theorem 1). This defines a rational invariant attached to singular point $x$, denoted here by $v - \text{ord}(d-1)(x)$.

If we fix two $n$-fold points $x$ and $y$, so that $x \in \overline{y}$, then it will be shown that $v - \text{ord}(d-1)(x) \geq v - \text{ord}(d-1)(y)$ (despite this property, the function is not upper semi-continuous). This inequality will be used in the proof of the two main objectives (1) and (2) in 1.4.

This invariant attached to the singularity has been largely studied in positive characteristic for the particular case of equations of the form $f_{\psi}(z) = z^n + a_{\psi} \in \mathcal{O}_{V(d-1)}[z]$ (the purely inseparable case), e.g., [10, 22, 20, 29]. This equation involves a particular transversal projection, say $V(d) \xrightarrow{\beta} V(d-1)$. Note that pure inseparability fails to hold if the projection is changed (pure inseparability is not a property of the singularities of a hypersurface). Our result shows that the rational number in (1.5.2), usually called the slope of the singularity, is independent of the projection and hence intrinsic of the singularity.

1.6. Organization and further comments.

Part I: $p$-presentations, adaptations and the tight monomial algebra.

The objective of this first part is the definition of the inductive function and the study of its main properties mentioned in 1.4. This leads to the two main Theorems stated in the last Section [7], we suggest a first look at this last section for an overall view of the preliminary results that are needed.

This first part is developed so as to introduce gradually the inductive function in positive characteristic, and to pave the way to the study of the strong monomial case in Part II. This part has been organized so as to present only those technical aspects which are essential in the first two parts, whereas other technical arguments are gathered in Part III.

Section 2 encompasses several notions used throughout the paper, such as Rees algebras and Rees algebras endowed with a suitable compatibility with differential operators. This will lead us to the definition of simple differential algebras, which will be essential for the definition of our invariants. The study of $n$-fold points of the hypersurface $X = V(f)$ is reformulated here in terms of the Rees algebra $\mathcal{O}_{V(d)}[f^W]$. This is our first example of simple algebra. Attached to this Rees algebra is a well-defined differential algebra, which in this case is of the form (1.2.1).

Simple algebras which are differential will lead us naturally to the study of monic polynomials (1.5.3), where now $n = p^e$ is a power of the characteristic.

We also discuss here the notion of elimination algebras, these are defined in terms of differential algebras and transversal projections. Elimination algebras will play a central role in our notion of local presentation and $p$-presentation.
We make use of a fundamental property of stability of transversal projections with monoidal transformations: To fix ideas set $X = \{ f = 0 \} \subset V^{(d)}$ and a transversal projection $V^{(d)} \xrightarrow{\pi} V^{(d-1)}$ as in (1.5.3). Consider now an arbitrary sequence of monoidal transformations

\[
\begin{array}{c}
X & \xrightarrow{\pi_{C_1}} & X_1 \\
V^{(d)} & \xrightarrow{\pi_{C_2}} & V_1^{(d-1)} \\
& \cdots & \\
& \xrightarrow{\pi_{C_r}} & V_r^{(d-1)}
\end{array}
\]

where each $X_i$ denotes the strict transform of $X_{i-1}$ and each $\pi_{C_i}$ is a monoidal transformation with center $C_i$ included in the $n$-fold points of $X_{i-1}$. The stability property of transversality is that (1.6.1) induces a sequence

\[
V^{(d-1)} \xleftarrow{\pi} V_1^{(d-1)} \xleftarrow{\pi} \cdots \xleftarrow{\pi} V_r^{(d-1)}
\]

together with projections $V_i^{(d)} \xrightarrow{\beta_i} V_i^{(d-1)}$ which are transversal to $X_i$ along the $n$-fold points (the $\beta_i$ are defined in an open neighborhood of the $n$-fold points of $X_i$ in $V_i^{(d)}$).

This will lead us to some form of transformations of the monic polynomial in (1.5.3):

\[
f^{(i)}(z_i) = z_i^n + a_1^{(i)} z_i^{n-1} + \cdots + a_n^{(i)} \in \mathcal{O}_{V_i^{(d-1)}}[z_i].
\]

The polynomials in (1.6.3) are not the strict transform of the first expression in (1.5.3). Changes of the transversal parameter $z_i$ would be required in the definition of each expression.

In Section 3 sequences as (1.6.1) are expressed as transformations of Rees algebras. In this context each transversal projection $\beta_i$ will define an elimination algebra on $V_i^{(d-1)}$. In this section, we also discuss a form of compatibility of elimination with monoidal transformations. This will lead to Theorem 3.7 in which monomial algebras appear in a natural manner (9).

One of the objectives of this first Part is to assign a monomial algebra, say $\mathcal{O}_{V_i^{(d)}}[I(H_1)^{h_1} \ldots I(H_r)^{h_r} W^s]$ (see (1.4 (1))), to a sequence of transformations of $X$ as (1.6.1). This sequence is formulated here as a sequence of transformations of Rees algebras. This monomial algebra will relate to the coefficients of $f^{(i)}(z_i) = z_i^n + a_1^{(i)} z_i^{n-1} + \cdots + a_n^{(i)} \in \mathcal{O}_{V_i^{(d-1)}}[z_i]$. In fact, we show that such expression can be chosen so each coefficient $a_i^{(r)}$ is divisible, in some weighted manner, by this monomial algebra (see Definition 3.9).

A first step in the definition of our inductive function $v - \text{ord}^{(d-1)}$ is addressed in Section 4 where a rational number is assigned to a $p$-presentation (slope at a point). A notion of well-adapted $p$-presentation at a point is introduced in Section 5. It will be ultimately shown, in a further section, that the slope of $p$-presentations which are well-adapted at a point $x$, is $v - \text{ord}^{(d-1)}(x)$ (the value of the inductive function at $x$). This highlights the importance of this notion in what follows.

Both Sections 4 and 5 are focused in giving, in an explicit manner, the value of the Main Inductive function at a singular point.

In Section 6 monoidal transforms of $p$-presentations are defined. This leads to the statement of the two main results of this first Part: Main Theorems 1 and 2, stated in Section 7. Main Theorem 1 (Theorem 7.2) asserts that the previously defined inductive function is independent of the chosen smooth projection $\beta$. Main Theorem 2 characterizes the monomial algebra, called here $\mathcal{M}_r W^s$, defined by the inductive functions. Proofs will be address in Part IV.

Part II: Strong monomial case.

In Part I we have defined the inductive functions, $v - \text{ord}^{(d-1)}$, and a monomial algebra, say $\mathcal{M}_r W^s$, for a given sequence of transformations (1.6.1). It can be shown that for any $n$-fold point $v - \text{ord}^{(d-1)}(x) \geq \text{ord}(\mathcal{M}_r W^s)(x)$, where the right hand side is the usual Hironaka’s order function. This will lead to our numerical characterization of the strong monomial case, expressed for any $n$-fold point $x$ by the condition $v - \text{ord}^{(d-1)}(x) = \text{ord}(\mathcal{M}_r W^s)(x)$.

It is proved in Theorem 8.13 that if the strong monomial case holds, then a combinatorial resolution of $\mathcal{M}_r W^s$ can be lifted to a simplification of the $n$-fold points. This settles 1.4 (2).
Part III: Embedded Resolution of 2-dimensional schemes.

Part I and Part II lead to resolution of singularities in small dimension; in this last Part III we address as a simply example the 2-dimensional case.

1.7. Final comments. The invariants studied in this paper make use of transversal projections $V^{(d)} \to V^{(d-1)}$ and of elimination algebras defined in $V^{(d-1)}$. There are other approaches in the definition of invariants along $n$-fold points of a hypersurface. The bibliography indicates some, but certainly not all the effort done in this way. An account on the problem, due to Hauser, appears in [21]. There is an alternative approach of Włodarczyk; his presentation in [33] includes an important study of pathologies in positive characteristic. There are also recent contributions by Kawanoue-Matsuki ([25], [26]), Hironaka ([24]), Cutkosky ([15]), and a fundamental contribution of Cossart-Jannsen-Saito in [11] which proves embedded resolution for 2-dimensional arithmetical schemes. We have profited from discussions with Encinas, Hauser, Kawanoue, Lipman, Matsuki, and from ideas of Ana Bravo which will be treated elsewhere.

Part I. Inductive functions and the tight monomial algebra.

2. Differential algebras, elimination and local presentations.

2.1. The initial motivation is the study of the highest multiplicity locus of an embedded hypersurface $X$. Here we begin in [22] by showing how we reformulate this study in terms of algebras. This reformulation enable us to consider algebras with more structure. In fact, algebras with a form of compatibility with differential operators are studied in [23], [24] and [25], where the notion of absolute and relative differential algebras are discussed.

It is in the context of differential algebras in which the fundamental notions of transversal projections and elimination algebras will be introduced (see [27] and [211] respectively).

The main objective of this section is to show that given a differential algebra, together with a transversal projection, the algebra can be entirely reconstructed in terms of two ingredients:

1. the elimination algebra, and
2. a monic polynomial.

This is the main result in this section, which is collected in Proposition 2.12. This form of presentation of the algebra will be essential throughout this work. In the case of characteristic zero the monic polynomial can be chosen of degree one. In the case of positive characteristic one can choose the monic polynomial so as to have as degree a power of the characteristic. This will lead to the definition of $p$-presentations in Definition 2.14.

The particular feature of positive characteristic is played by the coefficients of this monic polynomial as will be shown in this development. The definition of the main invariant will rely entirely on these two ingredients.

2.2. Rees algebras and the resolution problem. Here we introduce the notion of Rees algebras which will play a prominent role in our development. Let $V^{(d)}$ be a smooth scheme over a perfect field $k$ of dimension $d$. The problem of resolution of singularities of a singular scheme embedded in $V^{(d)}$ can be stated in terms of Rees algebras over $V^{(d)}$. These are algebras of the form $G = \bigoplus_{n \in \mathbb{N}} I_n W^n$, where $I_0 = \mathcal{O}_{V^{(d)}}$ and each $I_n$ is a coherent sheaf of ideals. Here $W$ stands for a dummy variable introduced simply to keep track of the degree. It will be assumed that, locally at any point of $V$, $G$ is a finitely generated $\mathcal{O}_{V^{(d)}}$-algebra.

A non-zero sheaf of ideals $J \subset \mathcal{O}_{V^{(d)}}$ defines an upper-semi-continuous function $\nu(J) : V^{(d)} \to \mathbb{Z}$, where $\nu_x(J)$ denotes the order of the stalk $J_x$ at the local regular ring $(\mathcal{O}_{V^{(d)},x}, \mathfrak{m}_x)$. Recall that the order of $J$ at $\mathcal{O}_{V^{(d)},x}$ is the highest integer $n$ so that $J_x \subset \mathfrak{m}_x^n$. The singular locus of $G$ is the closed set

$$\text{Sing}(G) = \{ x \in V^{(d)} \mid \nu_x(I_n) \geq n \text{ for each } n \in \mathbb{N} \}.$$ 

In the setting of [12] in which $X = V(f)$, we will first attach to this the algebra $G = \mathcal{O}_{V^{(d)}}[fW^n]$. The set $\text{Sing}(G)$ consists of the points of multiplicity $n$ of the hypersurface $X = V(f)$.

A monoidal transformation $V^{(d)} \xrightarrow{\nu} V^{(d)}$ along the closed smooth center $C \subset \text{Sing}(G)$, defines a new Rees algebra, $G_1 = \bigoplus_{n \in \mathbb{N}} I_n^{(1)} W^n$, called the transform of $G$. The transformation is denoted by
A sequence of transformations will be denoted by:

\[
\begin{array}{c}
\mathcal{G} & \xrightarrow{\pi G} & V(d)^t \\
\mathcal{G}_1 & \xrightarrow{\pi C_1} & V_1^{(d)} \\
\vdots & \ddots & \vdots \\
\mathcal{G}_r & \xrightarrow{\pi C_r} & V_r^{(d)}
\end{array}
\]

and herein we always assume that the exceptional locus of the composite morphism \(V^{(d)} \leftarrow V_r^{(d)}\) is a union of hypersurfaces with only normal crossings.

The sequence \((2.2.2)\) is said to be a resolution of \(\mathcal{G}\) if \(\text{Sing}(\mathcal{G}_r) = \emptyset\). For \(\mathcal{G} = \mathcal{O}_{V^{(d)}}[f W^n]\), a resolution \((2.2.2)\) defines a simplification of \(n\)-fold points as in \((1.1)\).

A Rees algebra \(\mathcal{G}\) is said to be simple at \(x \in \text{Sing}(\mathcal{G})\) if there is an index \(n \in \mathbb{N}\) so that \(\nu_\pi(I_x) = n\). It is said to be simple if this condition holds for any \(x \in \text{Sing}(\mathcal{G})\). Such is the case, for instance, with Log-resolutions of ideals on smooth schemes.

Given a ring \(S[X]\), a morphism of \(S\)-algebras, say \(Tay : S[X] \to S[X, T]\), is defined by setting \(Tay(X) = X + T\) (Taylor expansion). Here

\[Tay(f(X)) = f(X + T) = \sum r^\Delta(f(X))T^r,\]

for some operator \(\Delta^r : S[X] \to S[X]\) defined from this morphism. It is well known that \(\{\Delta^0, \Delta^1, \ldots, \Delta^r\}\) is a basis of the free module of \(S\)-differential operators of order \(r\). The same applies here for \(\mathcal{O}_{V^{(d-1)}}[z]\), the set \(\{\Delta^0, \Delta^1, \ldots, \Delta^r\}\) spans the sheaf of differential operators of order \(r\) relative to the smooth morphism \(\beta : V^{(d)} \to V^{(d-1)}\).

Throughout this paper, we will slightly abuse of notation, here \(\beta : V^{(d)} \to V^{(d-1)}\) is called a local projection, and the function \(z\) is said to be a section of \(\beta\).

Let \(\mathcal{G} = \bigoplus_{n \geq 0} I_n W^n\) be a Rees algebra on a \(d\)-dimensional smooth scheme \(V^{(d)}\). We always assume that \(I_0 = \mathcal{O}_{V^{(d)}}\) and that \(\mathcal{G}\) is locally finite generated \(\mathcal{O}_{V^{(d)}}\)-algebra. Namely that

\[\mathcal{G} = \mathcal{O}_{V^{(d)}}[f_n W^n, \ldots, f_n W^n, \{\subset \mathcal{O}_{V^{(0)}}[W]\}\]

locally at any point of \(V^{(d)}\).

Given two such algebras \(\mathcal{G}_1\) and \(\mathcal{G}_2\), \(\mathcal{G}_1 \odot \mathcal{G}_2\) will denote the smallest algebra containing \(\mathcal{G}_1\) and \(\mathcal{G}_2\). In terms of local generators, if \(\{f_1 W^{n_1}, \ldots, f_r W^{n_r}\}\) generates \(\mathcal{G}_1\) and \(\{g_1 W^{m_1}, \ldots, g_s W^{m_s}\}\) generates \(\mathcal{G}_2\), then \(\mathcal{G}_1 \odot \mathcal{G}_2\) is generated by \(\{f_1 W^{n_1}, \ldots, f_r W^{n_r}, g_1 W^{m_1}, \ldots, g_s W^{m_s}\}\).

A function \(\text{ord}(\mathcal{G})(-): V^{(d)} \to \mathbb{Q}\) is defined

\[(2.3.1)\]

\[\text{ord}(\mathcal{G})(x) = \min_{n \geq 0} \left\{ \frac{\nu_\pi(I_n)}{n} \right\}\]

where \(\nu_\pi\) denotes the order at the local regular ring \(\mathcal{O}_{V^{(d)}, x}\). It takes only finitely many values. In terms of this function, the singular locus is \(\text{Sing}(\mathcal{G}) = \{x \in V^{(d)} \mid \text{ord}(\mathcal{G})(x) \geq 1\}\).

**Remark 2.4.** It is a general fact that objects treated by resolution techniques are gathered in equivalence classes. Such is the case, for instance, with Log-resolutions of ideals on smooth schemes. If two ideals have the same integral closure, they undergo the same Log-resolution; so ideals are considered up to integral closure. A similar situation applies here, where the objects are algebras: two algebras with the same integral closure will not be distinguishable. For instance, if \(\mathcal{G}\) and \(\mathcal{G}'\) are two algebras on \(V^{(d)}\) with the same integral closure, then they define the same functions
ord(\mathcal{G}) = \text{ord}(\mathcal{G}')$ (in particular, $\text{Sing}(\mathcal{G}) = \text{Sing}(\mathcal{G}')$, Proposition 4.4, [34]). The reader should keep aware of this fact, as it also affects the notation. The expression $fW^r \in \mathcal{G} = \bigoplus_{n \geq 0} I_n W^n$ means that $f' \in I_r$, for some positive integer $r$.

A Rees algebra can be defined by fixing an ideal $I$ and a positive integer $s$, say $\mathcal{O}_V[IW^s]$, which we denote simply as $IW^s$. Moreover, any Rees algebra is of this kind up to integral closure (Remark 1.3 [16]). In this case, $fW^r \in \mathcal{O}_V[IW^s]$ means that $f' \in I^r$.

2.5. An algebra $\mathcal{G} = \bigoplus_{n \geq 0} I_n W^n$ over $V(d)$ is said to be a differential algebra if $D_r(I_n) \subset I_{n-r}$ for any $r < n$ and for any differential operator $D_r$ of order $r$, whenever we restrict to an affine open subset of $V(d)$.

$\mathcal{G}$ is said to be an absolute differential algebra, if this property holds for all $k$-linear differential operators. When a smooth projection $V(d) \xrightarrow{\beta} V(d-1)$ is fixed and the previous property holds for differential operators which are $\mathcal{O}_{V(d-1)}$-linear, or say, $\beta$-relative operators, then $\mathcal{G}$ is said to be a $\beta$-relative differential algebra, or simply $\beta$-differential.

If $\mathcal{G}$ is an absolute differential algebra, then it is also a $\beta$-relative differential algebra for any smooth morphism $V(d) \xrightarrow{\beta} V(d-1)$ defined over $k$. The $\beta$-relative structure has an advantage: The transform of an absolute differential algebra is not absolute differential, but the notion of $\beta$-differential algebra will turn out to be well suited with transformations.

If $\mathcal{G}$ is not a differential algebra, then it has a natural extension to a differential algebra (Theorem 3.4, [34]). The same holds if $\mathcal{G}$ is not a $\beta$-differential algebra. These natural extensions are compatible with integral closure: if $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same integral closure, then the same holds for their extensions to differential algebras (to $\beta$-differential algebras) Theorem 6.14, [34].

Remark 2.6. When $\mathcal{G}$ is a $\beta$-differential Rees algebra, then, locally, there is a finite set of elements of $\mathcal{G}$, say $\{f_1W^{n_1}, \ldots, f_sW^{n_s}\}$, so that 

$$\mathcal{G} = \mathcal{O}_{V(d)}[f_nW^{n_i}, \Delta^{(\alpha)}(f_i)W^{n_i-\alpha_i}]_{1 \leq \alpha_i \leq n_i-1, 1 \leq i \leq s},$$

with $\Delta^{(\alpha)}$ as in 2.3. Conversely, these local presentations characterize $\beta$-differential algebras (Theorem 2.9, [34]).

2.7. Transversal projections. The graded algebra of the maximal ideal $\mathfrak{M}_x$ of a point $x \in V(d)$, say $\text{Gr}_x(\mathcal{O}_{V(d)}[x])$, is isomorphic to a polynomial ring. When $x$ is a closed point, it is a polynomial ring in $d$-variables, which is the coordinate ring associated to the tangent space of $V(d)$ at $x$, namely $\text{Spec}(\text{Gr}_x(\mathcal{O}_{V(d)}[x])) = \mathcal{T}_{V(d)}[x]$. The initial ideal or tangent ideal of $\mathcal{G}$ at $x \in \text{Sing} \mathcal{G}$, say $I_n(x, \mathcal{G})$, is the ideal of $\text{Gr}_x(\mathcal{O}_{V(d)}[x])$ generated by the elements $I_n(x, I_n)$ for all $n \geq 1$, where $I_n(x, I_n)$ is the class of $I_n$ at $\mathfrak{M}_x/I_n\mathfrak{M}_x^{n+1}$. Observe that $I_n(x, \mathcal{G})$ is zero unless $\text{ord}(\mathcal{G})(x) = 1$. The zero set of the tangent ideal in $\text{Spec}(\text{Gr}_x(\mathcal{O}_{V(d)}[x]))$ is the tangent cone of $\mathcal{G}$ at $x$, denoted by $\mathcal{C}_\mathcal{G}[x]$.

Given a vector space $V$, a vector $v \in V$ defines a translation, say $tr_v(w) = w + v$ for $w \in V$. There is a largest linear subspace, denoted by $\mathcal{L}_V[x]$, so that $\mathcal{C}_\mathcal{G}[x]$ is invariant under translations of $\mathcal{L}_V[x]$, that is, $tr_v(\mathcal{C}_\mathcal{G}[x]) = \mathcal{C}_\mathcal{G}[x]$ for any $v \in \mathbb{L}_V$. This subspace $\mathcal{L}_V[x]$ is called the linear space of vertices.

Definition 2.8. (Hironaka’s $\tau$-invariant). $\tau_{\mathcal{G}, x}$ will denote the minimum number of variables required to express generators of the tangent ideal $I_n(x, \mathcal{G})$. This algebraic definition can be reformulated geometrically: $\tau_{\mathcal{G}, x}$ is the codimension of the linear subspace $\mathcal{L}_V[x]$ in $\mathcal{T}_{V(d)}[x]$.

2.9. Fix now a closed point $x \in V(d)$. Let $V(d) \xrightarrow{\beta} V(d-1)$ be smooth and set $\beta(x) = y \in V(d-1)$. A regular system of parameters $\{y_1, \ldots, y_s\}$ at $\mathcal{O}_{V(d-1)}[y]$, extends to $\{y_1, \ldots, y_s, z\}$, a regular system of parameters at $\mathcal{O}_{V(d)}[x]$. Here $x$ is a point of $\mathcal{T}_{\beta^{-1}(y)}$, and the tangent space of this subscheme at $x$, say $\mathcal{T}_{\beta^{-1}(y)}[x]$, is identified with the subscheme in $\mathcal{T}_{V(d)}[x]$ defined by the linear forms $\langle I_n(y_1), \ldots, I_n(y_s) \rangle \subset \text{Gr}_x(\mathcal{O}_{V(d)}[x])$ (i.e., a one dimensional subspace in $\mathcal{T}_{V(d)}[x]$).

Definition 2.10. A local projection $\beta : V(d) \rightarrow V(d-1)$ is said to be transversal to $\mathcal{G}$ at $x \in \text{Sing} \mathcal{G}$ if $\mathcal{C}_\mathcal{G}[x] \cap \mathcal{T}_{\beta^{-1}(y)}[x] = \mathcal{O}_x$, the origin of $\mathcal{T}_{V(d)}[x]$. The local projection is said to be transversal to $\mathcal{G}$ if it is so at any point of $\text{Sing} \mathcal{G}$. Transversality is an open condition so we are led to consider this condition only at closed points.
2.11. Elimination algebras. Let \( x \in \text{Sing}(\mathcal{G}) \) be a closed point in \( V(d) \), so \( y = \beta(x) \) is closed in \( V(d-1) \). A regular system of parameters \( \{y_1, \ldots, y_{d-1}\} \subset \mathcal{O}_{V(d-1),y} \) extends to a regular system of parameters \( \{y_1, \ldots, y_{d-1}, z\} \) at \( \mathcal{O}_{V(d),x} \). In this case, \( z \) defines a section of \( \beta : V(d) \to V(d-1) \) after suitable restrictions.

Take \( \mathcal{G} \) to be a simple algebra, and let \( \beta : V(d) \to V(d-1) \) be transversal to \( \mathcal{G} \). Fix a closed point \( x \in \text{Sing}(\mathcal{G}) \). Weierstrass Preparation Theorem ensures that, taking restrictions in étale topology, \( \mathcal{G} \) has the same integral closure as an algebra \( \mathcal{O}_{V(d),\beta}[f_1(z)W^{m_1}, \ldots, f_s(z)W^{m_s}] \), where each

\[
(2.11.1) \quad f_i(z) = z^{n_i} + a^{(i)}_{n_i-1}z^{n_i-1} + \cdots + a^{(i)}_0 \in \mathcal{O}_{V(d),\beta}[z]
\]
is a monic polynomial of degree \( n_i \in \mathbb{Z}_{\geq 0} \) (4.7 [35]).

The following properties are known to hold within this setting:

P0) the restriction of \( \beta \) to \( \text{Sing}(\mathcal{G}) \), say \( \beta : \text{Sing}(\mathcal{G}) \to \beta(\text{Sing}(\mathcal{G})) \), is a set theoretical bijection and two corresponding points have the same residue fields. Namely, \( k(x) \cong k(\beta(x)) \) (1.15 and Theorem 4.11 [35], or 7.1 [9]).

If \( \mathcal{G} \) is a \( \beta \)-relative differential algebra, then a Rees algebra on \( V(d-1) \), say \( \mathcal{R}_{\mathcal{G},\beta} \subset \mathcal{O}_{V(d-1),\beta}[W] \), is defined. It is called the elimination algebra of \( \mathcal{G} \), and has the following properties:

P1) \( \beta(\text{Sing}(\mathcal{G})) \subset \text{Sing}(\mathcal{R}_{\mathcal{G},\beta}) \); moreover if \( C \) is a closed and smooth scheme included in \( \text{Sing}(\mathcal{G}) \), then \( \beta(C) \subset V(d-1) \) is smooth, isomorphic to \( C \), and \( \beta(C) \subset \text{Sing}(\mathcal{R}_{\mathcal{G},\beta}) \) (Theorem 9.1 [9]).

P2) (Theorem 5.5, [35]) Fix two projections:

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\beta} & V(d) \\
\downarrow & & \downarrow \\
V(d) & \xrightarrow{\beta'} & V(d-1) \\
\mathcal{R}_{\mathcal{G},\beta} & \xrightarrow{\beta'} & \mathcal{R}_{\mathcal{G}',\beta'}
\end{array}
\]

where both \( \beta \) and \( \beta' \) are transversal to \( \mathcal{G} \). This defines an algebra \( \mathcal{R}_{\mathcal{G},\beta} \) over \( V(d-1) \) and an algebra \( \mathcal{R}_{\mathcal{G}',\beta'} \) over \( V(d-1) \). At any point \( x \in \text{Sing}(\mathcal{G}), \) \( \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(\beta(x)) = \text{ord}(\mathcal{R}_{\mathcal{G}',\beta'})(\beta'(x)). \)

P3) (1.15 [35]) If \( \text{ord}(\mathcal{R}_{\mathcal{G},\beta})(y) > 0 \) at a point \( y \in V(d-1) \), the restriction of (2.11.1) to the fiber over \( y \) (to \( \beta^{-1}(y) \)), say \( \mathcal{F}_i(Z) = Z^{n_i} + a^{(i)}_{n_i-1}Z^{n_i-1} + \cdots + a^{(i)}_0 \in k(y)[Z] \); is a power of a purely inseparable polynomial. Namely, \( \mathcal{F}_i(Z) = (Z^{p^{n_i}} + b_i)^{m_i} \) at \( k(y)[Z] \). Moreover, there is at most one point \( x \in V(d) \) so that \( \beta(x) = y \) and \( \text{ord}(\mathcal{G})(x) > 0 \).

A particular feature of characteristic zero is that \( z \) can be chosen to be of maximal contact. This is not always the case in positive characteristic, and the relative differential structure will partially fill in this gap. The previous formulation, in which the algebra is generated by monic polynomials, holds locally. In this local description, \( z \) is a section of \( \beta : V(d) \to V(d-1) \) as defined in 2.3.

2.12. (Local presentation) Set \( x \in \text{Sing}(\mathcal{G}) \) a closed point and \( V(d) \xrightarrow{\beta} V(d-1) \) transversal to \( \mathcal{G} \) at \( x \). Assume that \( \mathcal{G} \) is a \( \beta \)-relative differential algebra, that there is an element \( f_nW^n \in \mathcal{G}, f_n \) of order \( n \) at \( \mathcal{O}_{V(d),x} \), and that \( f_n = f_n(z) \) is a monic polynomial of degree \( n \) in \( \mathcal{O}_{V(d-1),\beta(z)}[z] \), where \( z \) is a \( \beta \)-section and an element at \( \mathcal{O}_{V(d),x} \). Then, at a neighborhood of \( x \), \( \mathcal{G} \) has the same integral closure as

\[
(2.12.1) \quad \mathcal{O}_{V(d)}[f_n(z)W^n, \Delta^{(\alpha)}(f_n(z))W^{n-\alpha}]_{\leq \alpha \leq n-1} \subset \mathcal{R}_{\mathcal{G},\beta},
\]

where \( \mathcal{R}_{\mathcal{G},\beta} \) is identified with \( \beta^*(\mathcal{R}_{\mathcal{G},\beta}) \), and \( \Delta^{(\alpha)} \) are as in 2.3 Moreover, \( \mathcal{R}_{\mathcal{G},\beta} \) is non-zero whenever \( \text{Sing}(\mathcal{G}) \) is not of co-dimension one locally at \( x \).

Proof. The last assertion follows from Theorem 4.11 i) [35]. Take \( f_n(z)W^n \in \{f_1W^{m_1}, \ldots, f_sW^{m_s}\} \) as in (2.11.1). For ease of notation we consider the case \( s = 2 \), i.e., \( \mathcal{G} = \mathcal{O}_{V(d)}[f_n(z)W^n, g_m(z)W^m] \).
Rees algebras are endowed with a natural graded structure. Elimination algebras are also Rees algebras. They are defined as a specialization of the so called universal elimination algebras, which are graded subalgebras in a polynomial ring.

Take variables $Z, Y_1, \ldots, Y_n$ and $V_1, \ldots, V_m$ over a field $k$, and set

$$F_n(Z) = (Z - Y_1) \cdot (Z - Y_2) \cdots (Z - Y_n).$$

This is the so called universal polynomial of degree $n$, and $f_n = f_n(z)$ can be obtained as a specialization of $F_n(Z)$. Similarly, let

$$G_m(Z) = (Z - V_1) \cdot (Z - V_2) \cdots (Z - V_m)$$

be the universal polynomial of degree $m$ which will specialize to $g_m(z)$.

The natural action of the permutation groups $S_n$ on $k[Y_1, \ldots, Y_n]$, and of $S_m$ on $k[V_1, \ldots, V_m]$, induces an action of the product $S_n \times S_m$ on $k[Z, Y_1, \ldots, Y_n, V_1, \ldots, V_m]$ by fixing $Z$. This group also acts on the subring

$$S = k[Z - Y_1, Z - Y_2, \ldots, Z - Y_n, Z - V_1, Z - V_2, \ldots, Z - V_m].$$

The subring of invariants of $S$, say $S_{\text{inv}}$, is

$$k[\Delta^\alpha(F_n(Z)), \Delta^\alpha(G_m(Z))]_{0 \leq \alpha \leq n-1, 0 \leq \alpha' \leq m-1},$$

where $\Delta^\alpha(F_n(Z))$ is an homogeneous polynomial of degree $n - \alpha$, obtained as in 2.3 Similarly $\Delta^\alpha(G_m(Z))$ is homogeneous of degree $m - \alpha'$.

The key observation to prove the assertion is that $S$ is spanned by two subrings: $k[Z - Y_1, \ldots, Z - Y_n]$ and $S'$, and $S_n \times S_m$ acts on both.

Recall that the subring of invariants in the first is $T = k[F_n(z)]W^n, \Delta^\alpha(F_n(z))W^{n-\alpha}]_{0 \leq \alpha \leq n-1},$ and the one of the second is the universal elimination algebra, say $R(\subset S').$

Thus both invariant algebras, $T$ and $R$, are included in $S_{\text{inv}}$. Let $\mathcal{T} \circ R$ denote the smallest algebra containing both rings. We now claim that $\mathcal{T} \circ R \subset S_{\text{inv}}$ is a finite extension of graded subalgebras of $S$. In order to prove this last assertion note that $S$ is a finite extension of both subalgebras.

The statement follows now from the previous observation. In fact, $\mathcal{G}$ and $2.12.1$ are obtained by specialization of the previous subrings. This specialization preserves the grading. On the other hand, integral extension of rings are preserved by specialization (change of base rings).

Remark 2.13. Fix a Rees algebra $\mathcal{G} = \bigoplus_{n \geq 0} I_nW^n$. If the setting of Proposition 2.12 holds at a closed point $x \in \operatorname{Sing}(\mathcal{G})$, then it holds globally after taking suitable restrictions of $V(d-1)$ to a neighborhood of $\beta(x)$, and of $V(d)$ to a neighborhood of $x$. Moreover, $z$ defines a $\beta$-section.

If the characteristic is zero $I_1$ has order one at $\mathcal{O}_V(d,x)$, and $z \in I_1$ can be chosen as an element of order one at this local ring. This is not always the case in positive characteristic. However, as $\mathcal{G}$ is a simple $\beta$-relative differential algebra, one can check that there is a power of the characteristic, say $p^r$, so that $I_{p^r}$ has order $p^r$ at $\mathcal{O}_V(d,x)$. Therefore the integer $n$ in the last Proposition can be chosen as a power of the characteristic. This power is defined in terms of $\mathcal{G}$ and the closed point $x \in \operatorname{Sing}(\mathcal{G})$. This leads to:

**Definition 2.14.** ($p$-Presentations). Fix, after suitable restriction in étale topology, a projection $V(d) \xrightarrow{\beta} V(d-1)$ transversal to a simple $\beta$-relative differential Rees algebra $\mathcal{G}$. Assume that $\operatorname{Sing}(\mathcal{G})$ has no components of co-dimension one.

Assume also that:
i) There is a $\beta$-section $z$.
i) There is an element $f_{p^e}(z)W^{p^e} \in \mathcal{G}$, where $f_{p^e}(z)$ is a monic polynomial of order $p^e$, say
$$f_{p^e}(z) = z^{p^e} + a_1 z^{p^e - 1} + \cdots + a_{p^e} \in \mathcal{O}_{V^{(d-1)}}[z],$$
where each $a_i$ is a global function on $V^{(d-1)}$.

The conditions of (2.12) hold for $\mathcal{G}$ and

$$\mathcal{O}_{V^{(d)}}[f_{p^e}(z)W^{p^e}, \Delta^{(a)}(f_{p^e}(z))W^{p^e-a}]_{1 \leq a \leq p^e-1} \oplus \beta^*(R_{\mathcal{G}, \beta}).$$

That is, $\mathcal{G}$ and (2.14.1) have the same integral closure.

In this case, we say that $\beta : V^{(d)} \to V^{(d-1)}$, the $\beta$-section $z$, and $f_{p^e}(z) = z^{p^e} + a_1 z^{p^e - 1} + \cdots + a_{p^e}$
define a $p$-presentation of $\mathcal{G}$. These data will be denoted by:

$$pP(\beta : V^{(d)} \to V^{(d-1)}, z, f_{p^e}(z) = z^{p^e} + a_1 z^{p^e - 1} + \cdots + a_{p^e}),$$
or simply $pP(\beta, z, f_{p^e}(z))$. Clearly (2.14.1) is expressed only in terms of $R_{\mathcal{G}, \beta}$ and $pP(\beta, z, f_{p^e}(z))$.

3. Monomial algebras and the behavior of elimination under monoidal transformations.

3.1. The definition of elimination algebras makes use of the notion of the relative differential structure. We now discuss some results that grow from a form of compatibility of the relative differential structure with monoidal transformations.

Recall that a sequence of transformations of $\mathcal{G}$ is a concatenation of transformations

$$\mathcal{G} \xrightarrow{\pi_1} \mathcal{G}_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_r} \mathcal{G}_r$$

where we always assume that the exceptional locus of $V^{(d)}$ union $V^{(d-1)}$ is a union of hypersurfaces with normal crossings. In the first part of this section we study the compatibility of transversality and elimination algebras with monoidal transformations. Sequences as (3.1.1) will also give rise to the definition of the so called monomial algebras (Definitions 3.5), and to a notion of monomial contact introduced in Definition 3.9. This notion appears in the formulation of Main Theorem 2.

3.2. Transversal projections are defined for simple algebras. When $\mathcal{G}$ is a simple algebra, we claim that all the $\mathcal{G}_i$ defined in (3.1.1) are also simple. It suffices to check this property locally. Fix a closed point $x \in C \subset \text{Sing}(\mathcal{G})$, where $C$ is a smooth center. There is an integer $n$ and an element $f_n \in \mathcal{I}_n$ so that $\nu_C(f_n) = n$. Note that $\nu_C(f_n) = n$ and $f_n$ is equimultiple at $C$ locally at $x$, so the strict transform of $f_n$ has multiplicity at most $n$ on points on the exceptional locus.

Take $\mathcal{G}$ to be a simple algebra on $V^{(d)}$, together with a transversal projection $\beta : V^{(d)} \to V^{(d-1)}$. Assume that $\mathcal{G}$ is a $\beta$-relative differential algebra. A notion of compatibility of this properties with monoidal transformations can be formulated as follows (3):

After suitable restrictions to an étale cover of $V^{(d)}$, the sequence (3.1.1) induces a diagram

$$\mathcal{G} \xrightarrow{\pi_1} \mathcal{G}_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_r} \mathcal{G}_r$$

where:

(i) Each vertical morphism $\beta_i : V^{(d)} \to V^{(d-1)}$ is transversal to $\mathcal{G}_i$, and each $\mathcal{G}_i$ is a $\beta_i$-differential algebra. These $\beta_i$ are defined only in a neighborhood of $\text{Sing}(\mathcal{G}_i)$.

(ii) The lower sequence induces transformations of the elimination algebra $R_{\mathcal{G}, \beta}$, and furthermore, each $(R_{\mathcal{G}, \beta})_i$ is the elimination algebra of $\mathcal{G}_i$ relative to $\beta_i : V^{(d)}_{\mathcal{G}_i} \to V^{(d-1)}_{\mathcal{G}_i}$, that is, $(R_{\mathcal{G}, \beta})_i = R_{\mathcal{G}_i, \beta_i}$ (Theorem 9.1 [9]).
Definition 3.3. A smooth morphism \( v_r(\beta) : V_r(\beta) \to V_r(\beta-1) \) is said to be \( r\)-transversal to \( G \) if there is a transversal morphism \( V(\beta) \to V(\beta-1) \), as in Definition 2.10, and a simple \( \beta \)-differential algebra \( G \) over \( O_{V(\beta)} \), so that \( G_r \) and \( \beta_r \) arise from a diagram as that in (3.2.1).

Remark 3.4. In characteristic zero, given a simple differential algebra \( G \), there are hypersurfaces of maximal contact at \( V(\beta) \). We fix one such hypersurface, and given a sequence of transformations of \( G \) (3.1.1), we consider the strict transforms of that fixed hypersurface. Here hypersurfaces of maximal contact will be replaced by transversal projections. We shall fix a transversal projection at \( V(\beta) \) and for any sequence (3.1.1) we will make use of the lifting of this fixed projection in (3.2.1).

Local \( p \)-presentations of \( G_r \) will be defined in terms of \( \beta_r \), where \( \beta_r \) arises from the fixed smooth transversal morphism \( \beta \). In Section 4 a notion of transformation of \( p \)-presentations will be defined. This together with the theorems on Section 5 will show that given a simple algebra \( G \), if \( V(\beta) \) can be covered by \( p \)-presentations of the form \( \beta(\beta,\pi,\pi,\pi) = \pi^p + a_1\pi^{p-1} + \cdots + a_p \), with the same exponent \( p \), then the same holds for \( G_r \) at \( V(\beta) \). Namely, that there is a covering of \( V(\beta) \) by \( p \)-presentations of the form \( \beta(\beta,\pi,\pi,\pi) = \pi^p + \tilde{a}_1\pi^{p-1} + \cdots + \tilde{a}_p \), where \( \beta_r \) is \( r \)-transversal, with the same exponent \( p \) on any such \( p \)-presentation.

Definition 3.5. Let \( E = \{H_1, \ldots, H_r\} \) be a set of smooth hypersurfaces with normal crossings. A monomial ideal supported on \( E \) will be an invertible sheaf of ideals of the form \( M = I(H_1)^{\alpha_1} \cdots I(H_r)^{\alpha_r} \), for some integers \( \alpha_i \geq 0 \).

A monomial algebra will be a Rees algebra of the form \( O_V[MW^s] \) for some positive integer \( s \). This algebra will be denoted by \( MW^s \). Locally at a point \( x \in V \), \( M_x \) is the ideal spanned by a monomial on a regular system of parameters of \( O_{V(\beta)} \cdot x \). Recall that Rees algebras are to be considered up to integral closures. Given \( f_n \in O_{V(\beta)} \cdot x \), \( f_nW^n \in MW^s \) if and only if \( f_n \) is divisible by \( M_n \) at \( O_{V(\beta)} \cdot x \) for any point \( x \).

3.6. Let \( \pi : V' \to V \) be a smooth morphism, then the pull-backs of the hypersurfaces of \( E \) have normal crossings at \( V' \) and a monomial ideal supported on \( E \) has a natural lifting to \( V' \).

In our setting, we fix a transversal smooth morphism \( \beta : V(\beta) \to V(\beta-1) \) as in Definition 2.10, a sequence (3.1.1) induces a diagram (3.2.1) with smooth morphisms \( \beta_i \) defined in a neighborhood of \( \text{Sing}(G_r) \). Note that at each such neighborhood, the exceptional hypersurfaces in \( V(\beta) \) are pull-backs of the exceptional hypersurfaces at \( V(\beta-1) \). In particular, a monomial algebra supported on the exceptional locus of the composite map \( V(\beta-1) \to V(\beta) \), say

\[
M_rW^s = I(H_1)^{\alpha_1} \cdots I(H_r)^{\alpha_r}W^s,
\]

can be naturally lifted to a monomial algebra supported on the exceptional locus of \( V(\beta) \to V(\beta-1) \).

Theorem 3.7 (Bravo-Villamayor [9]). Let \( G \) be a simple differential algebra and assume that \( \text{Sing}(G) \) has no component of co-dimension one. There is a sequence of transformations (3.1.7), so that for any local transversal projection \( \beta : V(\beta) \to V(\beta-1) \) (defined by restriction to an étale covering of \( V(\beta) \), the induced sequence (3.2.1) is such that \( (\mathcal{R}_{G,\beta})_r \) is a monomial algebra supported on the exceptional locus. Furthermore, the monomial algebra \( (\mathcal{R}_{G,\beta})_r \) is independent of \( \beta \).

In what follows, we can take, étale locally, a sequence (3.2.1) as in the formulation of the Theorem 3.7 (Main Theorem in [9]). So here, \( (\mathcal{R}_{G,\beta})_r \subset O_{V(\beta-1)}[W] \) is monomial and supported on the exceptional locus, and so is its pull-back to \( V(\beta) \). The same holds if we enlarge the sequence of transformations as this condition is stable.

We identify \( (\mathcal{R}_{G,\beta})_r \) with its pull-back, say

\[
(\mathcal{R}_{G,\beta})_r = I(H_1)^{\alpha_1} \cdots I(H_r)^{\alpha_r}W^s = NW^s.
\]

It will be shown that locally at any closed point of \( \text{Sing}(G_r) \), there is a \( \beta_r \)-section \( z' \), a monic polynomial, say \( f_p(z) \), so that \( G_r \) has the same integral closure as:

\[
O_{V(\beta)}[f_p(z)W^s, \Delta^s(f_p(z))W^{p-s-\alpha}][1 \leq \alpha \leq p \cdot s-1] \odot N_rW^s.
\]
3.8. The outcome of Theorem \(3.7\) in the case of fields of characteristic zero, is known as the reduction to the monomial case. In that context it is simple to extend \(3.1.1\) to a resolution. This is not the case in positive characteristic, however the following definition will lead us to the study of the role of the exceptional divisors.

**Definition 3.9.**

1) We say that a monomial algebra \(\mathcal{M}, W^s\) \((3.6.1)\) has **monomial contact** with \(G_r\) if locally at any closed point \(x \in \text{Sing}(G_r)\) there is a \(\beta_r\)-section \(z\) of order one at \(\mathcal{O}_{V^s}^{(x)}\), so that

\[G_r \subset (z)W \cap \mathcal{M}, W^s.\]

2) A local \(p\)-presentation of \(G_r\), say \(pP(\beta_r, z, f_{p-r}(z))\) (with \(f_{p-r}(z) = z^{p-r} + a_1 z^{p-r-1} + \cdots + a_{p-r}\)), is said to be **compatible with the monomial algebra** \(\mathcal{O}_{V^s}^{(a-1)}[\mathcal{M}, W^s]\) locally at \(x \in \text{Sing}(G_r)\) if the previous condition holds for the \(\beta_r\)-section \(z\). This, in turn, is equivalent to two conditions:

i) \((\mathcal{R}_{G, \beta})_r \subset \mathcal{O}_{V^s}^{(a-1)}[\mathcal{M}, W^s],\)

ii) \(a_i W^i \in \mathcal{O}_{V^s}^{(a-1)}[\mathcal{M}, W^s]\), for \(1 \leq i \leq p^r\).

3.10. We will show that given a simple algebra \(G\) and a sequence of transformations as in \((3.1.1)\), there is a monomial algebra \(\mathcal{M}, W^s\) supported on the exceptional locus which has monomial contact with \(G\). That is, locally at any point \(x \in \text{Sing}(G_r)\) there is a \(\beta_r\)-section \(z\) of order one at \(x\), so that \(G_r \subset (z)W \cap \mathcal{M}, W^s\). Main Theorem 2 will show that this monomial algebra will be defined in terms of the sequence \((3.1.1)\), with independence of the choice of \(\beta\) (of \((3.2.1)\)).

4. Invariants defined in terms of \(p\)-presentations.

4.1. Fix a transversal smooth projection \(\beta : V^d \longrightarrow V^{d-1}\) (Definition \(2.10\)) and a simple \(\beta\)-differential algebra \(G\). In Definition \(2.14\) we introduced the notion of \(p\)-presentation, say \(pP = pP(\beta, z, f_p(z))\). The aim of this Section is to define two functions:

(1) a function \(S(pP)\) \(V^{d-1} \longrightarrow \mathbb{Q}\) (Definition \(4.2\)).

(2) a function \(\beta - \text{ord}^{(d-1)}(G) : V^{d-1} \longrightarrow \mathbb{Q}\) (Definition \(4.11\)).

There are many \(p\)-presentations \(pP\) which make use of the fixed projection \(\beta\). Each \(p\)-presentation will define a function \(S(pP)\). The value of the new function \(\beta - \text{ord}^{(d-1)}(G)\) at a given point \(y \in V^{d-1}\) will be the biggest value of the form \(S(pP)(y)\) among \(p\)-presentations making use of \(\beta\).

Over fields of characteristic zero, the function \(\beta - \text{ord}^{(d-1)}(G)\) coincides with the upper-semicontinuous function \(\text{ord}(\mathcal{R}_{G, \beta})\) (see \(2.3.1\)). The situation in positive characteristic is quite different, for example \(\beta - \text{ord}^{(d-1)}(G)\) is not upper-semi-continuous. Theorem \(4.6\) features some pathologies in this way.

The function in (2) is a first step in the definition of our inductive function \(v - \text{ord}^{(d-1)}\) in Section 7.

In this section we simply fix a transversal projection \(\beta\) and study different rational numbers, attached to a point, defined by choosing different transversal sections \(z = 0\). We focus here, essentially, on how the function in (1) varies for different choices of \(z\).

**Definition 4.2.** Fix \(G, \beta : V^d \longrightarrow V^{d-1}\), a \(\beta\)-section \(z\), and \(f_p(z)\) as in \(2.14\). Namely, fix a \(p\)-presentation \(pP(\beta, z, f_p(z))\) with \(f_p(z) = z^{p-r} + a_1 z^{p-r-1} + \cdots + a_{p-r}\) as in \(2.14.2\), so that \(G\) has the same integral closure as

\[\mathcal{O}_{V^d}[f_p(z)W^s, \Delta(\alpha)(f_p(z))W^s-a_1]_{1 \leq \alpha \leq p-r-1} \cap \mathcal{R}_{G, \beta} = 0.\]

Define \(S(pP)(-) : V^{d-1} \longrightarrow \mathbb{Q}\),

\[S(pP)(y) := \min_{1 \leq j \leq p^r} \left\{ \frac{\nu_y(a_j)}{j}, \text{ord}(\mathcal{R}_{G, \beta})(y) \right\},\]

called the **slope of \(G\) relative to \(pP = pP(\beta, z, f_p(z))\)** at \(y \in V^{d-1}\).

**Remark 4.3.** The function \(\text{ord}(\mathcal{R}_{G, \beta})(-) : V^{d-1} \longrightarrow \mathbb{Q}\) takes values with denominators in \(\frac{1}{n}\mathbb{Z}\), for some integer \(n > 0\). Thus the same holds for the slope function: it takes values in \(\frac{1}{p^r}\mathbb{Z}\).

Moreover, both functions take only finitely many values.
Remark 4.4. Given a $p$-presentation $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p, x}(z))$, other $p$-presentations can be defined, for example by changing the section $z$. Here, given $x_0 \in V^{(d)}$ we study conditions on $z$ for which $S_l(p\mathcal{P})(\beta(x_0)) > 0$.

The element $z$ in the $p$-presentation in Definition 4.2 defines a closed set, say $\overline{V} = \{z = 0\} \subset V^{(d)}$, which is a section of $\beta$. In particular, any point $y \in V^{(d-1)}$ can be identified with a point in $\overline{V}$, say $x \in V \subseteq V^{(d)}$, namely $x = \beta^{-1}(y) \cap \{z = 0\}$.

The value of the function $S_l(p\mathcal{P})$ at a point $y \in V^{(d-1)}$ provides information of $\mathcal{G}$, locally at $x$.

In this Remark we show that $S_l(p\mathcal{P})(y) > 0$ if and only if $\text{ord}(\mathcal{G})(x) > 0$. Here, we discuss some equivalent formulations of this condition.

Let $k(y)$ be the residue field of the local ring $\mathcal{O}_{V^{(d-1), y}}$ and let $Z$ denote the restriction of $z$ to $\beta^{-1}(y)$, the fiber over $y$. Fix, as above, $x = (\beta^{-1}(y) \cap \{z = 0\})$. So $x$ is the unique point dominating $y$ for which $z$ is a non-invertible element at $\mathcal{O}_{V^{(d), x}}$ and $z$ is of order one at this local ring.

Here, $f_{p, x}(z)$ is a global function of $V^{(d)}$, and the restriction

$$\overline{f}_{p, x}(z) : V((f_{p, x}(z))) \longrightarrow V^{(d-1)}$$

is finite and flat. On the other hand, $V((f_{p, x}(z))) \cap \beta^{-1}(y)$ can be identified with the subscheme in $\text{Spec}(k(y)[Z])$ defined by the monic polynomial,

$$\overline{f}_{p, x}(z) = Z^{p^s} + \overline{a}_1 Z^{p^{s-1}} + \cdots + \overline{a}_{p^s} \in k(y)[Z].$$

Moreover, $x \in V((f_{p, x}(z)))$ if and only if $\overline{a}_{p^s} = 0$. In fact $Z$ is the class of $z$ on the fiber.

The following are equivalent conditions for the section $z$ and the point $y$:

1. $S_l(p\mathcal{P})(y) > 0$.
2. $\overline{f}_{p, x}(z) = Z^{p^s}$ and $\text{ord}(\mathcal{R}_{\mathcal{G}, \beta})(y) > 0$.
3. $\text{ord}(\mathcal{R}_{\mathcal{G}, \beta})(y) > 0$, the induced finite map,

$$\overline{f}_{p, x}(z) : V((f_{p, x}(z))) \longrightarrow V^{(d-1)},$$

has a unique point, say $x$, dominating $y$, and $z$ is a non-invertible at $\mathcal{O}_{V^{(d), x}}$.

4. $\text{ord}(\mathcal{R}_{\mathcal{G}, \beta})(y) > 0$, $V((f_{p, x}(z))) \cap \beta^{-1}(y)$ is a unique point, say $x$, the local rings $\mathcal{O}_{V^{(d), x}}$ and $\mathcal{O}_{V^{(d-1), y}}$ have the same residue field, say $k(x) = k(y)$, and if $\{y_1, \ldots, y_s\}$ is a regular system of parameters at $\mathcal{O}_{V^{(d-1), y}}$ then $\{y_1, \ldots, y_s, z\}$ is a regular system of parameters at $\mathcal{O}_{V^{(d), x}}$.

Remark 4.5.

1. We shall prove in Proposition 4.8) that if $x \in \text{Sing}(\mathcal{G})$ (i.e., if $\text{ord}(\mathcal{G})(x) \geq 1$), all conditions in d) will hold at $y = \beta(x)$, whenever a $\beta$-transversal section $z$ is chosen with order one at $\mathcal{O}_{V^{(d), x}}$. Moreover, in such case $S_l(p\mathcal{P})(y) \geq 1$.

2. To prove that (c) implies (d) note that if (c) holds, then $Z$ divides $\overline{f}_{p, x}(z)$. As there is only one factor, then $\overline{f}_{p, x}(z) = Z^{p^s}$, and so the point $x$ must be rational over $k(y)$.

Theorem 4.6. Fix $\mathcal{G}$ and $p\mathcal{P} = p\mathcal{P}(\beta, z, f_{p, x}(z))$ as in 2.14. If $S_l(p\mathcal{P})(y) = \nu_y(a_j)/j$ for some index $j \in \{1, \ldots, p^s - 1\}$, then $S_l(p\mathcal{P})(y) = \nu_y(a_{p^s})/p^s$. In particular,

$$S_l(p\mathcal{P})(y) = \min \left\{ \frac{\nu_y(a_{p^s})}{p^s}, \text{ord}(\mathcal{R}_{\mathcal{G}, \beta})(y) \right\}.$$  

Proof. Let $n \in \{1, \ldots, p^s - 1\}$ be the smallest index for which $S_l(p\mathcal{P})(y) = \nu_y(a_n)/n$. That is,

$$\frac{\nu_y(a_i)}{i} > \frac{\nu_y(a_n)}{n} \quad \text{for} \quad i \leq n - 1 \quad \text{and} \quad \frac{\nu_y(a_{\ell})}{\ell} \geq \frac{\nu_y(a_n)}{n} \quad \text{for} \quad \ell \geq n + 1.  

Recall the definition of the $\beta$-differential operators $\Delta^{(s)}$ in 2.3. As $\mathcal{G}$ is assumed to be a $\beta$-differential algebra, then $\Delta^{(p^s-n)}(f_{p, x}(z))W^n \in \mathcal{G}$.

Note that

$$\Delta^{(p^s-n)}(f_{p, x}(z))W^n = (c_1 a_1 z^{n-1} + \cdots + c_{n-1} a_{n-1} z + a_n)W^n \in \mathcal{G}$$

for some elements $c_i \in k$ for $i = 1, \ldots, n - 1$. 


Let $\Delta^{p-n}(f_{p^n}(z))W^n$ denote the class of $\Delta^{p-n}(f_{p^n}(z))W^n$ in $\mathcal{O}_{V^{(d)}}(f_{p^n}(z))[W]$. The scheme $\mathcal{O}_{V^{(d)}}(f_{p^n}(z))[W]$ is a finite and free extension of $\mathcal{O}_{V^{(d-1)}}[W]$. The norm of the element 

$$\Delta^{p-n}(f_{p^n}(z))W^n = (c_1a_1z^{n-1} + \cdots + c_{n-1}a_{n-1}z^n)W^n$$

over $\mathcal{O}_{V^{(d-1)}}[W]$ is an element of the elimination algebra of $f_{p^n}(z)$, and hence of $\mathcal{R}_G$, see $[35]$. Denote this element by $\mathcal{G}(a_1, \ldots, a_{n-1})$. In addition, in this case $t = np^n$, and $\mathcal{G}(V_1, \ldots, V_{p^n}) \in k[V_1, \ldots, V_{p^n}]$ is a weighted homogeneous of degree $t = p^s$ provided each $V_i$ is given weight $i$.

Note that,

1. $G(a_1, \ldots, a_{n-1}) = a_1^{p^n} + \mathcal{G}(a_1, \ldots, a_{n-1})$.
2. $\mathcal{G}(a_1, \ldots, a_{n-1}) \in (a_1, \ldots, a_{n-1})$.

To check the last assertion set formally $a_1 = 0, \ldots, a_{n-1} = 0$, in which case $\Delta^{p-n}(f_{p^n}(z))W^n = a_nW^n$, which has norm $a_n^{p^n}W^{np^n}$.

Here $\mathcal{G}$ is a weighted homogeneous polynomial of degree $np^n$, and each monomial in $\mathcal{G}$ is of the form $a_1^{p^n} \cdots a_{n-1}^{p^n}$ with $\sum_{j=1}^{p^n} \alpha_j = np^n$, and $\alpha_j \neq 0$ for some $j < n$.

We claim that $\nu_p(a_1^{p^n} \cdots a_{n-1}^{p^n}) > \nu_p(a_{n-1}^{p^n}) = p^s \nu_p(a_n)$ for any monomial in $\mathcal{G}$. In fact:

$$\nu_p(a_1^{p^n} \cdots a_{n-1}^{p^n}) = \sum_{j=1}^{p^n} \alpha_j \nu_p(a_j) > \sum_{j=1}^{p^n} \alpha_j \frac{\nu_p(a_n)}{n} = np^n \nu_p(a_n) = \nu_p(a_n),$$

where the inequality follows from the hypotheses in $[4.6.1]$. In particular, $\nu_p(\mathcal{G}) > \nu_p(a_n)$.

This proves that the order of $GW^{np^n}(\mathcal{R}_G) = \frac{n\nu_p(a_{p^n})}{np^n} = \frac{\nu_p(a_{p^n})}{n}$. Hence $\text{ord}(\mathcal{R}_G) = \frac{n\nu_p(a_{p^n})}{np^n}$.

Finally, since $\text{SL}(p\mathcal{P})(y) \leq \text{ord}(\mathcal{R}_G)(y)$, it follows that $\text{SL}(p\mathcal{P})(y) = \text{ord}(\mathcal{R}_G)(y)$.

**Remark 4.7.** Let $p\mathcal{P}$ be a $p$-presentation defined in a neighborhood of a closed point $x \in V^{(d-1)}$ and assume $x \in \mathcal{P}$ for some $y \in V^{(d-1)}$. In this case,

$$\text{SL}(p\mathcal{P})(y) \leq \text{SL}(p\mathcal{P})(x).$$

Recall that $\text{SL}(p\mathcal{P})(y) = \{\nu_\alpha(a_{p^n})/p^n, \text{ord}(\mathcal{R}_G)(y)\}$. Since $p\mathcal{P}$ is defined in a neighborhood of $x$, then $\nu_\alpha(a_{p^n}) \leq \nu_\alpha(a_{p^n})$. The upper-semicontinuity of $\text{ord}(\mathcal{R}_G)(y)$ implies that $\text{ord}(\mathcal{R}_G)(y) \leq \text{ord}(\mathcal{R}_G)(x)$. Thus $\text{SL}(p\mathcal{P})(y) \leq \text{SL}(p\mathcal{P})(x)$.

**Proposition 4.8.** Fix $G$ and $\beta : V^{(d)} \rightarrow V^{(d-1)}$ together with a $p$-presentation $p\mathcal{P} = p\mathcal{P}(\beta, \tau, f_{p^n}(z))$ as in $[2.14]$

i) Suppose that $\text{SL}(p\mathcal{P})(y) > 0$ at $y \in V^{(d-1)}$ and let $x$ be the unique point in $V((f_{p^n}))$ mapping to $y$ (see Remark 4.4). Then,

$$x \in \text{Sing}(G) \text{ if and only if } \text{SL}(p\mathcal{P})(y) \geq 1.$$

ii) If $\beta^{-1}(y) \cap \text{Sing}(G) \neq \emptyset$, then $\beta^{-1}(y) \cap \text{Sing}(G)$ is a unique point, say $q$, and:

iia) If $\text{SL}(p\mathcal{P})(y) > 0$, then $q$ is the unique point in $V((f_{p^n}))$ that maps to $y$.

iib) If $\text{SL}(p\mathcal{P})(y) = 0$, then the class of $a_{p^n}$ is a $p^s$-th power in $k(y)$, say $\tau_\alpha = \alpha_{p^n}$, and the class of $a_i$ is zero for $i = 1, \ldots, p^s - 1$. Namely,

$$f_{p^n}(z) = Z^{p^n} + \alpha_{p^n} \in k(y)[Z].$$

**Proof.**

i) Fix a regular system of parameters $\{y_1, \ldots, y_s\}$ at $\mathcal{O}_{V^{(d-1)}}(y)$. In this case, $\{y_1, \ldots, y_s, z\}$ is a regular system of parameters at $\mathcal{O}_{V^{(d)}}(x)$, so $f_{p^n}(z) = z^{p^n} + a_1z^{p^n-1} + \cdots + a_{p^n} \in m_x^{p^n}$ if and only if $a_i \in m_y^n$. The equivalence now follows straightforward.

ii) Note that $\text{Sing}(G) \subset V((f_{p^n}))$. Moreover, $\text{Sing}(G) \subset \mathcal{F}_{p^n}$, the closed set of points of multiplicity $p^n$ of the hypersurface $V((f_{p^n}))$. A theorem of Zariski states that $\beta$ induces a set theoretical bijection: $\beta : \mathcal{F}_{p^n} \rightarrow \beta(\mathcal{F}_{p^n})$, and matching points have the same residue field. This proves property P0 in $[2.11]$ (see [7], 8.4). In particular $\beta^{-1}(y) \cap \text{Sing}(G)$ is a unique point.
iii) As \( q \in V((f_{\rho^n})) \), the assertion follows from the equivalence of a) and c) in \( \text{def:z:adap:x} \).

ib) In this case, \( y = \beta(q) \in \text{Sing}(\mathcal{R}_{G,\beta}) \) (see \( \text{P3} \)) in \( \text{2.3} \), so \( \text{ord}(\mathcal{R}_{G,\beta})(y) \geq 1 \). On the other hand, as \( q \in F_{\rho^n}: k(q) = k(y) \), which together with Theorem \( \text{4.6} \) imply that

\[
\overline{f_{\rho^n}}(z) = Z^{\rho^n} + \overline{a_{\rho^n}} \in k(y)[Z],
\]

in \( \text{4.4} \), and that this purely inseparable polynomial is a \( \rho^n \)-th power of a monic polynomial of degree 1, say \( Z^{\rho^n} + \overline{a_{\rho^n}} = (Z + \alpha)^{\rho^n} \) in \( k(y)[Z] \).

\[ \blacksquare \]

**Corollary 4.9.** Fix two \( p \)-presentations for \( G \) on \( V^{(d)} \). Say, \( p\mathcal{P} \), defined in terms of \( \beta: V^{(d)} \rightarrow V^{(d-1)} \), a \( \beta \)-section \( z \) and a monic polynomial \( f_{\rho^n}(z) \); and another \( p \)-presentation \( p\mathcal{P}' \) defined by \( \beta': V^{(d)} \rightarrow V^{(d-1)}, a \beta' \)-section \( z' \), and a polynomial \( f'_{\rho^n}(z') \).

Fix points \( y \in V^{(d-1)}, y' \in V^{(d-1)} \), and assume that:

1) \( S\beta\beta(p\mathcal{P})(y) > 0 \) and \( S\beta\beta(p\mathcal{P}')(y') > 0 \).
2) There is a point \( q \in V^{(d)} \) which is the unique point mapping to both (see \( \text{4.4} \)). Namely, \( \beta(q) = y \) and \( \beta'(q) = y' \).

Then,

\[ S\beta\beta(p\mathcal{P})(y) \geq 1 \text{ if and only if } S\beta\beta(p\mathcal{P}')(y) \geq 1. \]

In fact, this condition holds when both \( y \) and \( y' \) are image of a point \( q \in \text{Sing}(G) \).

**4.10.** In what follows we fix the simple algebra \( G \) on a smooth scheme \( V^{(d)} \), together with a transversal morphism \( \beta: V^{(d)} \rightarrow V^{(d-1)} \), and define different \( p \)-presentations of the form \( p\mathcal{P}(\beta, z, f_{\rho^n}(z)) \), for different choices of sections \( z \).

Let us denote by \( F(G, \beta) \) the set of all such \( p \)-presentations. Namely,

\[ F(G, \beta) = \{ p\mathcal{P}(\beta, z, f_{\rho^n}(z)) \} \text{ for which \( \text{2.14} \) holds} \]

There is a natural notion of restriction on local presentations. Let \( U^{(d-1)} \) be an open subset in \( V^{(d-1)} \), and set \( U^{(d)} \) as the inverse image of \( U^{(d-1)} \). There is a natural restriction of \( G \), say \( G|_{\beta} \), of \( \beta \), say \( \beta|_{\beta} : U^{(d)} \rightarrow U^{(d-1)} \), and of the \( p \)-presentation \( p\mathcal{P} \), so that \( \text{2.14} \) holds at the restriction.

For each open \( U^{(d-1)} \subset V^{(d-1)} \) we take all \( p \)-presentations \( F(G|_{\beta}, \beta|_{\beta}) \).

Finally, fix a point \( y \in V^{(d-1)} \), and set

\[ F(G, \beta, y) = \bigcup F(G|_{\beta}, \beta|_{\beta}) \]

where the union is over all restrictions \( U^{(d-1)} \subset V^{(d-1)} \) containing \( y \).

**Definition 4.11.** Fix \( \beta: V^{(d)} \rightarrow V^{(d-1)} \) and \( G \) as in \( \text{2.3} \). Define the \( \beta \)-order at \( y \in V^{(d-1)} \) as

\[ \beta - \text{ord}^{(d-1)}(G)(y) = \max_{p\mathcal{P} \in F(G, \beta, y)} \{ S\beta\beta(p\mathcal{P})(y) \}. \]

5. Well adapted \( p \)-presentations.

**5.1.** Assume that \( \beta - \text{ord}^{(d-1)}(G)(y) > 0 \), and let \( p\mathcal{P} \) be a \( p \)-presentation involving \( \beta \). Here we sketch a criteria which will allow us to decide when, for a given point \( y \in V^{(d-1)} \), a \( p \)-presentation \( p\mathcal{P} \) is such that \( \beta - \text{ord}^{(d-1)}(G)(y) = S\beta\beta(p\mathcal{P})(y) \). So well adapted \( p \)-presentations at a singular point will ultimately be giving us the value of the inductive function at such point (see \( \text{7.3} \)).

The starting point of this discussion grows from the observation that when \( S\beta\beta(p\mathcal{P})(y) > 0 \), the following cases can occur:

A) \( S\beta\beta(p\mathcal{P})(y) = \text{ord}(\mathcal{R}_{G,\beta})(y) \)

B) \( S\beta\beta(p\mathcal{P})(y) = \frac{\nu_{a_{\rho^n}}(p\mathcal{P})}{\rho^n} < \text{ord}(\mathcal{R}_{G,\beta})(y) \) (see Theorem \( \text{4.6} \), and

B1) \( \frac{\nu_{a_{\rho^n}}(p\mathcal{P})}{\rho^n} \notin \mathbb{Z}_{>0} \).

B2) \( \frac{\nu_{a_{\rho^n}}(p\mathcal{P})}{\rho^n} \in \mathbb{Z}_{>0} \) and \( \text{In}_{a_{\rho^n}} \) is not a \( \rho^n \)-th power at \( \text{Gr}_{a_{\rho^n}}(\mathcal{O}_{V^{(d-1)}, y}) \).

B3) \( \frac{\nu_{a_{\rho^n}}(p\mathcal{P})}{\rho^n} \in \mathbb{Z}_{>0} \) and \( \text{In}_{a_{\rho^n}} \) is a \( \rho^n \)-th power at \( \text{Gr}_{a_{\rho^n}}(\mathcal{O}_{V^{(d-1)}, y}) \).
We shall prove that a new \( p \)-presentation \( pP' \) can be defined with the condition \( Sl(pP')(y) > Sl(pP)(y) \), only in case B3). This leads to the cleaning process developed in Proposition 5.3.

This cleaning process relies on suitable changes of the transversal section \( z \). The finiteness of this process will be address in Remark 5.6. In Proposition 5.7, we show that these change of \( z \), in this cleaning process, can be done so as to be compatible with the notion of monomial contact; a property that will be used in the proof of Main Theorem 2.

Proposition 5.8 will be useful in the study of \( p \)-presentations and its compatibility with monomial transformations.

5.2. Let \( pP = pP(β, z, f_\nu(z)) \) be a \( p \)-presentation and fix \( y ∈ V^{(d−1)} \). Suppose \( Sl(pP)(y) > 0 \). We study changes of the \( p \)-presentation \( pP \) obtained by changing the \( β \)-section \( z \) by another of the form \( uz + α \). Here \( u \) and \( α \) are in \( O_{V^{(d−1)},y} \) and \( u \) is a unit, so the change is a composition of \( z_1 = uz \) and \( z_2 = z + α \). The function \( u \) is a unit (invertible) at any point in an open neighborhood of \( y \), say \( U^{(d−1)} \). This is to be interpreted as a new \( p \)-presentation, defined at the restriction of both \( G \) and \( V^{(d)} \) over \( U^{(d−1)} = β^{-1}(U^{(d−1)}) \) as in 4.10.

A change of the form \( z_1 = uz \), set \( pP_1 \) with

\[
 f'(z_1) = u^\nu f_{\nu}(z) = z_1^\nu + au_{\nu}z_1^{\nu−1} + \cdots + u^\nu a_{\nu} ∈ O_{V^{(d−1)},y}(z).
\]

Clearly, \( Sl(pP)(y) = Sl(pP_{1})(y) \) and also Cases A), B1), B2), and B3) in 5.1 are preserved. Henceforth we study only changes of the form \( z' = z + α \). At \( O_{V^{(d−1)},y}(z) = O_{V^{(d−1)},y}(z') \),

\[
 (5.2.1) \quad f'_{\nu}(z) = f_{\nu}(z') = z^\nu + a_1z^{\nu−1} + \cdots + a_\nu ∈ O_{V^{(d−1)},y}(z'),
\]

Define, as before, a new presentation, say \( pP' \), with these data at a suitable restriction to a neighborhood of \( y \).

Proposition 5.3. (Cleaning process). Fix the setting and notation as above, where the function \( β − ord^{(d−1)}(G)(y) > 0 \) and where \( pP \) is such that \( Sl(pP)(y) > 0 \). Assume that \( Sl(pP)(y) = \frac{τ_{\nu}(α_{p})}{p^\nu} < ord(R_{G,β})(y) \). There will be a change of the form \( z' = z + α \), defining a new presentation \( pP' \) as in 5.2, so that

\[
 Sl(pP)(y) < Sl(pP')(y)
\]

if and only if case B3) holds in 5.1 for \( pP \).

Proof. Theorem 4.6 ensures that if \( Sl(pP)(y) = \frac{τ_{\nu}(α_{p})}{p^\nu} < ord(R_{G,β})(y) \), then

\[
 (5.3.1) \quad \frac{τ_{\nu}(α_{p})}{p^\nu} < \frac{τ_{\nu}(α_{i})}{p^i} \quad \text{for } i = 1, \ldots, p^\nu − 1.
\]

Set \( z' = z + α \) as above. If \( τ_{\nu}(α) < \frac{τ_{\nu}(α_{p})}{p^\nu} \) then the previous inequalities applied to 5.2.2 show that \( τ_{\nu}(a'_{\nu}) = τ_{\nu}(α_{p}) \), so \( Sl(pP')(y) < Sl(pP)(y) \).

Assume that \( τ_{\nu}(α) ≥ \frac{τ_{\nu}(α_{p})}{p^\nu} \). For each summand in 5.2.2 of the form \( a_1α^\nu−i, i = 1, \ldots, p^\nu − 1, \)

\[
 (5.3.2) \quad τ_{\nu}(a_1α^\nu−i) = (p^\nu − i)τ_{\nu}(α) + τ_{\nu}(a_1) > (p^\nu − i)τ_{\nu}(a_1) + \frac{τ_{\nu}(α_{p})}{p^\nu} ≥ (p^\nu − i) \frac{τ_{\nu}(a_{p})}{p^\nu} + i \frac{τ_{\nu}(α_{p})}{p^\nu} = τ_{\nu}(a_{p}).
\]

Therefore 5.2.2 can be expressed as

\[
 (5.3.3) \quad a'_{\nu} = α^\nu + A + a_{p},
\]

where \( τ_{\nu}(α^\nu) ≥ τ_{\nu}(a_{p}) \) and \( τ_{\nu}(A) > τ_{\nu}(a_{p}) \).

On the other hand,

\[
 a'_{\nu} = Δ^{(p^\nu−n)}(f_{\nu})(α) = c_1α^{n−1}a_1 + \cdots + c_{n−1}αa_{n−1} + a_n,
\]

where \( c_j ∈ k \) for \( j = 1, \ldots, n−1 \).
For each summand of the form \( a_j \alpha_n^{-j}, j = 1, \ldots, n, \)
\[ \nu_g(a_j \alpha_n^{-j}) = (n - j) \nu_g(\alpha) + \nu_g(a_j) > 0 \]
(5.3.4) 
\[ > (n - j) \nu_g(\alpha) + j \frac{\nu_g(a_{p_r})}{p^r} \geq (n - j) \frac{\nu_g(a_{p_r})}{p^r} + j \frac{\nu_g(a_{p_r})}{p^r} = \frac{n\nu_g(a_{p_r})}{p^r}. \]
In particular, \( \nu_g(a_{p_r}) > \frac{\nu_g(a_{p_r})}{p^r} \).

One can easily check now that if B1) holds, then \( f'_{p_r}(z') = z^{-p^r} + a'_1 z^{-p^r-1} + \cdots + a'_{p_r} \) in (5.2.1) is also in case B1), and \( SL(pP)(y) = SL(pP')(y). \)

The same arguments apply if B2) holds, namely \( f'_{p_r}(z') \) is also in case B2), and \( SL(pP)(y) = SL(pP')(y). \)

On the contrary, in case B3) it suffices to choose \( \alpha \) so that \( \nu_g(\alpha p^r + a_{p_r}) > \nu_g(a_{p_r}) \) to get \( SL(pP)(y) < SL(pP')(y). \)

Remark 5.4. In Proposition 5.3 we assumed that \( \nu \beta - ord^{(d-1)}(G)(y) > 0 \), and \( pP \) was such that \( SL(pP)(y) > 0 \). Suppose that \( \nu \beta - ord(G)(y) > 0 \) and \( SL(pP)(y) = 0 \). In this case, \( \nu_g(a_{p_r}) = 0 \) and \( f'_{p_r}(z) = z^{-p^r} + \pi_{p_r} \) is a \( p^r \)-th power.

Namely, \( \alpha_i = 0, i = 1, \ldots, r - 1 \) (see Theorem 4.6).

If \( z^{-p^r} + \pi_{p_r} = (Z + \delta)^{p^r} \in k(y)[Z] \) for some \( \delta \in k(y) \), then a change of the form \( z' = z + \alpha \), where \( \alpha \in O_{V^{(d-1)}_y} \) maps to \( \delta \) in \( k(y) \), will define a new presentation, say \( pP' \), and \( SL(pP')(y) > 0 \). This is always the case when \( y \) is the image of a point \( x \in Sing(G) \) (see Proposition 4.8 iib).

Def: a well-adapted \( \mu \)-presentation at a point preserves the compatibility with \( O_{V^{(d-1)}_y} \) as in 3.9.

Remark 5.6. Finiteness of the cleaning process.

When Case B3) occurs, \( \nu_g(a_{p_r}) = \ell p^r \) for some integer \( \ell \geq 1 \), and \( \text{In}_g(a_{p_r}) = F^{\ell p^r} \) for some homogeneous polynomial \( F \) of degree \( \ell \) at \( \text{Gr}_y(O_{V^{(d-1)}_y}) \). In this case, we define \( z' = z + \alpha \) for some \( \alpha \in O_{V^{(d-1)}_y} \) such that \( \text{In}_g(\alpha) = F \). Thus \( SL(pP)(y) < SL(pP')(y). \)

If this new presentation \( pP' = pP'(\beta, z', f'_{p_r}(z')) \) is within Case A), B1) or B2) then stop. If, on the contrary, \( f'_{p_r}(z') \) is in Case B3), then
i) \( \nu_g(a_{p_r}) = \ell' p^r \) with \( \ell' > \ell \), and
ii) \( \text{In}_g(a_{p_r}) = (F')^{\ell' p^r} \) for some homogeneous element \( F' \) of degree \( \ell' \) at \( \text{Gr}_y(O_{V^{(d-1)}_y}) \).

So again we can set \( z'' = z' + \alpha' \) for some \( \alpha' \in O_{V^{(d-1)}_y} \) with \( \text{In}_g(\alpha') = F' \); and \( SL(pP')(y) < SL(pP'')(y) \). This shows that with this procedure of modification of the transversal section, locally over \( y \), the slope will increase every time we come to Case B3). Finally, Remark 4.3 guarantees that Case B3) can arise only finitely many times throughout this procedure. So ultimately the procedure leads to a well-adapted \( \mu \)-presentation.

Proposition 5.7. Assume that \( pP \) is compatible with a monomial algebras \( O_{V^{(d)}_y}[MW^*] \) as in 3.9.

Then the cleaning process to obtain a well-adapted \( \mu \)-presentation at a point preserves the compatibility with \( O_{V^{(d)}_y}[MW^*] \).

Proposition 5.8. Simultaneous adaptation.

Let \( pP = pP'(\beta, z, f_{p_r}(z)) \) be a \( \mu \)-presentation compatible with a monomial algebra \( MW^* \). Let \( y \) and \( x \) be points in \( V^{(d-1)} \), so that \( x \in C = \overline{y} \), and assume that \( O_{C,x} \) is regular. Then,
A) It can be assumed that \( pP \) is well-adapted to \( \mathcal{G} \) at \( y \), defined in a neighborhood of \( x \), and compatible with \( MW^* \).
B) There is a \( \mu \)-presentation which is well-adapted to \( \mathcal{G} \) both at \( y \) and \( x \), and also compatible with \( MW^* \).
Proof of Propositions 5.7 and 5.8. Once we fix a p-presentation, say \( pP = pP(\beta, z, f_{p^e}) \) and a point \( y \in V^{(d-1)} \), cleaning applies either in the case of Remark 5.4 or of Remark 5.6. In both cases, the condition is given by the fact that \( I_{y_p}(a_{p^e}) \) is a \( p^e \)-th power, cleaning consists in finding \( \alpha \in \mathcal{O}_{V^{(d)}, y} \) so that \( (I_{y_p}(\alpha))^{p^e} = I_{y_p}(a_{p^e}) \).

We face now the proof of Proposition 5.8. Fix a p-presentation \( pP \) locally defined at \( \mathcal{O}_{V^{(d-1)}, x} \). Set \( p \subset \mathcal{O}_{V^{(d-1)}, x} \) the regular prime ideal corresponding to \( y \) (so that localization at \( p \) is \( \mathcal{O}_{V^{(d-1)}, y} \)). Cleaning is necessary at \( \mathcal{O}_{V^{(d-1)}, y} \) if and only if \( I_{y_p}(a_{p^e}) \) is a \( p^e \)-th power in \( gr_y(\mathcal{O}_{V^{(d-1)}, y}) \). Since \( x \) is a smooth point at \( y \), then \( gr_y(\mathcal{O}_{V^{(d-1)}, y}) \) is a regular ring and \( I_{y_p}(a_{p^e}) \in gr_y(\mathcal{O}_{V^{(d-1)}, y}) \) and by localization at \( gr_y(\mathcal{O}_{V^{(d-1)}, y}) \), we get \( I_{y_p}(a_{p^e}) \in gr_y(\mathcal{O}_{V^{(d-1)}, y}) \). Hence \( I_{y_p}(a_{p^e}) \) is a \( p^e \)-th power if and only if \( I_{y_p}(a_{p^e}) \) is a \( p^e \)-th power. This ensures that the element \( \alpha \), used in the cleaning process at \( y \), can be chosen to be an element in \( \mathcal{O}_{V^{(d-1)}, x} \) and hence, the cleaning process at \( y \) can be done so as to obtain a new \( p \)-presentation with coefficients in \( \mathcal{O}_{V^{(d-1)}, x} \).

Now, we prove that the the cleaning process at \( x \) is possible without affecting the fact that the presentation is already well-adapted at \( y \).

Consider a regular system of parameters \( \{y_1, \ldots, y_l, y_{l+1}, \ldots, y_{d-1}\} \) at \( \mathcal{O}_{V^{(d-1)}, x} \) so that \( p = \langle y_1, \ldots, y_l \rangle \). There are two cases to consider, case \( SL(pP)(y) = \text{ord}(\mathcal{R}_{\mathcal{G}, \beta})(y) \) and case \( SL(pP)(y) = \nu_y(a_{p^e}) < \text{ord}(\mathcal{R}_{\mathcal{G}, \beta})(y) \). Assume that the latter case holds and set \( \nu_y(a_{p^e}) = \frac{n}{p^e} \). Theorem 4.6 ensures that \( \nu_y(a_{p^e}) < \nu_y(a_1) \) for \( j = 1, \ldots, p^e - 1 \).

At the completion, \( a_{p^e} \) is a sum of monomials of the form \( y_1^{\alpha_1} \cdots y_l^{\alpha_l} y_{l+1}^{\alpha_{l+1}} \cdots y_{d-1}^{\alpha_{d-1}} \) with \( \alpha_1 + \cdots + \alpha_\ell \geq n \). We can identify \( I_{y_p}(a_{p^e}) \) with a sum of some of these terms. If there is an element \( \alpha \in \mathcal{O}_{V^{(d-1)}, x} \) so that \( (I_{y_p}(\alpha))^{p^e} = -I_{y_p}(a_{p^e}) \), then \( \alpha \in \langle y_1, \ldots, y_l, \frac{1}{p^e} \rangle \) and, in particular, \( \nu_y(\alpha) \geq \frac{n}{p^e} \).

A change of variables of the form \( z \mapsto z + \alpha \) is so that the new independent coefficient is of the form:

\[ a'_{p^e} = \alpha^{p^e} + a_1 \alpha^{p^e-1} + \cdots + a_{p^e}, \]

where \( \nu_y(a_{p^e}) > \frac{n}{p^e} - (p^e - 1) \frac{n}{p^e} = n, \) so \( \nu_y(a_{p^e}) > \min\{\nu_y(a_{p^e}), \nu_y(a_{p^e})\} \geq n \) and the new presentation is still well-adapted at \( y \).

In case \( SL(pP)(y) = \text{ord}(\mathcal{R}_{\mathcal{G}, \beta})(y) = \frac{n}{p^e} \), the same arguments lead to the existence of \( \alpha \in \mathcal{O}_{V^{(d-1)}, x} \) so that \( \nu_y(\alpha) = \frac{n}{p^e} \) which again ensures that the change of variables \( z \mapsto z + \alpha \) (needed for the cleaning process at \( x \)) does not affect the slope at \( y \).

To prove Proposition 5.7 just notice that such change can be achieved with \( \alpha W \in \mathcal{O}_{V^{(d-1)}}, [MW^*] \) for \( \alpha W \in \mathcal{O}_{V^{(d-1)}}, [MW^*] \) (\( n = 1, \ldots, p^e \)).

6. Transformations of \( p \)-presentations.

6.1. In the previous sections, invariants were defined in terms of \( p \)-presentations. In this section we discuss a form of compatibility of these invariants when applying a monoidal transformation along a smooth center \( C \).

The starting point will be a notion of transformation of \( p \)-presentations in 6.2. A monoidal transformation is defined by a smooth center \( C \) and introduces an exceptional hypersurface, say \( H \). The aim of the section is to relate the value of the slope \( SL \) at the generic point of the exceptional hypersurface with the value of \( SL \) at the generic point of the center \( C \) (see Proposition 6.6). This result will be an essential ingredient for the proofs of Main Theorems in this work.

6.2. Take a \( p \)-presentation \( pP = pP(\beta, z, f_{p^e}) \) of a simple \( \beta \)-differential algebra \( \mathcal{G} \) on \( V^{(d)} \). Namely, a smooth morphism \( V^{(d)} \to V^{(d-1)}, \) a \( \beta \)-section \( z \) and a monic polynomial \( f_{p^e}(z) = z^{p^e} + a_1 z^{p^e-1} + \cdots + a_{p^e} \). Assume that \( C \) is a closed and smooth center, and that \( z \in I(C) \). Locally at a closed point \( x \in C \), there is a regular system of parameters \( \{z, x_1, \ldots, x_{d-1}\} \) and, after restriction to a suitable
neighborhood of $x$, $I(C) = (z, x_1, \ldots, x_\ell)$. Consider the commutative diagram

\[
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\pi_C} & \mathcal{G}_1 \\
V^{(d)} & \xleftarrow{\delta} & V_1^{(d)} \\
\downarrow{\beta} & & \downarrow{\beta_1} \\
V^{(d-1)} & \xrightarrow{\pi_{\beta(C)}} & V_1^{(d-1)} \\
\mathcal{R}_{\mathcal{G}, \beta} & \xrightarrow{(\mathcal{R}_{\mathcal{G}, \beta})_1 = \mathcal{R}_{\mathcal{G}_1, \beta_1}} & \\
\end{array}
\]

and recall that $\text{Sing}(\mathcal{G}_1) \subset V_1^{(d)}$ can be covered by affine charts $U_{x_i}$,

\[U_{x_i} = \text{Spec} \left( \mathcal{O}_{V^{(d)}} \left( \frac{z}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_i}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_\ell}{x_i}, \frac{x_{\ell+1}}{x_i}, \ldots, x_{d-1} \right) \right),\]

for $i = 1, \ldots, \ell$; and also $V_1^{(d-1)}$ is covered by charts $U'_{x_i}$

\[U'_{x_i} = \text{Spec} \left( \mathcal{O}_{V^{(d-1)}} \left( \frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_i}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_\ell}{x_i}, \frac{x_{\ell+1}}{x_i}, \ldots, x_{d-1} \right) \right).\]

Note that the strict transform of $z$, say $z_1 = \frac{z}{x_i}$, is a transversal parameter for $U_{x_i} \to U'_{x_i}$.

The hypersurface defined by $f_{p'}$ at $V^{(d)}$ has multiplicity $p'$ along points of $C$. Let

\[f^{(1)}_{p'}(z_1) = z_1^{p'} + a_1^{(1)} z_1^{p'-1} + \cdots + a_r^{(1)}\]

denote the strict transform of $f_{p'}(z)$. These data define, locally, a $p$-presentation of $\mathcal{G}_1$, say $pP_1 = pP_1(\beta_1, z_1, f^{(1)}_{p'})$, which we call the transform of $pP = pP(\beta, z, f_{p'})$.

**Remark 6.3.** (1) In the previous discussion we have assumed that $z \in I(C)$. If $C$ is irreducible, this condition will hold for any $p$-presentation $pP(\beta, z, f_{p'})$ well-adapted at $\xi_{\beta(C)}$, (the generic point of $\beta(C)$ in $V^{(d-1)}$). In fact, after a suitable restriction to a neighborhood of the closed point $x \in C$, the simultaneous cleaning procedure at $\beta(x)$ and $\xi_{\beta(C)}$ will allow us to modify $z$ so that $z \in I(C)$ (see Proposition 5.3).

(2) Note that the exponent $p'$, which is the degree of the monic polynomial, is also preserved by transformations of $p$-presentations.

**Remark 6.4.** A point $y \in V_1^{(d-1)}$ has an image in $V^{(d-1)}$, say $\pi_{\beta(C)}(y)$. If $y$ is not in the exceptional locus of $\pi_{\beta(C)}$, there is an open neighborhood, say $U$, of $\pi_{\beta(C)}(y)$ over which both $\pi_C$ and $\pi_{\beta(C)}$ are the identity map. Thus the restriction of both $p$-presentations $pP$ and $pP_1$ to $U$ coincide.

In particular, if $pP$ is well-adapted to $\mathcal{G}$ at $\pi_{\beta(C)}(y)$, then the same holds for $pP_1$ at $y$. Therefore,

\[\text{Sl}(pP_1)(y) = \text{Sl}(pP)(\pi_{\beta(C)}(y))\]

whenever $y \in V_1^{(d-1)}$ is not on the exceptional locus.

**Remark 6.5.** Let $x \in \text{Sing}(\mathcal{G})$ be a closed point so that $\tau_{\mathcal{G}, x} = 1$ and let $C$ be a permissible center containing $x$. Fix a $p$-presentation, say $pP(\beta, z, f_{p'})$ and denote by $y$ the generic point of $\beta(C)$. Assume that $pP$ is simultaneously well-adapted at $\beta(x)$ and $y$. Then, $x' \in \{ z_1 = 0 \}$ for any $x' \in \text{Sing}(\mathcal{G}_1)$ mapping to $x$, where $z_1$ denotes the strict transform of $z$.

**Proposition 6.6.** Let $C$ be a permissible center passing through a closed point $x \in \text{Sing}(\mathcal{G})$ and assume that $\tau_{\mathcal{G}, x} = 1$. Fix a $p$-presentation $pP(\beta, z, f_{p'})$. Let $y$ denote the generic point of $\beta(C)$ and assume that $pP$ is well-adapted to $\mathcal{G}$ at $\beta(x)$ and at $y$. Define a monomial transformation with center $C$. Then the transform $pP_1$ is well-adapted to $\mathcal{G}_1$ at $\xi_H$, (the generic point of the exceptional hypersurface $H \subset V^{(d-1)}$). Moreover,

\[\text{Sl}(pP_1)(\xi_H) = \text{Sl}(pP)(y) - 1.\]
Proof. Set

\[
\begin{align*}
V(d) & \xrightarrow{\pi_c} V_1(d) \\
\beta & \downarrow \\
V(d-1) & \xrightarrow{\pi_{\beta}(c)} V_1(d-1) \\
y = \xi_{\beta(C)} & \xrightarrow{} \xi_H
\end{align*}
\]

where \( H \) is the exceptional hypersurface, and

\[
f_{p^e}(z_1) = z_1^{p^e} + a_1^{(1)}z^{p^e-1} + \cdots + a_{p^e}^{(1)}
\]
is the strict transform of \( f_{p^e}(z) \). At points of \( U_{x_1}, z_1 = \frac{1}{x_1} \) and the coefficients \( a_n^{(1)} \) factor as

\[
a_n^{(1)} = \nu_{\xi_H}(a_n) - n a_n = x_n^{r_n} a_n',
\]

where \( a_n' \) denotes the strict transform of \( a_n \) and \( r_n = \nu_{\xi_H}(a_n) \), for \( n = 1, \ldots, p^e \). Different cases can arise under these assumptions, we classify them as in 6.1:

(A) Suppose that \( Sl(pP)(y) = \) ord\((R_{G,\beta})(y)\) and, in particular, that \( \frac{\nu_{\xi_H}(a_n)}{p^e} \geq \) ord\((R_{G,\beta})(y)\). At the points of \( \text{Sing}(\mathcal{G}_1) \cap U_{x_1}, \)

\[
\frac{\nu_{\xi_H}(a_n^{(1)})}{p^e} = \nu_{\xi_H}(a_n) - 1 \geq \text{ord}(R_{G,\beta})(y) - 1 = \text{ord}((R_{G,\beta})_1)(\xi_H).
\]

Thus \( pP_1 \) is well-adapted to \( \mathcal{G}_1 \) at \( \xi_H \).

(B) Suppose that

\[
\text{Sl}(pP)(y) = \nu_{\xi_H}(a_n^{p^e}) < \text{ord}(R_{G,\beta})(y).
\]

(B.1) Assume now that \( \nu_{\xi_H}(a_n^{p^e}) \notin \mathbb{Z}_{>0} \). In this case, \( \nu_{\xi_H}(a_n) > p^e \) and in addition \( \nu_{\xi_H}(a_j) > j \) for \( j = 1, \ldots, p^e - 1 \). In particular, \( \nu_{\xi_H}(a_j) > j \) for \( j = 1, \ldots, p^e \) and hence \( \text{In}_{\xi}(f_{p^e}) = Z^{p^e} \).

Lemma 6.5 (2) applies so \( \text{Sing}(\mathcal{G}_1) \cap H_{i+1} \subset \{z_1 = 0\} \). Under these assumptions,

\[
\frac{r_n^{p^e}}{p^e} = \nu_{\xi_H}(a_n^{p^e}) - 1 < \frac{\nu_{\xi_H}(a_j)}{j} - 1 = \frac{r_j}{j}
\]

for \( j = 1, \ldots, p^e - 1 \), and \( \frac{r_n^{p^e}}{p^e} \notin \mathbb{Z}_{>0} \), so locally at any closed point \( x' \in \text{Sing}(\mathcal{G}_1) \) mapping to \( x \), \( pP_1 = pP_1(\beta_1, z_1, f_{p^e}^{(1)}) \) is of the form B1) and therefore well-adapted to \( \mathcal{G}_1 \) at \( \xi_H \).

(B.2) Suppose that \( \nu_{\xi_H}(a_n^{p^e}) = r \in \mathbb{Z}_{>0} \) and \( t_{G,\xi} = 1 \). The singular locus at any exceptional point mapping to \( x \) is contained in the strict transform of \( z \) as indicated in 6.5. Consider \( \text{In}_{\beta(C)}(a_{p^e}) \in \text{Gr}_{\xi_{\beta(C)}}(\mathcal{V}(U_{x-1})) = \mathcal{O}_{\beta(C)}[X_1, \ldots, X_\ell] \), and set

\[
\text{In}_{\beta(C)}(a_{p^e}) = \sum_{|\alpha|=rp^e} b_\alpha M^\alpha,
\]

which, by assumption, is not a \( p^e \)-th power.

(B.2.a) First assume that \( \text{In}_{\beta(C)}(a_{p^e}) \notin \mathcal{O}_{\beta(C)}[X_1^{p^e}, \ldots, X_\ell^{p^e}] \). In this case, there is a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_\ell) \) so that and with at least two integers which are not multiple of \( p^e \), say for example \( \alpha_1, \alpha_2 \notin 0 \) \((p^e)\) and \( b_\alpha \neq 0 \).

We claim now that \( a_{p^e}^\alpha \) restricted to the exceptional hypersurface \( x_1 = 0 \), say \( a_{p^e}^\alpha \), is not a \( p^e \)-th power. In fact, the assumption implies that \( a_{p^e}^\alpha \) is not \( p^e \)-th power. This ensures that \( pP_1 \) is in the case B2) and hence \( pP_1 \) is well-adapted to \( \mathcal{G}_1 \) at \( \xi_H \).

(B.2.b) Suppose now that any \( M^\alpha \) is a \( p^e \)-th power whenever \( |\alpha| = rp^e \). Notice here that

\[
\text{Sl}(pP)(y) = \frac{\nu_{\xi_H}(a_{p^e})}{p^e} = r, \text{ so by assumption some } b_\alpha \text{ is not a } p^e \text{-th power.}
\]

With the same notation as before, consider the monoidal transformation with center \( C \). Then \( a_{p^e}^\alpha \) is not a \( p^e \)-th power. So again \( pP_1 \) is well-adapted at \( \xi_H \).
7. On the two Main Theorems

7.1. Fix a smooth scheme $V^{(d)}$ and a simple algebra $G$ which we assume to be an absolute differential algebra. This ensures that $G$ is a $\beta$-differential algebra for any smooth transversal morphism $\beta : V^{(d)} \to V^{(d-1)}$. It is under this last condition that a function $\beta - \text{ord}^{(d-1)}(G) : V^{(d-1)} \to \mathbb{Q}$ was defined in [4.11]. The same holds for any other transversal $\beta' : V^{(d)} \to V^{(d-1)}$.

A sequence (3.1.1) of permissible transformations of $G$ induces two diagrams

\[
\begin{aligned}
G & \xrightarrow{\pi_{c_1}} G_1 & \cdots & \xrightarrow{\pi_{c_r}} G_r \\
V^{(d)} & \xrightarrow{\beta} V_1^{(d)} & \cdots & \xrightarrow{\beta} V_r^{(d)} \\
V^{(d-1)} & \xrightarrow{\pi_{\beta(c_1)}} V_1^{(d-1)} & \cdots & \xrightarrow{\pi_{\beta(c_r)}} V_r^{(d-1)} \\
\mathcal{R}_{G, \beta} & \xrightarrow{(\mathcal{R}_{G, \beta})_1} (\mathcal{R}_{G, \beta})_r & \cdots & \xrightarrow{(\mathcal{R}_{G, \beta})_r} (\mathcal{R}_{G, \beta})_{r-1}
\end{aligned}
\]

Theorem 7.2. (Main Theorem 1). Assume that the previous setting holds. Then, for any point $q \in \text{Sing}(G_r)$,

$$\beta_r - \text{ord}(G_r)(\beta(q)) = \beta_r - \text{ord}(G_r)(\beta'(q)).$$

Corollary 7.3. The previous result enables us to define a function along $\text{Sing}(G_r)$:

$$v - \text{ord}^{(d-1)}(G_r)(-): \text{Sing}(G_r) \to \mathbb{Q}.$$

7.4. Recall that the exceptional locus of the composite map $V^{(d)} \hookrightarrow V^{(d)}$ in (3.1.1), say $\{H_1, \ldots, H_r\}$, is a set of hypersurfaces at $V_r^{(d)}$ and it is assumed that the union has only normal crossings.

We now attached to the sequence (3.1.1) a monomial algebra supported on the exceptional locus:

$$\mathcal{M}_r W^s = \mathcal{O}_{V^{(d)}} [I(H_1)^{h_1} \cdots I(H_r)^{h_r} W^s],$$

with exponents $h_i \in \mathbb{Z}_{\geq 0}$ defined so that:

$$q_{H_i} := \frac{h_i}{s} = v - \text{ord}^{(d-1)}(G_{(i-1)})(\xi_{C_i}) - 1$$

where $\xi_{C_i}$ denotes the generic point of each center $C_i$ ($i = 1, \ldots, r$).

Here $s$ is a positive integer so that $\{q_{H_1}, \ldots, q_{H_r}\} \subseteq \mathbb{Q}$. As Rees algebras are considered up to integral closure, $\mathcal{M}_r W^s$ is independent of the choice of $s$ and will be called the tight monomial algebra of $G_r$ or the tight monomial algebra defined by (3.1.1).

Theorem 7.5. (Main Theorem 2). Fix a sequence of permissible transformations as (3.1.1). Let $\mathcal{M}_r W^s$ denote the tight monomial algebra defined in (3.1.1). Then, at any closed point $x \in \text{Sing}(G_r)$, $\mathcal{M}_r W^s$ has monomial contact with $G_r$, i.e., there is a $\beta_r$-transversal section $z$ of order one at $\mathcal{O}_{V^{(d)}, x}$ for which

$$G_r \subset (z)W \oplus \mathcal{M}_r W^s.$$
We begin this part by showing that the inequality

\[ v - \text{ord}^{(d-1)}(G_r)(x) \geq \text{ord}(\mathcal{M}_r W^*)(x) \]

holds at any closed point \( x \in \text{Sing}(G_r) \). The main objective is to study the case in which equality is achieved at any closed point of \( \text{Sing}(G_r) \). This will be called strong monomial case in Definition 8.4 and we prove that:

1. The strong monomial case is stable under transformations (Proposition 8.12).
2. It parallels the so called monomial case in characteristic zero. Namely that if \( G_r \) is in the strong monomial case, then a combinatorial resolution leads to an enlargement of \( G \) (Theorem 8.13).

Remark 8.2. Let \( pP(\beta_r, z, f_{p^e}) \) be a \( p \)-presentation compatible with \( \mathcal{M}_r W^* \) and well-adapted to \( G_r \) at \( x = \beta_r(x) \ (x \in \text{Sing}(G_r)) \). We denote \( x = \beta_r(x) \) along this section. In this case, \( z \) must be an element of order one at the local ring \( \mathcal{O}_{V(x)} \) and \( f_{p^e}(z) = z^{p^e} + a_1 z^{p^{e-1}} + \cdots + a_{p^e} \), where \( a_j w^j \in \mathcal{M}_r W^* \) for \( j = 1, \ldots, p^e \). In addition, \( (\mathcal{R}_G)_{r} \subset \mathcal{M}_r W^* \).

We now prove (8.1.2) for any \( \mathcal{P}(\beta_r, z, f_{p^e}) \) as above. It follows from the previous discussion that \( \text{ord}((\mathcal{R}_G)_{r}(x)) \geq \text{ord}(\mathcal{M}_r W^*)(x) \) and \( \frac{\text{ord}(a_{p^e})}{p^e} \geq \text{ord}(\mathcal{M}_r W^*)(x) \) for \( j = 1, \ldots, p^e \). In particular,

\[ v - \text{ord}^{(d-1)}(G_r)(x) = \min \left\{ \frac{\text{ord}(a_{p^e})}{p^e}, \text{ord}((\mathcal{R}_G)_{r}(x)) \right\} \geq \text{ord}(\mathcal{M}_r W^*)(x). \]

8.3. We shall say that a Rees algebra is within the monomial case when its elimination algebra is monomial, as stated in Theorem 3.7. We shall assume here that \( G_r \) is in the monomial case. Namely, that \( (\mathcal{R}_G)_{r} = \mathcal{N}_r W^* \) is a monomial algebra. Without lost of generality fix \( s \in \mathbb{Z} \) as in (7.4), so

\[ \mathcal{N}_r W^* = I_1 H_1^{\alpha_1} \cdots I_h H_h^{\alpha_h} W^* \quad \text{and} \quad \mathcal{M}_r W^* = I_1 H_1^{h_1} \cdots I_h H_h^{h_h} W^* \quad \text{(see 7.4),} \]

and note that the monomial \( \mathcal{M}_r \) divides \( \mathcal{N}_r \) (i.e., \( \alpha_i \geq h_i \) for any \( i = 1, \ldots, r \)).

Definition 8.4. \( G_r \) is said to be within the strong monomial case at a closed point \( x \in \text{Sing}(G_r) \) if

\[ v - \text{ord}^{(d-1)}(G_r)(x) = \text{ord}(\mathcal{M}_r W^*)(x). \]

We say that \( G_r \) is within the strong monomial case if this condition holds at any closed point \( x \in \text{Sing}(G_r) \).

The following provides a characterization of this case.

Theorem 8.5. (Characterization of the strong monomial case). Fix a \( p \)-presentation \( pP(\beta_r, z, f_{p^e}) \) well-adapted to \( G_r \) at \( x = \beta_r(x) \) for a closed point \( x \in \text{Sing}(G_r) \) and compatible with \( \mathcal{M}_r W^* \). The algebra \( G_r \) is in the strong monomial case at \( x \) if and only if one of the following conditions holds in an open neighborhood, either

1. \( (\mathcal{R}_G)_{r} = \mathcal{M}_r W^* \), or
2. The algebras \( \mathcal{O}_{V(\beta_r)}[a_{p^e} W^e] \) and \( \mathcal{O}_{V(\beta_r)}[\mathcal{M}_r W^*] \) have the same integral closure.

The first condition holds if and only if \( v - \text{ord}^{(d-1)}(G_r)(x) = \text{ord}((\mathcal{R}_G)_{r})(x) \).

Proof. (i) Fix \( x \in \text{Sing}(G_r) \) and denote by \( E_x = \{ H_i, \ldots, H_{i_x} \} \) the set of exceptional hypersurfaces containing \( x \). Let \( \Lambda_x = \{ i_1, \ldots, i_{\ell_x} \} \) be the indexes-set of \( E_x \).

If \( v - \text{ord}^{(d-1)}(G_r)(x) = \text{ord}((\mathcal{R}_G)_{r})(x) \) and \( G_r \) is within the strong monomial case at \( x \in \text{Sing}(G_r) \), then

\[ \sum_{i \in \Lambda_x} \alpha_i = \text{ord}((\mathcal{R}_G)_{r})(x) = \text{ord}(\mathcal{M}_r W^*)(x) = \sum_{i \in \Lambda_x} h_i. \]

Since \( \alpha_i = h_i \) for any \( i \), then \( h_i = \alpha_i \) for any \( i \in \Lambda_x \). So \( \mathcal{M}_r W^* = (\mathcal{R}_G)_{r} \) at \( x \).

Conversely, if \( \mathcal{M}_r W^* = (\mathcal{R}_G)_{r} \) locally at \( x \), then the inequality \( v - \text{ord}^{(d-1)}(G_r)(x) \geq \text{ord}(\mathcal{M}_r W^*)(x) \) in (8.1.2) must be an equality since \( v - \text{ord}^{(d-1)}(G_r)(x) = \min \left\{ \frac{\text{ord}(a_{p^e})}{p^e}, \text{ord}((\mathcal{R}_G)_{r})(x) \right\} \).

(ii) Suppose now that \( v - \text{ord}^{(d-1)}(G_r)(x) = \frac{\text{ord}(a_{p^e})}{p^e} < \text{ord}((\mathcal{R}_G)_{r})(x) \). We claim that for any index \( j = 1, \ldots, p^e - 1 \), the coefficients are of the form \( a_j^{p^e} = \mathcal{M}_r \cdot b_j \) where \( b_j \) is not a unit. In fact, assume that for one index \( j_0 < p^e \) it holds that \( a_{j_0}^{p^e} = \mathcal{M}_r \cdot u \) where \( u \) is a
Remark 8.6. Let $\mathcal{G}_r$ be in the strong monomial case at the closed point $x \in \text{Sing}(\mathcal{G}_r)$. Then

1. In case (i), it can be proven that the transversal parameter $z$ defines a hypersurface of maximal contact. In particular, there exists a open neighborhood of $x$ where $(\mathcal{R}_{\mathcal{G},r})_r = N_r W^s = M_r W^s$.

2. In case (ii), the monomial algebra can be described as $\mathcal{M}_r W^p = (\mathcal{M}_s W^p)^{\mathcal{G}_r}$ (i.e., $s = p^e$) where $\mathcal{M}_r$ is not a $p^e$-th power.

This last assertion follows since $pP^e$ is adapted to $\mathcal{G}_r$ at $x$ and it is compatible with $\mathcal{M}_r W^p$ (see Section 4).

That is, $\text{In}_{\mathcal{G}_r}(a_{p^e})$ is not a $p^e$-th power and $a_{p^e} W^p = M_r W^p$ ($a_{p^e} = M_r$).

**Lemma 8.7.** Let $\mathcal{G}_r$ be in the strong monomial case, and set $(\mathcal{R}_{\mathcal{G},r})_r = N_r W^s$. Fix $y \in \text{Sing}(\mathcal{G}_r)$.

(A) If $v - \text{ord}^{d-1}(\mathcal{G}_r)(y) = \text{ord}((\mathcal{R}_{\mathcal{G},r})_r)(\beta_r(y))$, then

(A1) there is a dense open set $U \subset \overline{\mathcal{G}}$ so that $v - \text{ord}^{d-1}(\mathcal{G}_r)(x) = \text{ord}((\mathcal{R}_{\mathcal{G},r})_r)(x)$ and $N_r W^s = M_r W^s$ at any $x \in U$ (where $x = \beta_r(x)$).

(A2) ord $((\mathcal{R}_{\mathcal{G},r})_r)(\beta_r(y)) = \text{ord}((\mathcal{M}_r W^s)\beta_r(y))$.

(B) If $v - \text{ord}^{d-1}(\mathcal{G}_r)(y) < \text{ord}((\mathcal{R}_{\mathcal{G},r})_r)(\beta_r(y))$, then at each closed point $x \in \overline{\mathcal{G}}$ (in $\text{Sing}(\mathcal{G}_r)$):

$$v - \text{ord}^{d-1}(\mathcal{G}_r)(x) = \frac{\nu_s(a_{p^e})}{p^e} < \text{ord}((\mathcal{R}_{\mathcal{G},r})_r)(x)$$

for any $pP^e(\beta, z, f_{p^e})$ well-adapted at $x$ and $\beta_r(y)$.

**Proof.** (A) Note that (A2) follows from (A1) (since $N_r W^s = M_r W^s$ at any $x \in U$).

Let $E_y = \{H_{j_1}, \ldots, H_{j_t}\}$ denote the exceptional exponents containing $y$ with index set $\Lambda_y = \{j_1, \ldots, j_t\}$. Consider a closed point $x$ so that $x \in \overline{\mathcal{G}} \setminus \bigcup_{i \in \Lambda_y} H_i$. Fix a $p$-presentation $pP^e(\beta, z, f_{p^e})$ well-adapted to $\mathcal{G}_r$ both at $\beta_r(x)$ and $\beta_r(y)$. Note that, for such $x$, and since $(\mathcal{R}_{\mathcal{G},r})_r$ is a monomial algebra supported on the exceptional components

$$\text{ord}((\mathcal{R}_{\mathcal{G},r})_r)(x) = \text{ord}((\mathcal{M}_{\mathcal{G}_r})_r)(\beta_r(y)) \leq \frac{\nu_{\beta_r(y)}(a_{p^e})}{p^e} \leq \frac{\nu_s(a_{p^e})}{p^e}.$$

So in this case, $v - \text{ord}^{d-1}(\mathcal{G}_r)(x) = \text{ord}((\mathcal{R}_{\mathcal{G},r})_r)(x)$ and since $\mathcal{G}_r$ is in the strong monomial case, $\text{ord}(\mathcal{M}_r W^s)(x) = v - \text{ord}^{d-1}(\mathcal{G}_r)(x) = \text{ord}((\mathcal{R}_{\mathcal{G},r})_r)(x)$. Finally, argue as in the Proof of Theorem 8.5 to conclude that $a_{i_s} = h_i$ for all $i \in \Lambda_y$. Hence $(\mathcal{G}_{\mathcal{G}_r})_r = N_r W^s = M_r W^s$. In particular,

$$\text{ord}((\mathcal{R}_{\mathcal{G},r})_r)(\beta_r(y)) = \text{ord}(\mathcal{M}_r W^s)(\beta_r(y)).$$

(B) Fix a closed point $x \in \overline{\mathcal{G}}$ and a $p$-presentation $pP^e(\beta, z, f_{p^e})$ well-adapted to $\beta_r(x)$ and $\beta_r(y)$. Assume on the contrary that $v - \text{ord}^{d-1}(\mathcal{G}_r)(x) = \text{ord}((\mathcal{R}_{\mathcal{G},r})_r)(x) = \text{ord}(\mathcal{M}_r W^s)(x)$. Then,
Corollary 8.8. Let \( G_r \) be within the strong monomial case at a closed point \( x \in \Sing(G_r) \). Let \( y \in \Sing(G_r) \) be a point so that \( x \in \overline{y} \). Then,

\[
v - \text{ord}^{(d-1)}(G_r)(y) = \text{ord}(M_r W^s)(\beta_r(y)).
\]

Proof. (A) When \( v - \text{ord}^{(d-1)}(G_r)(y) = \text{ord}((R_{C, \beta})_r) (\beta_r(y)) \) the assertion is (A2) in Lemma 8.7.

(B) Assume that \( v - \text{ord}^{(d-1)}(G_r)(y) = \text{ord}((R_{C, \beta})_r)(\beta_r(y)) \). By Lemma 8.7 (B), at a closed point \( x \in \overline{y} \), the algebras generated by \( a_{p^e} W^{p^s} \) and \( M_r W^s \) have the same integral closure, so

\[
\frac{\nu_{p^e}(a_{p^e})}{p^e} = \text{ord}(M_r W^s)(\beta_r(y)).
\]

Remark 8.9. Let \( G_r \) be in the strong monomial case. Then the functions \( v - \text{ord}^{(d-1)}(G_r)(-) \) and \( \text{ord}(M_r W^s)(-) \) take the same value at any point of \( \Sing(G_r) \). In particular, whenever \( G_r \) is in the strong monomial case, the function \( v - \text{ord}^{(d-1)}(G_r) \) is upper-semicontinuous.

Corollary 8.11. Assume that \( G_r \) is in the strong monomial case. Let \( C \subset \Sing(G_r) \) be an irreducible permissible center. Let \( V_r^{(d)} \subset C \subset V_{r+1}^{(d)} \) be the monoidal transformation with center \( C \). Denote by \( M_r W^s \) the transform of \( M_r W^s \) and by \( M_{r+1} W^s \) the tight monomial algebra in \( V_{r+1}^{(d)} \). Then,

\[
M_{r+1} W^s = M_r W^s + 1.
\]

Proof. By definition of the tight monomial algebras,

\[
M_{r+1} W^s = I(H_1)^{h_1} \ldots I(H_r)^{h_r} I(H_{r+1})^{h_{r+1}} W^s,
\]

where \( H_j \) are the strict transforms of the previous exceptional hypersurfaces \( j = 1, \ldots, r \) and \( H_{r+1} \) is the new exceptional hypersurface. Recall that \( h_{r+1} = q_{H_{r+1}} \cdot s \) where \( q_{H_{r+1}} = v - \text{ord}(G_r)(\xi_C) - 1 \). On the other hand,

\[
M_r W^s = I(H_1)^{h_1} \ldots I(H_r)^{h_r} I(H_{r+1})^{h_{r+1}} W^s,
\]

where \( \gamma = \text{ord}(M_r W^s)(y) - s \) and \( y \) denote the generic point of \( C \). Corollary 8.8 asserts that in the strong monomial case \( v - \text{ord}^{(d-1)}(G_r)(\xi_C) = \text{ord}(M_r W^s)(\xi_C) \). Thus,

\[
M_r W^s = M_{r+1} W^s.
\]

Proposition 8.12. \( (\tau = 1\text{-stability of the strong monomial case}) \). Assume that \( G_r \) is within the strong monomial case. Let \( C \) be a permissible center, and let \( x \in C \) be a closed point so that \( \tau_{G_r, x} = 1 \). Consider the monoidal transformation of center \( C \), say \( V_{r+1}^{(d)} \subset C \subset V_{r+1}^{(d)} \). Then the transform of \( G_r \), say \( G_{r+1} \), is within the strong monomial case.

Proof. Fix a \( p \)-presentation \( pP(\beta, z, f_{p^e}) \) well-adapted to \( G_r \) at \( x = \beta_r(x) \), and \( \xi_{\beta_r(C)} \) and compatible with \( M_r W^s \). Set \( f_{p^e}(z) = z^{p^e} + a_1 z^{p^e-1} + \cdots + a_{p^e} \).

Proposition 6.6 asserts that \( pP_1 \) is well-adapted to \( G_{r+1} \) at \( \xi_{H_{r+1}} \). The main part of the proof is devoted to show that \( pP_1 \) is also well-adapted to \( x' = \beta_{r+1}(x') \) for any closed point \( x' \in \Sing(G_{r+1}) \) mapping to \( x \). Set

\[
\tau_{G_r, x} = 1, \text{ then } x' \in \{ z_1 = 0 \} \text{ for any closed point } x' \in \Sing(G_{r+1}) \cap H_{r+1} \text{ mapping to } x.
\]

In particular, if \( pP_1 \) is well-adapted to \( G_{r+1} \) at \( x' \), then \( v - \text{ord}^{(d-1)}(G_{r+1})(x') = \min\{ \frac{\nu_{p^e}(a_{p^e})}{p^e}, \text{ord}((R_{C, \beta})_{r+1})(x') \} \), where \( a_{p^e}^{(1)} \) denotes the independent term of the strict transform \( f_{p^e}^{(1)}(z_1) \).
As \( G_r \) is in the strong monomial case at \( x \), then either \( M_r W^s = (R_{G_r})_r = NW^s \) or the algebras spanned by \( a_{p^e} W^{p^e} \) and \( M_r W^s \) have the same integral closure. Corollary 8.11 asserts that the transform of \( M_r W^s \) is equal to \( M_{r+1} W^s \). Hence, in \( V_{r+1}^{[d]} \), conditions (i) or (ii) in Theorem 8.5 are preserved. Thus, \( G_{r+1} \) is also in the strong monomial case at \( x' \).

Fix a closed point \( x' \in \text{Sing}(G_{r+1}) \cap H_{r+1} \) mapping to \( x \) (i.e., \( \pi(x') = x \)). Recall that, under the hypothesis \( H_{r+1} \), Lemma 6.6 asserts that \( x' \in \{ z_1 = 0 \} \), where \( z_1 \) denotes the strict transform of \( z \). We prove now that \( pP_1 \) is well-adapted to \( G_{r+1} \) at \( x' \).

Firstly, reduce the tight monomial algebra \( M_r W^s \) to be of the form

\[
I(H_1)^{h_1} \cdots I(H_r)^{h_r} W^s \quad \text{with } 0 < h_i < s.
\]

In order to achieved this, it suffices to consider a finite sequence of permissible transformations with centers of codimension two.

We divide the proof in the following cases:

1. Assume that \( v - \text{ord}^{(d-1)}(G_r)(x) = \text{ord}((R_{G_r})_r)(x)(\leq \text{ord}(M_r W^s)(x)) \). By Theorem 8.5, \( (R_{G_r})_r = N_r W^s = M_r W^s \) and, in particular, \( \text{ord}((R_{G_r})_r)(\xi) = \text{ord}(M_r W^s)(\xi) \). Note that \( M_{r+1} W^s = (R_{G_r})_{r+1} \), and again by Theorem 8.5, \( G_{r+1} \) is in the strong monomial case (in particular \( pP_1 \) is adapted to \( G_{r+1} \) at \( x' \)).

2. Suppose now that \( v - \text{ord}^{(d-1)}(G_r)(x) = \frac{\nu^{(a_{p^e})}}{p^e} < \text{ord}((R_{G_r})_r)(x) \). In this case, the algebras spanned by \( a_{p^e} W^{p^e} \) and \( M_r W^s \) have the same integral closure (see Theorem 8.5). In particular, we can take \( s = p^e \) and suppose that \( a_{p^e} W^{p^e} = u M_r W^{p^e} \), where \( u \) is a unit.

The equality \( a_{p^e} W^{p^e} = u M_r W^{p^e} \) implies that

\[
\frac{\nu_{\beta_r(C)}(a_{p^e})}{p^e} = \text{ord}(M_r W^s)(\xi_{\beta_r(C)}((\leq \text{ord}(R_{G_r})_r)(\xi_{\beta_r(C)})),
\]

so \( v - \text{ord}^{(d-1)}(G_r)(\xi) = \frac{\nu_{\beta_r(C)}(a_{p^e})}{p^e} \).

2.A If \( \frac{\nu_{\beta_r(C)}(a_{p^e})}{p^e} \notin \mathbb{Z}_{>0} \), then \( h_{r+1} = \nu_{\ell_{r+1}}(a_{p^e}) = \nu_{\beta_r(C)}(a_{p^e}) - p^e(\neq 0 \mod p^e) \). Here \( \text{In}_{W'}(a_{p^e}^{(1)}) \) is not a \( p^e \)-th power since \( h_{r+1} \neq 0 \mod p^e \). Hence \( pP_1 \) is adapted to \( G_{r+1} \) at \( x' \).

Before we address the proof of the remaining cases, we introduce some notation: Fix a regular system of parameters at the local regular ring \( O_{V^{[d]}_r} \), say \( \{ z, x_1, \ldots, x_{d-1} \} \), such that the tight monomial algebra is locally generated by a monomial in \( x_1, \ldots, x_r \), and the permissible center is \( I(C) = (z, x_1, \ldots, x_r, y_1, \ldots, y_m) \), where \( \ell \leq r \) and \( y_j = x_{r+j} \) for \( j = 1, \ldots, m \). In such setting,

\[
a_{p^e} = u x_1^{h_1} \cdots x_{r+1}^{h_{r+1}} \cdots x_r^{h_r}.
\]

2.B Assume that \( \frac{\nu_{\beta_r(C)}(a_{p^e})}{p^e} \in \mathbb{Z}_{>0} \) and that \( \ell < r \). In this case, one can check that \( \text{In}_{W'}(a_{p^e}^{(1)}) \) is not a \( p^e \)-th power. In fact at each chart

\[
a_{p^e}^{(1)} = u x_1^{h_1} \cdots x_{r+1}^{h_{r+1}} \cdots x_r^{h_r} \quad \text{in the } U_{x_1} \text{-chart},
\]

\[
a_{p^e}^{(1)} = u y_1^{h_1} \cdots y_{r+1}^{h_{r+1}} \cdots y_r^{h_r} \quad \text{in the } U_{y_1} \text{-chart},
\]

and \( 0 < h_{r+1} < p^e \) (i.e., \( h_{r+1} \neq 0 \mod p^e \)).

2.C Assume that \( \frac{\nu_{\beta_r(C)}(a_{p^e})}{p^e} \in \mathbb{Z}_{>0} \) and \( \ell = r \).

Note here that \( \ell = r \geq 2 \), since \( M_r W^{p^e} \) is not a \( p^e \)-th power, and that \( h_1 + \cdots + h_r \equiv 0 \mod p^e \). We divide the proof that \( pP_1 \) is well-adapted at \( x' \) in two cases:

2.C.1 Firstly suppose that \( \frac{\nu_{\beta_r(C)}(a_{p^e})}{p^e} < \text{ord}((R_{G_r})_r)(\xi_{\beta_r(C)}). \) After a finite number of monoidal transformations, at centers of codimension 2, we can assume that \( h_{r+1} = 0 \). Thus, the independent term, say \( a_{p^e}^{(1)} W^{p^e} \), is

\[
a_{p^e}^{(1)} = u (x_2^{r_2})^{h_2} \cdots (x_r^{r_r})^{h_r} \quad \text{in the } U_{x_1} \text{-chart},
\]

\[
a_{p^e}^{(1)} = u (y_1^{r_1})^{h_1} \cdots (y_r^{r_r})^{h_r} \quad \text{in the } U_{y_1} \text{-chart}.
\]
Both cases are analogous, so it suffices to consider the problem at the $U_{x_{1}}$-chart. The main different with the previous case appears when $a_{p_{\ell}}^{(1)}$ is a unit. We address now this case. Let
\[ f_{p_{\ell}}^{(1)} = x_{1}^{p_{\ell}} + a_{1}^{(1)} x_{1}^{p_{\ell} - 1} + \cdots + a_{p_{\ell}}^{(1)} \]
be the strict transform of $f_{p_{\ell}}$. By hypothesis, we assume that $\frac{\nu_{\xi_{\beta}(\mathbb{C})}}{p_{\ell}}(a_{p_{\ell}}) < \text{ord}((\mathcal{R}_{\mathbb{C}, \beta})_{r})(\xi_{\beta}(\mathbb{C}))$, so $\text{ord}((\mathcal{R}_{\mathbb{C}, \beta})_{r+1})(\xi_{\beta}(\mathbb{C})) > 0$, and hence $x_{1}$ divides $a_{j}^{(1)}$ for $j = 1, \ldots, p_{\ell} - 1$ (see Theorem 4.6).

We claim that if $x' \in \{x_{1} = 0\} \cap \{\frac{x_{2}}{x_{1}} \neq 0, \ldots, \frac{x_{r}}{x_{1}} \neq 0\}$, then $x' \not\in \text{Sing}(\mathcal{G}_{1})$. Let $f_{p_{\ell}}^{(1)} = x_{1}^{p_{\ell}} + a_{p_{\ell}}^{(1)}$ be the restriction to $x_{1} = 0$, where $p_{\ell}^{(1)} = p_{\ell}^{(1)} h_{j_{1}} \cdots \frac{x_{r}}{x_{1}} h_{j_{r}}$. Consider the Taylor expansion of the monomial given by $\pi_{p_{\ell}}^{(1)}$ at any closed point of $U_{x_{1}} \cap \{x_{1} = 0\}$. The differential operators $\Delta^{\alpha}$ that appears in this expansion, are the differential operators relative to the ring
\[ \mathcal{O}_{\beta}(\mathbb{C}) \left[ \frac{\pi_{1}}{x_{1}}, \ldots, \frac{\pi_{r}}{x_{1}}, \frac{y_{1}}{x_{1}}, \ldots, \frac{y_{m}}{x_{1}} \right] . \]

Note here that $\pi \in \mathcal{O}_{\beta}(\mathbb{C})$, so in particular, $\Delta^{\alpha}(\pi_{p_{\ell}}^{(1)}) = \pi \Delta^{\alpha}((\frac{x_{2}}{x_{1}})^{h_{2}} \cdots (\frac{x_{r}}{x_{1}})^{h_{r}})$.

Since it is assumed that $h_{j} < p_{\ell}$, it follows that
\[ \Delta^{\alpha_{j}}(\pi_{p_{\ell}}^{(1)}) = \pi(\frac{x_{2}}{x_{1}})^{h_{j}} \cdots (\frac{x_{r}}{x_{1}})^{h_{j}}(\frac{x_{j+1}}{x_{1}})^{h_{j+1}} \cdots (\frac{x_{r}}{x_{1}})^{h_{r}} , \]
for $\alpha_{j} = (0, \ldots, h_{j}, \ldots, 0) \in \mathbb{N}^{d-1}$. Moreover, $\Delta^{\alpha_{j}}(\pi_{p_{\ell}}^{(1)})(x') = 0$ for $j = 2, \ldots, r$ if and only if $x' \in \left\{ \frac{x_{r}}{x_{1}} = 0 \right\}$. In this case, $\text{Im}_{x'}(\alpha_{p_{\ell}}^{(1)})$ is not a $p_{\ell}$-th power and $p_{\ell}^{(1)}$ is well-adapted at $x'$.

(2.C.2) Suppose finally that $\frac{\nu_{\xi_{\beta}(\mathbb{C})}}{p_{\ell}}(a_{p_{\ell}}) = \text{ord}((\mathcal{R}_{\mathbb{C}, \beta})_{r})(\xi_{\beta}(\mathbb{C}))$ within the case $\ell = r$. Since $\text{ord}(\mathcal{M}_{r} W^{s})(\xi_{\beta}(\mathbb{C})) = \text{ord}(\mathcal{N}_{r} W^{s})(\xi_{\beta}(\mathbb{C}))$, then $h_{i} = \alpha_{i}$ for $i = 1, \ldots, r$.

By assumption, $\text{ord}(\mathcal{M}_{r} W^{s})(x) < \text{ord}(\mathcal{N}_{r} W^{s})(x)$. Thus, there must be an exceptional hypersurface, say $H$, so that $x \in H$, $H$ is not a component of $\mathcal{M}_{r}$, and $H$ is a component of $\mathcal{N}_{r}$. That is, $H \neq H_{j}$ for $j = 1, \ldots, r$ and $\text{ord}(\mathcal{N}_{r} W^{s})(\xi_{H}) > 0$.

After a finite number of monoidal transforms at centers of codimension 2, the new exceptional hypersurface is not a component of the independent term, say $a_{p_{\ell}}^{(1)}$, which takes similar expressions as those in (2.C.1) both in $U_{x_{1}}$-charts or in $U_{x_{r}}$-charts. Moreover, the strict transform of $H$ is a component of the elimination algebra.

Taking restriction of the equation to $H$, the setting is analogous as that treated in case (2.C.1). Hence, the same argument applies to show that $p_{\ell}^{(1)}$ is well-adapted at $x'$.

There is a well-known notion of resolution of simple Rees algebras by decreasing induction in $\tau$. The following Theorem, under the assumption that once $\mathcal{G}_{r}$ is in the strong monomial case, leads to a reduction of the resolution problem as it guarantees that $\tau$ increases.

**Theorem 8.13.** Let $\mathcal{G}_{r}$ be within the strong monomial case. Then, any combinatorial resolution of $\mathcal{M}_{r} W^{s}$ can be lifted to a sequence of transformations of $\mathcal{G}_{r}$, say
\[ (8.13.1) \quad V_{r}^{(d)}(x) \xrightarrow{\pi_{r+1}} V_{r+1}^{(d)}(x) \xrightarrow{\pi_{r+2}} \cdots \xrightarrow{\pi_{N}} V_{N}^{(d)}(x) \]
for any $x' \in \text{Sing}(\mathcal{G}_{r})$ is a closed point so that $\tau_{\mathcal{G}_{r}, x'} = 1$, then $\pi^{-1}(x') \cap \text{Sing}(\mathcal{G}_{N}) = \emptyset$. Hence, $\tau_{\mathcal{G}_{N}, x'} \geq 2$ for any $x' \in \text{Sing}(\mathcal{G}_{N})$.

**Proof.** Recall that $\mathcal{M}_{r} W^{s}(\subset \mathcal{O}_{\mathcal{G}_{r}}^{(d)})$ is the pull-back of a monomial algebra, say $\mathcal{M}_{r} W^{s}$ again, in $\mathcal{O}_{\mathcal{G}_{r}}^{(d-1)}$. What we mean here is that a combinatorial resolution of $\mathcal{M}_{r} W^{s}$ in dimension $d - 1$ can be lifted to a permissible sequence in dimension $d$.

Fix a closed point $x \in \text{Sing}(\mathcal{G}_{r})$ so that $\tau_{\mathcal{G}_{r}, x} = 1$. Proposition 8.12 ensures that after a permissible sequence of monoidal transformation as (8.13.1), the transform $\mathcal{G}_{N}$ is in the strong monomial case.
In particular, \( v - \text{ord}(d-1)(G_t)(x') = \text{ord}(M_\ast W^\ast)(x') \) for any closed point \( x' \in \text{Sing}(G_t) \) mapping to \( x \). Moreover, by assumption \( \text{ord}(M_\ast W^\ast)(x') < 1 \). That is, \( \pi^{-1}(x) \cap \text{Sing}(G_t) = \emptyset \).

\( \Box \)

Part III. Embedded Resolution of 2-dimensional schemes.

Here we address the proof of embedded resolution of 2-dimensional schemes. We discuss the case of a hypersurface embedded in a 3-dimensional smooth scheme. The extension to resolution of arbitrary 2-dimensional schemes follows easily from our invariants.


We take as starting point a diagram (3.2.1) in the setting of Theorem 3.7, in which \( V_r^{(d-1)} \) is a 2-dimensional smooth scheme, say \( V_r^{(2)} \). Let us fix notation as in 8.3, i.e., \( (R_{\beta,r})_r = I(H_1)^{\alpha_1} \ldots I(H_r)^{\alpha_r} W^\ast \) for some integers \( \alpha_i \geq 0 \). Assume, in addition, that the tight monomial algebra, defined in (7.4.1), fulfills the conditions in 8.12.1, that is, \( M_r W^\ast = I(H_1)^{h_1} \ldots I(H_r)^{h_r} W^\ast \) with \( 0 \leq h_i < s \). Recall that this condition can be achieved by blowing-up permissible centers of codimension 2. In this case, we shall say that \( M_r W^\ast \) is reduced.

This assumption guarantees that \( \text{Sing}(M_r W^\ast) \subset V_r^{(2)} \) has no components of codimension 1, and hence neither does \( \beta_r(\text{Sing}(G_r)), \) which is, therefore, a finite set of closed points.

Throughout this section, we will always assume that the tight monomial algebra is reduced, that is, every quadratic transformation will be followed by a finite procedure of blow-ups at centers of codimension 2 so as to guarantee that the new tight monomial algebra is reduced.

We shall fix a \( p \)-presentation \( pP(\beta, z, f_p(z)) \) well-adapted to \( G_r \) at any isolated closed point \( x \in \text{Sing}(G_r) \). Here, \( f_p(z) = z^p + a_1 z^{p-1} + \cdots + a_p \in O_{V_r}(z) \). Recall that \( v - \text{ord}(d-1)(G_r)(x) = \min \{ \frac{v_{x}(a_p)}{p^p}, \text{ord}((R_{\beta,r})_r(x)) \} \) for \( x = \beta_r(x) \). We shall fix this notation (\( x = \beta_r(x) \)) along this section.

Since \( (R_{\beta,r})_r \), is, by assumption, monomial, then the singular locus of \( G_r \) is entirely included in a union of the exceptional hypersurfaces. We begin by fixing a suitable stratification of the exceptional hypersurfaces in the 2-dimensional scheme \( V_r^{(2)} \) and the successive quadratic transformations defined over it.

**Definition 9.2.** Let us say that an isolated closed point \( x \in \text{Sing}(G_r) \) is green if

\[
v - \text{ord}(d-1)(G_r)(x) = \text{ord}((R_{\beta,r})_r(x)).
\]

\( x \) is said to be purple if

\[
v - \text{ord}(d-1)(G_r)(x) = \frac{v_{x}(a_p)}{p^p} < \text{ord}((R_{\beta,r})_r(x)).
\]

In the same manner, exceptional hypersurfaces will be distinguished in terms of the exponents that arise in \( M_r W^\ast \) and \( (R_{\beta,r})_r \).

**Definition 9.3.** We will say that the exceptional hypersurface \( H_i \) is green if \( h_i = \alpha_i \) and \( H_i \) is purple if \( h_i < \alpha_i \).

As new quadratic transformations over \( V_r^{(2)} \) are defined, new exceptional hypersurfaces will be introduced, each of which will be either green or purple as above.

**Remark 9.4.** Note that the closed point \( x \in \text{Sing}(G_r) \) is green (purple) if and only if the exceptional hypersurface introduced by a quadratic transformation at \( x \) is green (purple).

Our stratification will be defined only in the union of purple hypersurfaces. In fact, if a closed point \( x \) is only included in green hypersurfaces, then \( \text{ord}(R_{\beta,r})(x) = \text{ord}(M_r W^\ast)(x) \) and a transversal parameter can be chosen as a hypersurface of maximal contact (see Remark 8.6 (1)).

9.5. Stratification at level \( r \).

According to the previous discussion, we draw attention to those points of \( \text{Sing}(G_r) \) with images in the union of purple lines. As these points are isolated, we may assume, after restriction, that the point is unique. The following two situations can arise:

- There exists a unique hypersurface \( H_1 \) so that \( x \in H_1 \).
• \(x\) is an intersection point of two hypersurfaces, say \(H_1\) and \(H_2\).

In the first case, we define an unique stratum which can be identified with an affine open subset of \(H_1\) containing the point \(x\), say \(A^1\). In the second case, we stratify \(H_1 \cup H_2\) in two strata: one affine open subset of \(H_1\) containing \(x\) and the affine subset \(H_2 \setminus \{x\}\).

Further quadratic transformations will be defined over level \(r\) (over \(V_r^{(2)}\)). This will introduce new exceptional components, say \(H_j = \mathbb{P}^1\). We are to stratify the union of new components which are purple (Definition 9.3). Each stratum, will be either

- an affine line \(A^1\), or
- a point.

A zero-dimensional stratum, a point, will be called an \textit{infinitesimal stratum}. These zero-dimensional strata will always arise as intersection of two exceptional hypersurfaces: one green and the other purple. However, the intersection point of a purple and a green line is not necessarily a zero-dimensional stratum. These strata will appear and be treated in detail in 9.6 Case C).


Case A) In this case, we consider a quadratic transformation at a point \(x\), which is purple, and is not a zero-dimensional stratum. The following sub-cases can arise:

Here, the stratification of the quadratic transformation at the point will not introduce new zero-dimensional components.

Along this section, horizontal lines will denote new exceptional components. Let \(H_1\) denote again the strict transform of \(H_1\). The new stratification is defined as follows:

A unique 1-dimensional stratum \(A^1\) is introduced after the quadratic transformation. This new affine stratum is defined as \(A^1 = \mathbb{P}^1 \setminus \{q\}\) where \(q = H_1 \cap \mathbb{P}^1\).

Case B) Here, we study the case of a quadratic transformation at a green point \(x\) which is in a purple line \(H_1\) and is not a zero-dimensional stratum. This can occur in the following sub-cases:
The new exceptional hypersurface introduced is green, and the new stratification is defined as before by taking strict transform of the previous stratum defined over $H_1$:

![Picture B1](image1)

![Picture B2](image2)

**Case C)** In this case, we consider a quadratic transformation with center a green point $x$ located in a purple line, say $H_1$, which is an intersection of two purple lines $H_1$ and $H_2$. As in the previous cases, we assume that the point is not a zero-dimensional stratum.

![Picture C](image3)

Note that the new (horizontal) exceptional line is green and the stratification is defined by
- The strict transform of the previous strata.
- A new zero-dimensional stratum defined as the intersection of $H_2$ and the new exceptional green line.

Note that this case, depicted below, introduces a zero-dimensional stratum $Q$.

![New C-stratification after the quadratic transformation](image4)

9.7. Quadratic transformations at infinitesimal points.
**Case D1**) Here we study a quadratic transformations at a purple point $Q$ which is also an infinitesimal stratum:

![Picture D1]

The new stratification after the quadratic transformation is defined by

a) The strict transform of the previous strata.

b) The stratification of the new exceptional line $P_1$ as a union of a 1-dimensional stratum $A^1$ and an infinitesimal stratum $Q' = P_1 \cap H_2$.

This new stratification is depicted as follows:

![New D1-stratification after the quadratic transformation]

**Case D2**) It is the case in which the center of the quadratic transformation is a green point $Q$ which is an infinitesimal stratum. Namely,

![Picture D2]

The new stratification is represented by

![New D2-stratification after the quadratic transformation]

so only a new zero-dimensional stratum $Q'$ is introduced at the purple locus.
9.8. Definition of the local data and local invariants.

We shall assign to each 1-dimensional stratum \(\mathbb{A}^1 = \text{Spec}(k[y])\) a polynomial \(g(y)\) in \(k[y]\) and to each zero-dimensional stratum \(Q\) an element of \(\mathcal{O}_{V^{(2)},x}\). Local invariants will be defined in terms of these data.

\(\bullet\) **Case** \(x \in \mathbb{A}^1\): Here we take an isolated point \(x \in \beta_{\ell'}(\text{Sing}(G_{r'}))\) in a 1-dimensional stratum \(\mathbb{A}^1\) (included in the union of purple hypersurfaces).

Let \(pP = pP(\beta_{\ell'}, z, f_{r'})\) be a well-adapted \(p\)-presentation at \(x\), where

\[
f_{r'}(z) = z^{r'} + a_1 z^{r'-1} + \cdots + a_{p'r'}.
\]

Fix a regular system of parameters \(\{x, y\}\) at \(\mathcal{O}_{V^{(2)},x}\), so that \(\{x = 0\}\) defines \(\mathbb{A}^1\) locally at \(x\).

Let us denote by \(\ell = q/p^e\), where \(q\) is the rational number attached to the exceptional hypersurface \(\{x = 0\}\) (see [7,4]). So there is a factorization of the form

\[
a_{p'r'} = x^\ell (g(y) + x\Omega(x,y)).
\]

The local data at \(x\) will be defined as the pair \(\left(\frac{\ell}{p^e}, g(y)W^{r'}\right)\).

**Definition 9.9.** At a point \(x \in \mathbb{A}^1\), we define the *order of \(g\) at \(x\)* as follows:

- If \(\ell \neq 0\), \(\text{ord}_x(g)\) is the usual order of \(g\) at \(\mathcal{O}_{x,\mathbb{A}^1}\).
- If \(\ell = 0\), \(\text{ord}_x(g)\) is the smallest power of \(y\) which appears in the Taylor expansion at the point that is not a \(p^r\)-th power.

\(\bullet\) **Case** \(x = Q\) an infinitesimal point: Set local coordinates so that \((R_{G,\beta}, r') = x^a y^b W^s\), assuming that \(\{x = 0\}\) denotes the green hypersurface through the point. Now, the rational number in [7.4] attached to \(\{x = 0\}\) is \(q = \frac{a}{b}\). Define the local invariant as the pair \(\left(\frac{a}{b}, y^b W^s\right)\).

**Definition 9.10.** Fix a point \(x \in \beta_{\ell'}(\text{Sing}(G_{r'}))\) with local data \((\frac{a}{b}, g(y)W^t)\). The local invariant assign to \(x\) will be defined as:

- If \(x \in \mathbb{A}^1\), then \(\text{inv}(x) = \frac{\text{ord}_x(g)}{p^e}\) (in this case \(t = p^e\)).
- If \(x = Q\), then \(\text{inv}(x) = \frac{\nu_s(g)}{s}\) (in this case \(t = s\), and \(g(y)W^t = y^b W^s\)).

9.11. Invariants and transformations.

We now study the behavior of the previous invariants under quadratic transformations, taking into account the distinction in the cases presented in 9.6.

**Case A)** Local coordinates \(\{x, y\}\) are chosen locally at \(\mathcal{O}_{V^{(2)},x}\) so that locally \(\mathbb{A}^1 = \{x = 0\}\).

In this case,

\[
v - \text{ord}(G_{r})(x) = \frac{\nu_s(a_{p'r'})}{p^e} < \text{ord}(R_{G_{r},\beta_{\ell'}})(x),
\]

and the initial form of \(a_{p'r'}\) in \(\text{Gr}_x(\mathcal{O}_{V^{(2)}})\) is not a \(p^r\)-th power.

The objective is to define local invariants after the quadratic transformation at \(x\). This leads to:

1. The definition of the invariants at the strict transform of \(H_1\).
2. The definition of the invariants in the new stratum \(\mathbb{A}^1\).

(1) **Local invariants at the strict transform of \(H_1\).**

Local factorization at \(x\) is \(a_{p'r'} = x^\ell (g(y) + x\Omega(x,y))\) at \(x_1\), the origin of the \(U_y\)-chart, (with local coordinates \(x_1 = \frac{y}{y}, y_1 = y\)), the factorization is given by

\[
a_{p'r'}^{(1)} = x_1^\ell y_1^{t-p^e}(g(y_1) + xy\Omega').
\]

In this case, the new local invariant is defined by setting \(g_1(y_1) = y^{\ell-p^e} g(y)\) and \(\ell_1 = \ell\).

As we assume that the tight monomial algebra \(\mathcal{M}_{r'}W^s\) is reduced (see [9.1]), it holds that \(\ell < p^e\) and hence

\[
\text{ord}_{x_1}(g_1) < \text{ord}_x(g).
\]
(2) Invariants along the new stratum $\mathbb{A}^1$.

Set $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{x_1\} = \text{Spec}(k[Y])$ the 1-dimensional stratum defined in Case A). We first assign a polynomial $g_1(Y)$ to $\mathbb{A}^1$.

Locally at $x$, the 1-dimensional stratum containing $x$ was defined by $\{x = 0\}$. The affine line $\mathbb{A}^1$ is defined as the intersection of the new exceptional hypersurface with the open chart $U_{x_1}$ (with coordinates $x_1 = x$ and $y_1 = \frac{x}{y}$).

Set $\text{Gr}_x(Q_{\ell,i}(x))$ is defined by $X = \text{In}_x(x)$ and $Y = \text{In}_x(y)$, and set $\text{In}_x(a_{\ell,i}) = \sum_{i+j=d} \lambda_{i,j} X^i Y^j$, the initial form of $a_{\ell,i}$ at $x$, where $d = \text{ord}(a_{\ell,i})$. Finally, define $h(Y) = \text{In}_x(a_{\ell,i})|_{X=1}$.

For any point in $\mathbb{A}^1$, we may assume that $Y$ vanishes at such point, after suitable change of coordinates, say $y_1$. The local invariant is $(\frac{\partial}{\partial x}, g_1(y_1))$, where $\frac{\partial}{\partial x}$ is the rational number attached to the new exceptional hypersurface as in [7.4] and $g_1$ is defined as before.

The change of variables done before does not affect the degree of the polynomial $g_1$. Namely $\text{deg}(h) = \text{deg}(g_1)$.

**Lemma 9.12.** In the previous setting, for any point $x' \in \mathbb{A}^1$ \begin{equation} \text{ord}_{x'}(g_1) \leq \text{ord}_x(g). \end{equation}

**Proof.** Viewing $a_{\ell,i}$ as a formal power series in the variables $x$ and $y$, [9.8.2] indicates that it is expressed as a sum of monomials of the form $x^s y^t$, with $s \geq t$. In addition, there is a monomial of the form $x^s y^t$, where $s = \text{ord}_x(g(y))$ appears in such formal expression.

This leads to the following conclusions:

1. $\text{In}_x(a_{\ell,i}) = \sum_{i \geq t} \lambda_{i,d-i} X^i Y^{d-i}$.
2. $d \leq \ell + M$.

From where it is inferred that if $\lambda_{i,d-i} \neq 0$, then $d - i \leq M$. In particular, $\sum_{i \geq t} \lambda_{i,d-i} Y^{d-i}$ is a polynomial of degree $M_1$ with $M_1 \leq \text{ord}_x(g)$. Note that $\text{ord}_{x'}(g_1) \leq M_1$. This already ensures the inequality (9.12.1).

**Lemma 9.13.** There does not exist an infinite sequence of quadratic transformations at points $x_0, x_1, \ldots$, where each $x_i$ maps to $x_{i-1}$ and such that

1. Case A) occurs at each point $x_i$.
2. $\text{ord}_{x_i}(g_1) = \text{ord}_{x_{i-1}}(g_1-1)$.

**Proof.** These conditions would contradict the assumption that $\beta_r(\text{Sing}(G_{r'}))$ has no 1-codimensional component. In fact, at the completion of the surface at the closed point $x_0$, there is a smooth curve whose successive strict transforms passes through the sequence $x_i$. Such curve would be a 1-dimensional component of the singular locus which contradicts the hypothesis.

**Lemma 9.14.** Equality at (9.12.1) can occur only finitely many times within case A).

**Proof.** Fix a point $x \in \beta_r(\text{Sing}(G_{r'}))$ which we may assume to be isolated and within case A). We claim that after a finite sequence of quadratic transformations over this point, any exceptional point $x_i$ for which condition A) holds and equality occurs at (9.12.1) must be in case A2).

To check this claim note first that under the assumption of the equality in (9.12.1), case A2) is stable. Namely, if case A2) holds at a point, and case A) holds at some exceptional point, then this exceptional point must be within case A2). On the other hand, assuming that $x$ is in condition A1) or A3), Lemma 9.13 ensures that after finitely many quadratic transformations, every point within case A) and for which equality holds, must be in case A2). In fact, otherwise the exceptional hypersurface $H_2$ would be a component of $\beta_r(\text{Sing}(G_{r'}))$, in contradiction with our hypothesis.

The previous finite sequence of quadratic transformations over the point $x$ introduces finitely many new exceptional components, say $H_{n_1}, \ldots, H_{n_1}$. The elimination algebra is of the form $(R_{G_{r'}})_{n_1} = I(H_{1})^{a_1} \ldots I(H_{n_1})^{a_{n_1}} W^s$, for some integers $\alpha_i > 0$. We claim now that after at most $\alpha_1 + \cdots + \alpha_{n_1}$ quadratic transformations, the inequality (9.12.1) will be strict at any point mapping to $x$ which fulfills condition A).

To check this, note first that locally at any point within condition A2), there is a regular system of parameters $\{x, y\}$ so that $(R_{G_{r'}})_{n_1} = x^a W^s$ with $a = \alpha_i$ for some $i \in \{1, \ldots, n_1\}$. Finally, we shall make use of the fact that a quadratic transformation at each point introduces a new hypersurface,
Lemma 9.15. Assume the previous setting. Then \( (R_{G,j})_{m+1} = x_1^{a_1-s}W^s \). The claim follows now from the inclusion \( \beta_j(\operatorname{Sing}(G_j)) \subset \operatorname{Sing}((R_{G,j})) \).

Case B) In this case, the stratification is defined by the strict transform of the previous strata. So attention should be drawn only at the unique point \( q \) of the strict transform of \( H_1 \).

This parallels the situation of Case A) (1) and the invariant strictly drops as in \((0.11.1)\).

Case C) If a point \( x \) is in case C), then \( x \) is an intersection of two purple exceptional hypersurfaces \( H_1 \) and \( H_2 \). Moreover, the point \( x \) is green and belongs to a 1-dimensional stratum included in \( H_1 \).

Therefore a quadratic transformation at such point \( x \), introduces a green exceptional hypersurface \( \mathbb{P}^1 \) and locally over \( x \) the new stratification is defined by

- The strict transform of the previous strata.
- A zero-dimensional stratum \( Q = \mathbb{P}^1 \cap H_2 \).

The invariants at the strict transform of \( H_1 \) are to be dealt with exactly as in \((9.11)\) Case A) (1). We therefore restrict attention to the data and invariants to be defined at \( Q \).

Let us fix locally at \( x \) coordinates \( x, y \) so that \( H_1 = \{ x = 0 \} \) and \( H_2 = \{ y = 0 \} \). Assume that a local presentation is given so that \( a_{p^c} \) is as in \((9.8.2)\). Therefore, the local invariant at \( x \) is \((\frac{a}{p^c}, g(y)W^s) \). Set \( (R_{G,j})_{r^c} = x^by^sW^s \). As we assume that the point \( x \) is green, then \( \frac{a+b}{p^c} \leq \frac{s(a_{p^c})}{p^c} \).

The point \( Q \) is the origin at the \( U_z \)-chart (with coordinates \( x_1 = x, y_1 = \frac{y}{z} \)). As the exceptional line is green, the exponents of the tight monomial algebra and the elimination algebra along this hypersurface coincide. So, after reduction, we may assume that at \( Q, (R_{G,j})_{r^c+1} = x^{a+b-sm}y_1^{s}W^s \), for a suitable integer \( m \geq 0 \) so that \( a+b-sm < s \).

Set \( g_1(y)W^s = y_1^{s}W^s \). According to \(9.8\) case \( x = Q \), the local data we assign to \( Q \) is \((\frac{a+b-sm}{s}, y_1^{s}W^s) \).

Lemma 9.15. Assume the previous setting. Then \( \frac{b}{s} = \operatorname{inv}(Q) < \operatorname{inv}(x) \).

Proof. Set \( M = \operatorname{ord}_x(g) \). Then

\[
\frac{\ell + M}{p^c} \geq \frac{\nu_s(a_{p^c})}{p^c} \geq \frac{a+b}{s} \geq \frac{\ell}{p^c} + \frac{b}{s}.
\]

The first inequality follows from the fact that \( x^f y^M \) is a monomial that appears in the formal expression of \( a_{p^c} \). The second inequality is due to the fact that \( x \) is a green point. Finally, the last inequality is a consequence of the fact that \( H_1 \) is purple and hence \( \frac{b}{s} > \frac{\ell}{p^c} \).

Case D1) In this case, we blow-up a point \( Q \), which is an infinitesimal stratum, and hence \( Q \) is the intersection of a purple hypersurface \( H_1 \) and a green hypersurface \( H_2 \). We assume here that \( Q \) is a purple point. It will introduce a exceptional line \( \mathbb{P}^1 \), which is purple, and it will give rise to two strata: and infinitesimal stratum \( Q' \) and an affine line \( \mathbb{A}^1 = \mathbb{P}^1 \setminus \{ Q' \} \). Fix local coordinates \( \{ x, y \} \) at \( Q \) so that \( \{ x = 0 \} \) defines the purple line \( H_1 \) and \( \{ y = 0 \} \) defines the green line \( H_2 \). Set \( (R_{G,j})_{r^c} = x^by^sW^s \).

Lemma 9.16. Fix a point \( x' \in \mathbb{A}^1 \). The invariant strictly drops, i.e., \( \operatorname{inv}(x') < \operatorname{inv}(Q) \).

Proof. At \( Q \) the second coordinate of the local data is given by \( g(x)W^s = x^aW^s \). So the local invariant at \( Q \) is \( \operatorname{inv}(Q) = \frac{\nu_s(g(x))}{s} = \frac{a}{s} \).

As we assume that \( Q \) is a purple point, \( \frac{\nu_s(a_{p^c})}{s} := \frac{d}{s} < \frac{a+b}{s} \), for \( a_{p^c} \) as in \((9.8.1)\). On the other hand, \( y = 0 \) defines the green line, so we claim that \( \frac{b}{s} < 1 \). In fact, green hypersurfaces are those for which the exponents of the elimination algebra and tight monomial algebra coincide by definition. Since we assume that the tight monomial algebra is reduced, then \( \frac{b}{s} < 1 \).

Let \( \operatorname{In}(a_{p^c}) = \sum_{i+j=d} \lambda_{ij} X^i Y^j \) denote the initial form of \( a_{p^c} \) at \( Q \). Note that \( \frac{d}{p^c} \geq \frac{b}{s} \).

From the previous inequalities, we obtain \( \frac{a+b}{s} > \frac{d}{p^c} = \frac{a+b}{p^c} \geq \frac{\ell}{p^c} + \frac{b}{s} \), and hence, \( \frac{a}{s} > \frac{b}{s} \).
At the $U_y$-chart (with coordinates $y_1 = y$, $x_1 = \frac{x}{y}$), $a^{(1)}_{p^e} = y_1^{d-p^e}(g_1(x_1) + y_1\Omega')$, where $g_1(x_1)$ is obtained by the global polynomial $\text{In}(Q(a^e_p)|Y=1)$. So the previous discussion shows that if $\lambda_{i,j} \neq 0$, then $i_{p^e} < a^e$. This, in turn, suffices to check that $\text{inv}(x') = \text{ord}_{x'}(g_1) < a^e = \text{inv}(Q)$.

Now, we study the invariant at $Q'$. Note that local coordinates at $Q'$ are given by $x_1 = x$, $y_1 = \frac{y}{x}$. So the elimination algebra is $(R_G,\beta)_{r+1} = x_1^{a+b-s}y_1^bW^s$ and the strict transform of the green hypersurface $H_2$ is given by $\{y_1 = 0\}$. Therefore, the second coordinate of the local data at $Q'$ is $g_1(x_1)W^s = x_1^{a+b-s}W^s$.

**Lemma 9.17.** With the previous setting, $\text{inv}(Q') < \text{inv}(Q)$.

**Proof.** Recall that the second coordinate of the local data at $Q$ is $g(x)W^s = x^aW^s$. By definition $\text{inv}(Q) = \frac{\nu_Q(g)}{s} = \frac{a}{s}$ and $\text{inv}(Q') = \frac{\nu_{Q'}(g_1)}{s} = \frac{a + b - s}{s}$. Since $H_2$ is a green hypersurface, then $\frac{b}{s} < 1$, from which the strict inequality is clear. ⚫

**Case D2)** This is, as in D1), the case of a quadratic transformation at a point $Q$ which is an infinitesimal stratum. It is the intersection of a purple hypersurface $H_1$ and a green hypersurface $H_2$. We assume now, in addition, that $Q$ is a green point. Since the new exceptional hypersurface is green, there is a unique stratum $Q'$ which will be infinitesimal.

Fix local coordinates $\{x, y\}$ at $Q$, so that $\{x = 0\}$ defines the purple line $H_1$ and $\{y = 0\}$ defines the green line $H_2$. Set $(R_G,\beta)_{r'} = x^a y^b W^s$. Recall that the second coordinate of the local data is $g(x)W^s = x^a W^s$.

Note that $x_1 = \frac{x}{y}$, $y_1 = y$ are local coordinates at $Q'$. The elimination algebra is $(R_G,\beta)_{r'+1} = x_1^a y_1^{a+b-s}W^s$. The new green exceptional hypersurface is defined by $y_1 = 0$, so the second coordinate of the local data is $g_1(x_1)W^s = x_1^a W^s$.

**Remark 9.18.** If case D2) holds, then

- $\nu_{Q'}(g_1) = \nu_Q(g)$ and hence $\text{inv}(Q') = \text{inv}(Q)$.
- On the other hand, $\frac{a+b-s}{s} = \frac{a}{s} + \left(\frac{b}{s} - 1\right) < \frac{a}{s}$.

So case D2) cannot occur in a successive manner more than finitely many times.
In the following Table we indicate, in a synthetic manner, why resolution is achieved.

<table>
<thead>
<tr>
<th>Initial stratification</th>
<th>After blow-up</th>
<th>Invariants</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>CASE A)</strong></td>
<td></td>
<td>$\mathbb{A}^1 = \mathbb{P}^1 \setminus {x_1}$</td>
</tr>
<tr>
<td>$H_1$</td>
<td>$H_1$</td>
<td>• Invariants strictly drop after finitely many quadratic transformations or at most remain equal at points which are in cases B) or C). (Lemma 9.12, Lemma 9.13, and Lemma 9.14).</td>
</tr>
<tr>
<td>$x$</td>
<td>$x_1$</td>
<td></td>
</tr>
<tr>
<td><strong>CASE B)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H_1$</td>
<td>$H_1$</td>
<td>• This case adds no new stratum.</td>
</tr>
<tr>
<td>$x$</td>
<td>$x_1$</td>
<td>• The invariant strictly drops, i.e., $\text{inv}(x) &lt; \text{inv}(x_1)$.)</td>
</tr>
<tr>
<td><strong>CASE C)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H_1$</td>
<td>$H_1$</td>
<td>$H_2$</td>
</tr>
<tr>
<td>$x$</td>
<td>$x_1$</td>
<td>$Q$</td>
</tr>
<tr>
<td>$H_2$</td>
<td></td>
<td>$\mathbb{A}^1 = \mathbb{P}^1 \setminus {Q}$</td>
</tr>
<tr>
<td><strong>CASE D1)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H_1$</td>
<td>$H_2$</td>
<td>$H_1$</td>
</tr>
<tr>
<td>$Q$</td>
<td>$Q'$</td>
<td>$\mathbb{A}^1 = \mathbb{P}^1 \setminus {Q'}$</td>
</tr>
<tr>
<td>$H_2$</td>
<td></td>
<td>$\mathbb{A}^1 = \mathbb{P}^1 \setminus {Q'}$</td>
</tr>
<tr>
<td><strong>CASE D2)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H_1$</td>
<td>$H_2$</td>
<td>$H_1$</td>
</tr>
<tr>
<td>$Q$</td>
<td>$Q'$</td>
<td>$\mathbb{A}^1 = \mathbb{P}^1 \setminus {Q'}$</td>
</tr>
<tr>
<td>$H_2$</td>
<td></td>
<td>$\mathbb{A}^1 = \mathbb{P}^1 \setminus {Q'}$</td>
</tr>
</tbody>
</table>

- Only adds a new infinitesimal stratum $Q'$. |
- The invariant remains equal, $\text{inv}(Q') = \text{inv}(Q)$. |
- Leads to resolution or Case D1) after finitely many quadratic transformations |
Part IV. Proofs of Theorems

APPENDIX A. PROOF OF MAIN THEOREM 1.

A.1. Hironaka’s weak equivalence. There are two natural operations on Rees algebras, that will play a central role in Hironaka’s notion of invariance. We introduce here these operations. Fix a smooth scheme $V^{(d)}$ and a set of smooth hypersurfaces with normal crossings, say $E = \{H_1, \ldots, H_r\}$. Let $\mathcal{G} = \bigoplus I_n W^n$ be a Rees algebra in $V^{(d)}$. Let now

\begin{equation}
V^{(d)} \leftarrow U
\end{equation}

be defined by:

\begin{enumerate}
  \item \text{(A)} An open set $U$ of $V^{(d)}$ defined in Zariski or étale topology.
  \item \text{(B)} The projection of $U = V^{(d)} \times \mathbb{A}^n_\mathbb{K}$ on the first coordinate. Here, $\mathbb{A}^n_\mathbb{K}$ denotes the $n$-dimensional affine scheme (with $n \in \mathbb{Z}_{\geq 1}$).
\end{enumerate}

In both cases, there is a natural notion of pull-backs of the Rees algebra $\mathcal{G}$ and the set $E$ to a Rees algebra $\mathcal{G}_U$ and to a set $E_U$. Here $E_U$ is the set of the pull-backs of the hypersurfaces in $E$. The Rees algebra $\mathcal{G}_U$ is defined as:

\begin{enumerate}
  \item \text{(A)} The restriction to $U$ in case (A), i.e., $\mathcal{G}_U = \bigoplus (I_n)_U W^n$
  \item \text{(B)} The total transforms of every ideal $I_n$, say $I^n_\mathbb{K}$, in case (B), i.e., $\mathcal{G}_U = \bigoplus I^n_\mathbb{K}$.
\end{enumerate}

In this case, to $V^{(d)} \leftarrow U$ we attach a pull-back of the form:

\begin{equation}
(V^{(d)}, E) \leftarrow \pi (U, E_U)
\end{equation}

\begin{equation}
\mathcal{G} \leftarrow \mathcal{G}_U
\end{equation}

Observe here that the singular locus of the Rees algebra $\mathcal{G}$ is compatible with pull-backs, i.e.,

\begin{equation}
\text{Sing}(\mathcal{G}_U) = \pi^{-1}(\text{Sing}(\mathcal{G})).
\end{equation}

**Definition A.2.** A local sequence of transformations of a Rees algebra and a set $E$ is a sequence

\begin{equation}
(V^{(d)}, E) \leftarrow \pi_1 (V^{(d)}_1, E_1) \leftarrow \pi_2 \cdots \leftarrow \pi_r (V^{(d)}_r, E_r)
\end{equation}

where each $V^{(d)}_i \leftarrow \pi_{i+1} (V^{(d)}_{i+1}, E_{i+1})$ is a pull-back or a usual transformation (a monoidal transformation along a center $C_i \subset \text{Sing}(\mathcal{G}_i)$ with normal crossing with the exceptional hypersurfaces in $E_i$) for $i = 0, \ldots, r - 1$. Here we denote $V^{(d)}_0 = V^{(d)}$.

**Definition A.3.** Fix two Rees algebras $\mathcal{G}$ and $\mathcal{G}'$ and a set of exceptional hypersurfaces $E$ in the smooth scheme $V^{(d)}$. We say that $\mathcal{G}$ and $\mathcal{G}'$ are weakly equivalent if:

\begin{enumerate}
  \item i) $\text{Sing}(\mathcal{G}) = \text{Sing}(\mathcal{G}')$.
  \item ii) Any local sequence of transformations of $\mathcal{G}$, say $[A.2.1]$, define a local sequence of transformations of $\mathcal{G}'$ (and vice versa), and $\text{Sing}(\mathcal{G}_i) = \text{Sing}(\mathcal{G}'_i)$ for $i = 0, \ldots, r$.
\end{enumerate}

**Remark A.4.** Note that if $\mathcal{G}$ and $\mathcal{G}'$ are weakly equivalent as before, then also their transforms $\mathcal{G}_i$ and $\mathcal{G}'_i$ are weakly equivalent. So the weak equivalence is preserved after any arbitrary local sequence of transformations. Note also that two algebras with the same integral closure are weakly equivalent.

A.5. On Main Theorem 1.

**Proposition A.6.** Fix a Rees algebra $\mathcal{G}$ and a presentation $\mathcal{P} = \mathcal{P}(\beta, z, f_\alpha(z))$. Let $H$ be a smooth irreducible hypersurface in $V^{(d-1)}$. Denote by $y$ the generic point of $H$ and assume that $\mathcal{P}$ is in normal form at $y$.

Then, $H$ is a component of $\beta(\text{Sing}(\mathcal{G}))$ if and only if $\text{Sl}_{y}(\mathcal{P}) \geq 1$.

**Proof.** See Proposition [4.8][j].
Theorem A.7. (Main Theorem 1). Fix a Rees algebra $\mathcal{G}$. Consider a point $x \in \text{Sing}(\mathcal{G})$ and a $p$-presentation well-adapted at $\beta(x)$. The value $\text{SL}(p\mathcal{P})(\beta(x))$ is completely determined by the weak equivalence class of $\mathcal{G}$.

Proof. Fix $\mathcal{P} = \mathcal{P}(\beta, z, f_{p^e}(z))$ well-adapted to $\mathcal{G}$ at $x = \beta(x)$. Suppose that $f_{p^e}(z) = z^{p^e} + a_1 z^{p^e-1} + \cdots + a_0$ and denote $r_j = v_{x}(a_j)$ for $j = 1, \ldots, p^e$. Theorem 4.6 asserts that $\text{SL}(\mathcal{P})(x) = \min\{v_{x}(a_0\cdot p^e)\}(x), \text{ord}^{(d-1)}(\mathcal{R}_{\mathcal{G}_p})(x)\}$.

Firstly assume that $v_{x}(a_0\cdot p^e) < \text{ord}^{(d-1)}(\mathcal{R}_{\mathcal{G}_p})(x) = \frac{2}{e}$. Multiply $V^{(d)}$ by an affine space of dimension 1, i.e., $V^{(d)} \times \mathbb{A}^1$. Locally, in a neighborhood of $(x, 0) \in V^{(d)} \times \mathbb{A}^1$ we can consider the pull-back of $f_{p^e}(z)$, together with a natural projection $V^{(d)} \times \mathbb{A}^1 \rightarrow V^{(d-1)} \times \mathbb{A}^1$, mapping $(x, 0)$ to $(x, 0)$. Consider also the natural pull-back of $\mathcal{R}_{\mathcal{G}_p}$ in $V^{(d-1)} \times \mathbb{A}^1$. Note that

$$\text{SL}(p\mathcal{P})(x, 0)) = \text{SL}(p\mathcal{P})(x) = \frac{r_{p^e}}{p^e} \geq 1$$

and $I_{H}(p\mathcal{P})(a_{p^e})$ can be naturally identified with $I_{H}(p\mathcal{P})(a_{p^e})$, so it is not a $p^e$-th power.

Fix coordinates $[z, x_1, \ldots, x_t, t]$ locally at $(x, 0)$, where $[z, x_1, \ldots, x_t]$ is the regular system of parameters at $C_{\mathcal{V}(x), x}$. Consider the monoidal transformation with center $q_0 = (x, 0)$ and let $q_1$ be the intersection of the new exceptional hypersurface, say $H_1$, and the strict transform of $x \times \mathbb{A}^1$.

The point $q_1$ can be identified with the origin of the $U_i$-chart, $(U_i = \text{Spec}[k[\frac{z}{e}, \frac{x_1}{e}, \ldots, \frac{x_t}{e}], e])$. Here

$$f_{p^e}^{(1)}(z_1) = z_1^{p^e} + t^{\nu} z_1^{p^e-1} + \cdots + t^{\nu} a'_{p^e} z_1^{p^e-1}, \quad (\mathcal{R}_{\mathcal{G}_p})(z_1) = I(H_1)^{\alpha-s}(\mathcal{R}_{\mathcal{G}_p})'$$

where $a'_{p^e}$ and $(\mathcal{R}_{\mathcal{G}_p})'$ can be identified with $a_{p^e}$ and $\mathcal{R}_{\mathcal{G}_p}$ in some sense.

This process can be iterated $N$-times, defining a sequence of monoidal transformations at $q_1, \ldots, q_N-1$, where each $q_j$ is the intersection of the new exceptional component, say $H_j$, with the strict transform of $x \times \mathbb{A}^1$. The transforms at the $U_i$-chart are given by

$$f_{p^e}^{(N)}(z_N) = z_N^{p^e} + t^{N(\nu)} a'_{p^e} z_N^{p^e-1} + \cdots + t^{N(\nu)} a'_{p^e} z_N^{p^e-1}, \quad (\mathcal{R}_{\mathcal{G}_p})(z_N) = I(H_N)^{(\alpha-s)}(\mathcal{R}_{\mathcal{G}_p})'$$

For a fixed $N \gg 0$, we now look for the highest number of monoidal transformations in codimension 2 (involving the local coordinate $t$). We claim that this well-defined number is characterized by $\text{SL}(p\mathcal{P})(x)$.

Firstly, consider a monoidal transformation along the center $(z_N, t)$. For simplicity, denote $z' = \frac{z}{\ell}$ and its successive transforms. In the $U_i$-chart, the transform is given by

$$f_{p^e}(z') = z'^{p^e} + t^{N(\nu)} a'_{p^e} z'^{p^e-1} + \cdots + t^{N(\nu)} a'_{p^e} z'^{p^e-1}, \quad (\mathcal{R}_{\mathcal{G}_p})(z') = I(H_{N+1})^{(\alpha-s)}(\mathcal{R}_{\mathcal{G}_p})'$$

After $\ell$ monoidal transformations along centers of codimension 2, say for simplicity $(z', t')$, the exponents of $t$ in each coefficient is $N(\nu - j) - \ell s$, and the one in the elimination algebra is $N(\alpha - s) - \ell s$, and hence $(z, t)$ is a permissible center whenever $N(\nu - j) - \ell s \geq j$ for all $j \in \{1, \ldots, p^e\}$ and $(N(\alpha - s) - \ell s) \geq s$. In particular, this condition requires that

$$\ell \leq \min_{1 \leq j \leq n} \left\{ \frac{N(\nu - j) - 1}{N(\nu - j) - \ell s} \right\} = \frac{N(\nu - j) - 1}{N(\nu - j) - 1} - 1$$

Whenever $N(\nu - j) \not\in \mathbb{Z}$, after applying $\lceil N(\nu - j) \rceil$ monoidal transformations along the centers of codimension 2, one gets $V((z, t)) \subset V(f_{p^e}(z))$ and $0 < \text{SL}(p\mathcal{P})(x) < 1$. So $H = \{ t \neq 0 \}$ cannot be a component of $\text{Sing}(\mathcal{G})$ (see Proposition 4.6).

Finally assume that $N(\nu - j) \in \mathbb{Z}$, denote $m = \frac{N(\nu - j)}{p^e}$ and consider $N(m - 1)$ monoidal transformations along $(z, t)$ and its transforms. Note that $\text{SL}(p\mathcal{P})(x) = \frac{r_{p^e}}{p^e} = 0 < \text{ord}^{(d-1)}(\mathcal{R}_{\mathcal{G}_p})(x)$ and $I_{H}(p\mathcal{P})(a_{p^e})$ can be naturally identified with $I_{H}(p\mathcal{P})(a_{p^e})$ is not a $p^e$-th power. Proposition 4.6 ensures that $H$ is not a component of $\text{Sing}(\mathcal{G})$.

The case $\text{SL}(p\mathcal{P}) = \text{ord}^{(d-1)}(\mathcal{R}_{\mathcal{G}_p})(x)$ is straightforward from the previous discussion. Finally, these arguments imply that $\text{SL}(p\mathcal{P})(x)$ is totally characterized by Hironaka's weak equivalence class of $\mathcal{G}$.\[\Box\]
Corollary A.8. Let \( G \) be a Rees algebra. Fix two transversal projections \( V^{(d)} \xrightarrow{\beta} V^{(d-1)} \) and \( V^{(d)} \xrightarrow{\beta'} V^{(d-1)} \). For any \( x \in \text{Sing}(G) \)

\[
\beta - \text{ord}(G)(\beta(x)) = \beta - \text{ord}(G)(\beta'(x)).
\]

Corollary A.9. Let \( G \) be a Rees algebra. Fix a \( p \)-presentation \( pP = pP(\beta, z, f_{p'}(z)) \) well-adapted to \( G \) at \( x \in \text{Sing}(G) \). Then,

\[
v - \text{ord}^{(d-1)}(G)(x) = \text{Sl}(pP)(\beta(x)).
\]

APPENDIX B. THE TIGHT MONOMIAL ALGEBRA AND PROOF OF MAIN THEOREM 2.

We address here the Proof of Main Theorem 2.

Theorem B.2. (Main Theorem 2). Fix a sequence of permissible transformations, say

\[
\begin{array}{lll}
\mathcal{G} & \mathcal{G}_1 & \mathcal{G}_r \\
V^{(d)} & \xrightarrow{\pi_1} V_1^{(d)} & \ldots \xrightarrow{\pi_r} V_r^{(d)}
\end{array}
\]

and let \( \mathcal{M}_rW^s \) denote the tight monomial algebra defined in 7.4 (1). Then, at any closed point \( x \in \text{Sing}(\mathcal{G}_r) \), \( \mathcal{M}_rW^s \) has monomial contact with \( \mathcal{G}_r \), i.e., for some \( \beta_r \)-transversal section \( z \) of order one at the point \( x \),

\[
\mathcal{G}_r \subset \langle z \rangle W \odot \mathcal{M}_rW^s.
\]

Proof. For any index \( i = 1, \ldots, r \), at any closed point \( x \in \text{Sing}(\mathcal{G}_i) \) there is a \( \beta_i \)-transversal parameter \( z_i' \) of order one, and \( \mathcal{G}_i \subset \langle z_i' \rangle W \odot \left( y_{j_1}^{h_{j_1}} \ldots y_{j_t}^{h_{j_t}} \right) W^s \), where \( \left( y_{j_1}^{h_{j_1}} \ldots y_{j_t}^{h_{j_t}} \right) W^s \) is the tight monomial algebra locally at \( \mathcal{O}_{V_i^{(d)}, x} \). Suppose that \( x \in C_i \subset V_i^{(d)} \) and that \( C_i \) is a permissible center. Then, there is a \( \beta_i \)-transversal parameter \( z \) of order one at \( x \), so that

1. \( z \in I(C)_i \),
2. \( \langle z \rangle W \odot \left( y_{j_1}^{h_{j_1}} \ldots y_{j_t}^{h_{j_t}} \right) W^s = \langle z \rangle W \odot \left( y_{j_1}^{h_{j_1}} \ldots y_{j_t}^{h_{j_t}} \right) W^s \),
3. there is a \( p \)-presentation \( pP(\beta_i, z, f_{p'}(i)) \) which is well-adapted to \( \mathcal{G}_i \) at \( \xi_{\beta_i}(C) \) and \( \beta_i(x) \) (where \( \xi_{\beta_i}(C) \) denotes the generic point of \( \beta_i(C) \)).

In fact, note that any \( p \)-presentation of the form \( pP(\beta_i, z, f_{p'}(i)) \) for \( \beta_i \) and \( z \) as before, is compatible with the monomial algebra \( \left( y_{j_1}^{h_{j_1}} \ldots y_{j_t}^{h_{j_t}} \right) W^s \) (see Definition 3.9). Proposition 5.8 B) ensures that such \( p \)-presentation, can be taken to be well-adapted to \( \mathcal{G}_i \) at \( \xi_{\beta_i}(C) \) and \( \beta_i(x) \); and also compatible with \( \left( y_{j_1}^{h_{j_1}} \ldots y_{j_t}^{h_{j_t}} \right) W^s \). This proves (2) and (3).

Note also that (1) follows from the fact that \( \beta_i(C) \) is a permissible center, and hence \( \text{Sl}(pP)(\xi_{\beta_i}(C)) \geq 1 \) (see Proposition 4.8 i)).

Assume by induction in \( r \) that, locally at any closed point \( x \in \text{Sing}(\mathcal{G}_r) \) the algebra \( \mathcal{M}_rW^s \) has monomial contact with \( \mathcal{G}_r \), i.e., for some \( \beta_r \)-transversal section \( z' \) of order one at the point,

\[
\mathcal{G}_r \subset \langle z' \rangle W \odot \mathcal{M}_rW^s.
\]

The condition is empty for \( r = 0 \).

We claim that there is a \( p \)-presentation which is compatible with the strict transform of the monomial algebra \( \mathcal{M}_rW^s \), and is well-adapted to \( \mathcal{G}_{r+1} \) at \( \xi_{H_r^{(d)}}(x') \) in a neighborhood of any closed point \( x' \in \text{Sing}(\mathcal{G}_{r+1}) \). That is, locally at any closed point \( x' \in \text{Sing}(\mathcal{G}_{r+1}) \), there is a \( p \)-presentation which is well-adapted simultaneously to every \( \xi_{H_i^{(d-1)}}(x) \) (\( i = 1, \ldots, r + 1 \)).

If \( x' \notin H_{r+1}^{(d)} \), then Remark 6.4 shows that there is an identification between the \( p \)-presentations \( pP \in \mathcal{G}_r \) and \( pP_1 \) in \( \mathcal{G}_{r+1} \) (in an open subset). Thus there is nothing to check in this case.

Suppose that \( x' \in \text{Sing}(\mathcal{G}_{r+1}) \) \( \cap H_{r+1}^{(d)} \).

Firstly, assume that \( \text{In}_{\mathcal{G}_r}(f_{p'}) = Z_{p'} \). Then \( \pi^{-1}(x) \cap \text{Sing}(\mathcal{G}_{r+1}) \subset \{ z_1 = 0 \} \), where \( z_1 \) denotes the strict transform of \( z \) (see Lemma 6.5). Moreover, \( pP_1 \) is well-adapted to \( \mathcal{G}_{r+1} \) at \( \xi_{H_r^{(d)}}(x') \) (see Proposition 6.6).
Let

$$f^{(1)}_p(z_1) = z_1^p + a_1(1) z_1^{p-1} + \cdots + a_p(1)$$

be the strict transform of $f_p(z) = z^p + a_1 z^{p-1} + \cdots + a_p$. Since $a_i W_i \in \mathcal{M}_i W^s$, then $a_i(1) W_i \in \mathcal{M}_i W^s$ for $i = 1, \ldots, p'$. Here $\mathcal{M}_i W^s$ denotes the strict transform of $\mathcal{M}_i W^s$. On the other hand, $a_i(1) W_i \in I(\mathcal{H}_{r+1}^{(d)})_{h+1} W^s$, since $p \mathcal{P}_1$ is well-adapted at $\xi^{(d)}_{H_{r+1}}$ (recall that $q_{H_{r+1}} = \frac{h_{r+1}}{s}$). Thus $a_i(1) W_i \in \mathcal{M}_{r+1} W^s$ (the new tight monomial algebra).

The same arguments hold for $(\mathcal{R}_{\mathcal{G}, \beta})_{r+1}$, and hence $p \mathcal{P}_1$ is compatible with $\mathcal{M}_i W^s$ and well-adapted to $\mathcal{G}_{r+1}$ at $\xi^{(d)}_{H_{r+1}}$. Therefore, $p \mathcal{P}_1$ is compatible with $\mathcal{M}_{r+1} W^s$. Hence $\mathcal{G}_{r+1} \subset \{z_1\} W \cap \mathcal{M}_{r+1} W^s$.

Assume now that $\text{In}_x(f_p) \neq Z^{p}$, then two different cases can occur:

Suppose firstly that $\tau_{\mathcal{G}, \beta} \geq 2$ and $\text{In}_x(f_p) = Z^p + A_p x$, where $A_p$ is not a $p$-th power. In this case, $\text{SL}(p \mathcal{P})(\mathcal{G}_{r+1}^{(d)}) = 1$, and by definition $h_{r+1} = 0$.

Let $x' \in \text{Sing}(\mathcal{G}_{r+1}) \cap H_{r+1}^{(d)}$ be a closed point such that $\pi_C(x') = x$. Assume that $\beta_{r+1}(x') \in V(\mathcal{M}_{r}^{(d-1)}) \cap V_{r+1}^{(d-1)}$, where $\mathcal{M}_{r}^{(d-1)}$ denotes the stric transform of $\mathcal{M}_r W^s$ in $V_{r+1}^{(d-1)}$. One can check that $x' \in \{ z_1 = 0 \}$, so all coefficients $a_i(1)$ vanish at $\beta_{r+1}(x')$ for $i = 1, \ldots, p'$. The same argument used before shows that $p \mathcal{P}_1$ is compatible with $\mathcal{M}_{r+1} W^s$.

Assume now that $\beta_{r+1}(x') \not\in V(\mathcal{M}_{r}^{(d-1)})$. Locally at $\beta_{r+1}(x')$, the monomial algebra is $\mathcal{M}_{r+1} W^s$ which has the same integral closures as $\mathcal{O}_{V_{r+1}^{(d-1)}}[W]$ and there is nothing to prove in this case.

Finally, suppose that $\text{In}_x(f_p(z)) = Z^p + A_1 Z^{p-j} + \cdots + A_{j-1} Z^{p-j} + A_j$ with $A_1 \neq 0$ and $j < p$. In this case, $\text{ord}(\mathcal{R}_{\mathcal{G}, \beta})(\mathcal{G}_{r+1}^{(d)}) = 1$ and hence $h_{r+1} = 0$. Similar arguments to the ones used before apply here to show the compatibility of the strict transform with the monomial algebra: whenever the point $x' \in V(\mathcal{M}_{r}^{(d-1)})$, then $x' \in \{ z_1 = 0 \}$. If not, the monomial algebra is locally of the form $\mathcal{O}_{V_{r+1}^{(d-1)}}[W]$.

This concludes the proof.

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