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Finite time singularities in a 1D model of the quasi-geostrophic equation

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Abstract

In this paper we study 1D equations with nonlocal flux. These models have resemblance of the 2D quasi-geostrophic equation. We show the existence of singularities in finite time and construct explicit solutions to the equations where the singularities formed are shocks. For the critical viscosity case we show formation of singularities and global existence of solutions for small initial data.

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1. The model equations

The 2D quasi-geostrophic equation (QG), which models the dynamics of the mixture of cold and hot air and the fronts between them, is given by

$$\begin{aligned} &\theta_t + (u\cdot\nabla)\theta = 0, \\ &u = \nabla^\perp\psi, \quad \theta = -(-\Delta)^{\frac{1}{2}}\psi, \\ &\theta(x,0) = \theta_0(x), \end{aligned} \tag{1.1}$$

where $\nabla^\perp = (-\partial_2, \partial_1)$. Here, $\theta(x,t)$ represents the temperature of the air. Besides its direct physical significance [16,21], the quasi-geostrophic equation has very interesting features of resemblance to the 3D Euler equation, being also the finite time blow-up for (QG) an outstanding open problem. With respect to that question there are pioneering studies due to Constantin, et al. [7]. In particular they obtained a finite time blow-up criterion, which says that the local smooth solution for initial data $\theta_0 \in H^k(\mathbb{R}^2)$, $k \geq 3$, blows up at T if and only if

$$\int_0^T \|\nabla^\perp\theta(t)\|_{L^\infty}\, dt = \infty.$$

There are many studies on the equations following that work [2,4,12,13,19,22,28]. Motivated mainly by Constantin et al. [6], we are concerned here on constructing and studying a 1D model equation of QG. In order to derive that model equation we first write QG in another equivalent form as follows: From the second equation of QG we have the representation

$$u = -\nabla^\perp(-\Delta)^{-\frac{1}{2}}\theta = -R^\perp\theta, \tag{1.2}$$

where we have used the notation, $R^\perp\theta = (-R_2\theta, R_1\theta)$ with R_j, $j = 1, 2$, for the 2D Riesz transform defined by (see e.g. [26])

$$R_j(\theta)(x,t) = \frac{1}{2\pi} PV \int_{\mathbb{R}^2} \frac{(x_j - y_j)\theta(y,t)}{|x-y|^3}\, dy.$$

Using representation (1.2), we find that (1.1) is transformed into

$$\theta_t + \mathrm{div}[(R^\perp\theta)\theta] = 0, \tag{1.3}$$

because $\mathrm{div}(R^\perp\theta) = 0$. To construct the 1D model, we consider the unknown function $\theta(x,t)$ defined for $(x,t) \in \mathbb{R}\times\mathbb{R}_+$ or $\mathbb{T}\times\mathbb{R}_+$, and replace the Riesz transform, $R^\perp(\cdot)$ in (1.3), by the Hilbert transform $H(\cdot)$ defined by

$$H\omega(x) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{\omega(y)}{x-y}\, dy,$$

or

$$Hf(x) = \frac{1}{2\pi} P.V. \int_{-\pi}^{\pi} \frac{f(x-y)}{\tan\frac{y}{2}}\, dy$$

in the periodic case. And finally we replace div$(\cdot)$ in (1.3) by ∂_x. Then Eq. (1.3) is transformed into

$$\theta_t + (H(\theta)\theta)_x = 0, \tag{1.4}$$
$$\theta(x,0) = \theta_0(x).$$

This was already studied in [1,18], which was proposed in a different physical situation. In [18] it was considered the following equation:

$$\theta_t + \delta(H(\theta)\theta)_x + (1-\delta)H(\theta)\theta_x = 0 \quad \text{with } 0 \le \delta \le 1 \tag{1.5}$$

and the existence of singularities for $0 < \delta < \frac{1}{3}$, $\delta = \frac{1}{2}$ and $\delta = 1$ was proved.[1] Also, in [18], the question of singularities of (1.5) for the other ranges of $0 < \delta \le 1$ was left open. In Theorem 2.1 below, we proved existence of singularities for the full range of $0 < \delta \le 1$, thus solving the problem. The proof of existence of singularities in the case $\delta = 0$ is solved in [11] using a different technique.

In the case of 2D viscous Quasi-geostrophic equation Constantin and Wu [8] showed that for $\alpha > 1$ the system

$$\left(\partial_t + u \cdot \nabla\right)\theta = -\kappa(-\Delta)^{\frac{\alpha}{2}}\theta, \tag{1.6}$$
$$u = \nabla^{\perp}\psi, \quad \theta = -(-\Delta)^{\frac{1}{2}}\psi$$

does not develop singularities in finite time. For the critical viscosity $\alpha = 1$ it is an open problem, considered as a model problem of the 3D Navier–Stokes equations (see [3,4,5,9,10,17,22,23,24,25], for more details).

In Sections 3 and 4 we study the following 1D dimensional model of the critical viscous QG:

$$\theta_t + (\theta H(\theta))_x = -\kappa H\theta_x,$$
$$\theta_0(x) = \theta(x,0),$$

where the critical viscosity term $-\kappa(-\Delta)^{\frac{1}{2}}\theta$ in (1.6) is replaced by $-\kappa(H\theta)_x$. We show that the solutions to this equation may also develop singularities with the same initial data as in the inviscid case for any $\kappa < \|\theta_0\|_{L^\infty}$. When the viscosity $\kappa \ge \|\theta_0\|_{L^\infty}$ then the solution remains smooth. In Section 4, we prove global existence of solutions for

[1] Notice that after the change $\theta \to -\theta$, Eq. (1.5) can be formulated as it originally appears in [18].

the periodic case with small initial data. If the second order viscosity term $\mu\theta_{xx}$ added

$$\theta_t + (\theta H(\theta))_x = \mu\theta_{xx},$$
$$\theta_0(x) = \theta(x, 0),$$

then explicit solutions can be constructed for all positive μ by applying the Hopf–Cole transform. See the appendix.

2. The formation of finite time singularities

2.1. Blow-up in finite time: periodic case

We shall consider periodic solutions of Eq. (1.5) with $\delta > 0$, where $\theta(x + 2\pi, t) = \theta(x, t)$. Our goal is to show that, for very general smooth initial data θ_0, there is no $C^1([-\pi, \pi] \times [0, T))$ solution of (1.5) with $\delta > 0$ for all time T.

First let us observe that

$$\text{(i)}\quad Hf(x) = \frac{1}{2\pi} P.V. \int_{-\pi}^{\pi} \frac{f(x-y)}{\tan \frac{y}{2}}\, dy$$

$$\text{(ii)}\quad \Lambda f(x) = Hf_x(x) = \frac{1}{2\pi} P.V. \int_{-\pi}^{\pi} \frac{f(x) - f(y)}{\sin^2 \frac{x-y}{2}} dy$$

(iii) If the real valued function $f \in C^1$ has a maximum (respectively a minimum) at x_0 then $\Lambda f(x_0) \geq 0$ (respect. $\Lambda f(x_0) \leq 0$).

Theorem 2.1. *Given a periodic non-constant initial data $\theta_0 \in C^1([-\pi, \pi])$ such that $\int_{-\pi}^{\pi} \theta_0(x)\, dx = 0$, there is no $C^1([-\pi, \pi] \times [0, \infty))$ solution to (1.5) with $\delta > 0$.*

Proof. Suppose the existence of such a solution $\theta(x, t)$. We have

$$\frac{d}{dt} \int_{-\pi}^{\pi} \theta(x, t)\, dx = -\delta \int_{-\pi}^{\pi} (\theta H\theta)_x\, dx - (1 - \delta) \int_{-\pi}^{\pi} \theta_x H\theta\, dx$$
$$= (1 - \delta) \int_{-\pi}^{\pi} \theta H\theta_x\, dx \geq 0.$$

Therefore

$$M(t) \equiv \max_x \theta(x, t) \geq 0,$$
$$m(t) \equiv \min_x \theta(x, t)$$

and, in $t=0$, we have the strict inequalities: $M(0)>0$, $m(0)<0$. Both $M(t)$, $m(t)$ are continuous Lipschitz functions and by H. Rademacher's theorem, they are differentiable at almost every point t.

Under the hypothesis of differentiability we may choose $x(t)$, $\overline{x}(t)$ such that

$$M(t)=\theta(x(t),t),$$
$$m(t)=\theta(\overline{x}(t),t)$$

for every $t\geq 0$. Let t_0 be a point of differentiability of $M(t)$. By compacity we may choose a sequence of positive numbers $h_j\to 0$ so that $x(t_0+h_j)$ converges to x_0. Then by continuity we will obtain that $M(t_0)=\theta(x_0,t_0)$.

Next, let us consider

$$\begin{aligned}
\frac{M(t_0+h_j)-M(t_0)}{h_j} &= \frac{\theta(x(t_0+h_j),t_0+h_j)-\theta(x_0,t_0)}{h_j}\\
&= \frac{\theta(x(t_0+h_j),t_0+h_j)-\theta(x(t_0+h_j),t_0)}{h_j}\\
&\quad+\frac{\theta(x(t_0+h_j),t_0)-\theta(x_0,t_0)}{h_j}\\
&\leq \frac{\theta_t(x(t_0+h_j),t_0+\overline{h}_j)\cdot h_j}{h_j}\\
&= -(1-\delta)\theta_x(x(t_0+h_j),t_0+\overline{h}_j)H\theta(x(t_0+h_j),t_0+\overline{h}_j)\\
&\quad-\delta\theta(x(t_0+h_j),t_0+\overline{h}_j)\Lambda\theta(x(t_0+h_j),t_0+\overline{h}_j)
\end{aligned}$$

for certain $\overline{h}_j$, $0\leq\overline{h}_j\leq h_j$.

Taking limit when $h_j\to 0$ we get the inequality

$$M'(t_0)\leq -\delta\theta(x_0,t_0)\Lambda\theta(x_0,t_0)\leq 0$$

and since this happens at almost every point t_0, we may conclude that $M(t)$ is a positive decreasing function. Furthermore, if we compute the derivative taking a sequence of negative h_j we will reverse the sign of the inequality. Therefore at each point of differentiability of the function M we will get the identity

$$M'(t_0)=-\delta\theta(x_0,t_0)\Lambda\theta(x_0,t_0).$$

By a completely analogous argument we obtain that the negative function $m(t)$ is also decreasing and satisfies:

$$m'(t)=-\frac{\delta}{2\pi}m(t)\int_{-\pi}^{\pi}\frac{\theta(\overline{x},t)-\theta(y,t)}{\sin^2\frac{\overline{x}-y}{2}}\,dy\leq 0$$

at almost every t, where $\overline{x}$ is a point such that $m(t) = \theta(\overline{x}, t)$. Furthermore, since $\int_{-\pi}^{\pi} \theta_0(x)\, dx \geq 0$ and $M(t) \leq M(0)$, $m(t) \leq m(0) < 0$ the set

$$\{y : \theta(y,t) \geq \frac{\theta(\overline{x},t)}{2}\}$$

has strictly positive measure greater than a universal constant. In particular, there exists a universal positive constant C so that:

$$\frac{\delta}{2\pi}\int_{-\pi}^{\pi} \frac{\theta(y,t)-\theta(\overline{x},t)}{\sin^2\frac{\overline{x}-y}{2}}\, dy \geq C|\theta(\overline{x},t)|.$$

But then one obtains the inequality

$$|m|'(t) \geq C|m(t)|^2$$

which implies the blow-up of $m(t)$ in finite time contradicting our hypothesis about the regularity of $\theta(x,t)$.

Since the Hilbert transform H maps the space $\Lambda^\alpha = \{f \in L^\infty, \sup \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty\}$, $0 < \alpha < 1$, into itself, we have the following:

Corollary 2.2. *There is no non-zero solution of* (1.5) *with* $\delta > 0$ *so that* $\int_{-\pi}^{\pi} \theta(x,0)dx = 0$ *and* $\theta(\cdot,t) \in C^{1,\alpha}$, *for any* $\alpha > 0$ *and for every* t, $0 \leq t \leq T(\theta_0)$.

Remark 2.3. If $\int_{-\pi}^{\pi} \theta(x,0)\, dx \geq 0$ and $\min_x \theta_0 < 0$, then Theorem 2.1 and Corollary 2.2 also apply.

In the next section, we present some explicit solutions whose singularities go beyond Theorem 2.1 for $\delta = 1$.

2.2. Construction of exact solutions for $\delta = 1$

2.2.1. Periodic case

Following [6] (see [1]) closely, we can transform (1.4) into an equation for complex-valued functions. Let us recall the formulas for the Hilbert transform (see e.g. [20]):

$$H(Hf) = -f, \tag{2.1}$$

$$H(fHg + gHf) = (Hf)(Hg) - fg, \tag{2.2}$$

$$(Hf)_x = H(f_x). \tag{2.3}$$

Then, applying H on both sides of the first equation of (1.4), we have

$$(H\theta)_t + \tfrac{1}{2}((H\theta)^2 - (\theta)^2)_x = 0. \tag{2.4}$$

Thus, if we introduce the complex valued function

$$z(x,t) = H\theta(x,t) + i\theta(x,t), \quad z_0(x) = H\theta_0(x) + i\theta_0(x), \tag{2.5}$$

then (1.4) are the imaginary and the real parts of the equation,

$$z_t + zz_x = 0, \tag{2.6}$$
$$z(x,0) = z_0(x).$$

This is the inviscid Burgers equation in complex variable form, which is actually a condensed form of a system of two equations in contrast to the real Burgers equation, which is a scalar equation. In this section, we are concerned with the solutions of the following complex inviscid Burgers equation:

$$z_t + zz_x = 0, \tag{2.7}$$

where

$$z(x,t) = u(x,t) + i\theta(x,t)$$

and $u(x,t) \equiv H\theta(x,t)$. Expanding Eq. (2.7) in its real and imaginary parts one gets the system

$$u_t + uu_x - \theta\theta_x = 0, \tag{2.8}$$

$$\theta_t + u\theta_x + \theta u_x = 0. \tag{2.9}$$

In order to solve it let us introduce the hodograph transformation. This transformation is commonly used in the analysis of problems in gas dynamics and was also introduced, in a completely different context [14,15], in order to construct explicit solutions developing singularities. It will be used here for the same purpose. In order to perform the hodograph transformation we consider $x(u,\theta)$ and $t(u,\theta)$ instead of $u(x,t)$ and $\theta(x,t)$. Having in mind the relations

$$u_x = Jt_\theta,$$
$$\theta_x = -Jt_u,$$
$$u_t = -Jx_\theta,$$
$$\theta_t = Jx_u,$$

where $J = (x_u t_\theta - x_\theta t_u)^{-1}$ we deduce by direct substitution that the following linear system is equivalent to (2.8), (2.9):

$$-x_\theta + ut_\theta + \theta t_u = 0, \tag{2.10}$$

$$x_u - ut_u + \theta t_\theta = 0. \tag{2.11}$$

as far as $J^{-1} \neq 0$. System (2.10), (2.11) can be written more compactly in the form:

$$-(x - tu)_\theta + (t\theta)_u = 0,$$
$$(x - tu)_u + (t\theta)_\theta = 0,$$

which leads to the following Cauchy–Riemann system for $\eta(u, \theta) \equiv -(x(u, \theta) - t(u, \theta)u)$ and $\xi(u, \theta) \equiv -t(u, \theta)\theta$:

$$\xi_u = \eta_\theta,$$
$$\xi_\theta = -\eta_u.$$

Hence, $f(z) = \xi(u, \theta) + i\eta(u, \theta)$ where $z = u + i\theta$ is an analytic function. From the initial data for (2.7) one gets $u(x, 0)+i\theta(x, 0)$ which represents a curve γ in the complex plane parameterized by x. On the other hand, at $t = 0$ one has $\eta(u, \theta) = x(u, \theta)$ and $\xi(u, \theta) = 0$ defining the values of η and ξ along γ. Therefore, to solve the initial value problem for (2.7) is equivalent to extend analytically a complex variable function with values given along a certain curve γ. Let us consider the example

$$f(z) = \ln z.$$

The function $f(z)$ is analytic in the whole complex plane except for a branch that we locate at $(u, 0)$ with $u > 0$. Writing $z = re^{i\varphi}$ we have

$$f(z) = \ln r + i\varphi = \ln\sqrt{u^2 + \theta^2} + i \arctan\frac{\theta}{u}. \tag{2.12}$$

The real part of $f(z)$ is zero along the circumference of radius 1: $\gamma = \{(u, \theta) : u^2 + \theta^2 = 1\}$. Parameterizing γ in the form $(u, \theta) = (\cos\varphi, \sin\varphi)$ one gets $\eta = \operatorname{Im} f(z) = \varphi$. Since along γ one has $\eta(u, \theta) = -x(u, \theta)$ it follows that $\varphi = -x$ which yields the following initial data for z:

$$z(x, 0) = \cos x - i \sin x.$$

This initial data is compatible with (2.4), since $H(\sin x) = -\cos x$. From (2.12) and the definition of η and ξ it follows

$$-t\theta = \ln\sqrt{u^2 + \theta^2}, \tag{2.13}$$

$$-(x - tu) = \arctan\frac{\theta}{u} \tag{2.14}$$

which define implicitly the real and imaginary parts $(u(x, t), \theta(x, t))$ of the solution to (2.7) at any given (x, t). From (2.14) one can get

$$\theta = -u \tan(x - tu)$$

which inserted in (2.13) yields

$$tu\tan(x-tu)=\ln\left|\frac{u}{\cos(x-tu)}\right|. \tag{2.15}$$

Expression (2.15) defines $u(x,t)$ implicitly. Notice that $u(x,0)=\cos x$ satisfies (2.15). Our aim now is to show that $u(x,t)$ develops shock-type singularities at finite time. Let us fix our attention to points in a neighborhood of $x=\frac{\pi}{2}$; that is, in points of the form $x=\frac{\pi}{2}+\delta x$ with $\left|\delta x\right|\ll 1$. From (2.15) we get

$$-tu\frac{\cos(\delta x-tu)}{\sin(\delta x-tu)}=\ln\left|\frac{u}{\sin(\delta x-tu)}\right|. \tag{2.16}$$

which allows the construction of local solutions $u(x,t)$ of (2.15) near $x=\frac{\pi}{2}$ in the form

$$u\left(\frac{\pi}{2}+\delta x,t\right)\simeq A(t)\delta x.$$

Inserting this into (2.16) and letting $\delta x\to 0$ it follows:

$$-tA(t)\frac{1}{1-tA(t)}=\ln\left|\frac{A(t)}{1-tA(t)}\right|. \tag{2.17}$$

It is easy to show that $A(t)$, defined implicitly by (2.17) in such a way that $A(0)=-1$ (notice that $u_x(\frac{\pi}{2},0)=-\sin\frac{\pi}{2}=-1$), decreases for $t>0$ and blows-up to $-\infty$ at $t=e^{-1}\simeq 0.36788$. Hence, our conclusion is that $u_x(\frac{\pi}{2},t)$ blows-up at finite time. This phenomena represents the formation of a shock at $x=\frac{\pi}{2}$. We also claim that $\theta_x(\frac{\pi}{2},t)$ blows up at the same time $t=e^{-1}$. Indeed, $\theta_x(x,t)=-u_x\tan(x-tu)-u\sec^2(x-tu)(1-tu_x)$, and at $x=\frac{\pi}{2}$ we have

$$\begin{aligned}\theta_x\left(\frac{\pi}{2},t\right) &= -u_x\cot(tu)-u\csc^2(tu)(1-tu_x)\\ &= -u_x\left[\frac{\frac{1}{2}\sin(2tu)-tu}{\sin^2(tu)}\right]-\frac{u}{\sin^2(tu)}\\ &\simeq \frac{2}{3}tuu_x-\frac{1}{t^2u}\end{aligned}$$

for $|t-e^{-1}|\ll 1$. Since $u_x(\frac{\pi}{2},t)\searrow -\infty$ and $u(\frac{\pi}{2},t)\to 0$ as $t\to e^{-1}$, we conclude that $\theta_x(\frac{\pi}{2},t)$ blows up at $t=e^{-1}$.

In Figs. 1 and 2, we represent the profiles for u and θ at five different times $t=0, 0.09, 0.18, 0.27, e^{-1}$. Observe the appearance of a discontinuity in the derivative with respect to x both for u and θ at $x=(2n+1)\frac{\pi}{2}$, $n\in\mathbb{Z}$.

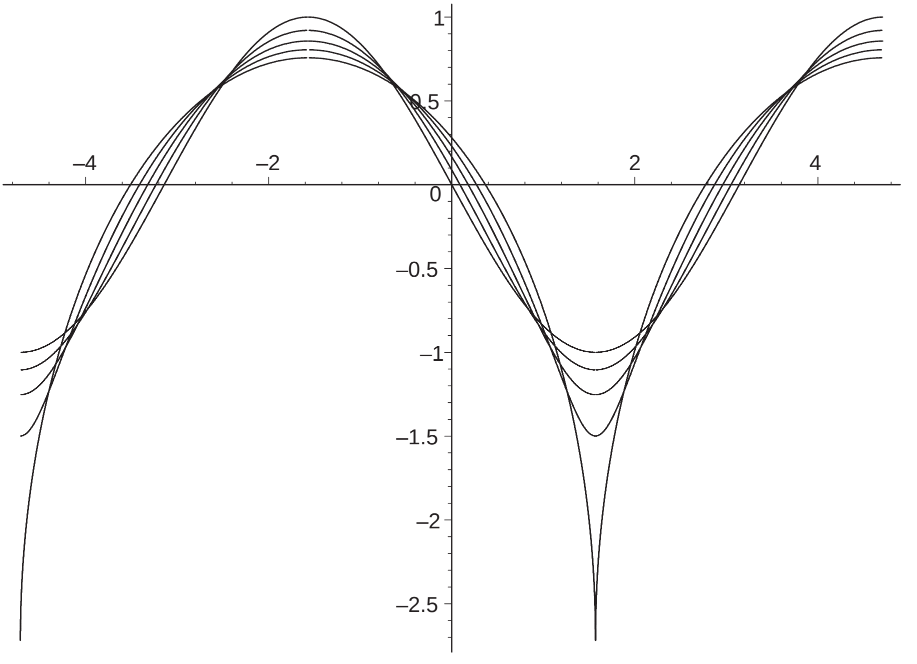

Fig. 1. θ at $t = 0,\ 0.09,\ 0.18,\ 0.27,\ e^{-1}$.

3. The critical viscous equation

If we add the first-order viscous term $(-\Delta)^{\frac{1}{2}}\theta$ to the 1D inviscid model, we get the following equation:

$$\theta_t + ((H\theta)\theta)_x = -\kappa H\theta_x.$$

Again, introducing $z = u + i\theta$ with $u = H\theta$ one gets the following viscous complex Burger's equation:

$$z_t + zz_x = -i\kappa z_x.$$

The use of the hodograph transformation allows us to obtain a system analogous to (2.10), (2.11):

$$-x_\theta + ut_\theta + \theta t_u = -\kappa t_u, \tag{3.1}$$

$$x_u - ut_u + \theta t_\theta = -\kappa t_\theta, \tag{3.2}$$

which can be written as the Cauchy–Riemann system

$$\xi_u = \eta_\theta,$$

$$\xi_\theta = -\eta_u.$$

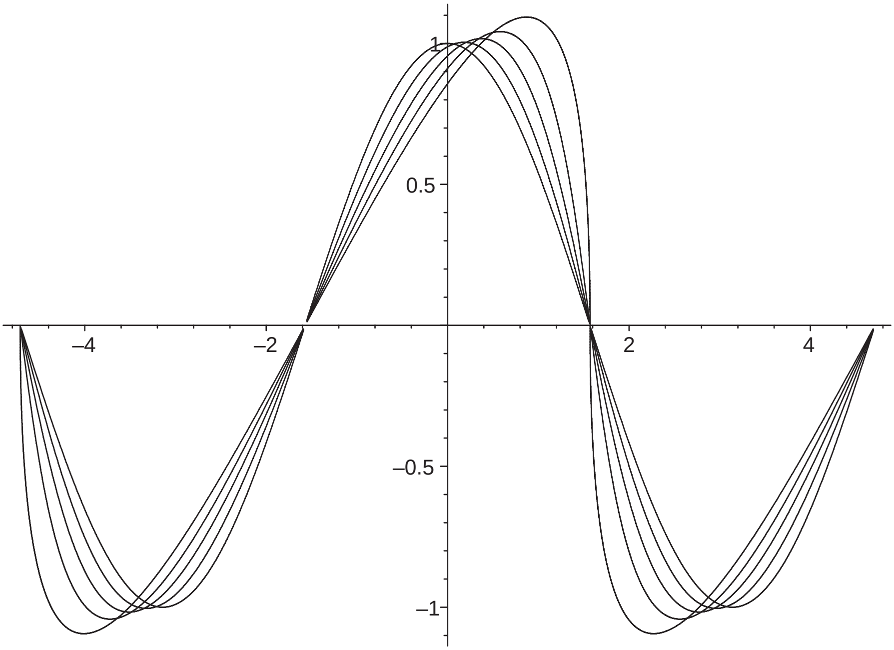

Fig. 2. u at $t = 0,\ 0.09,\ 0.18,\ 0.27,\ e^{-1}$.

for $\eta(u,\theta) \equiv -(x(u,\theta) - t(u,\theta)u)$ and $\xi(u,\theta) \equiv -t(u,\theta)(\theta+\kappa)$. With the same example studied in the previous section, $f(z) = \ln z$, one would get the following implicit equations defining $u(x,t)$ and $\theta(x,t)$:

$$-t(\kappa+\theta) = \ln\sqrt{u^2+\theta^2}, \tag{3.3}$$

$$-(x - tu) = \arctan\frac{\theta}{u}. \tag{3.4}$$

The initial data are also $u(x,0) = \cos x$, $\theta(x,0) = -\sin x$. Fixing our attention to a neighborhood of $x = \frac{\pi}{2}$, writing $u(\frac{\pi}{2} + \delta x, t) \simeq A(t)\delta x$ and letting $\delta x \to 0$ one gets, analogously to (2.17), the following equation for $A(t)$:

$$-\kappa t - tA(t)\frac{1}{1-tA(t)} = \ln\left|\frac{A(t)}{1-tA(t)}\right|. \tag{3.5}$$

If we had $A(t) \to -\infty$ at $t \to T^-$, then Eq. (3.5) would converge to the equation

$$-\kappa T + 1 = \ln\frac{1}{T},$$

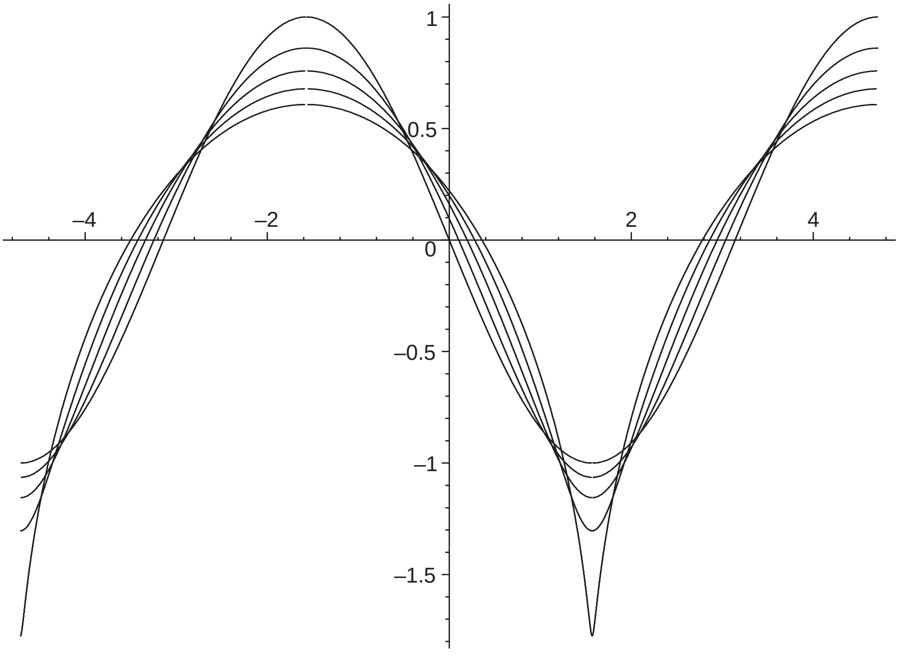

Fig. 3. θ at $t = 0,\ 0.11,\ 0.22,\ 0.33,\ 0.45$ for $\kappa = 0.5$.

which can be written in the form

$$g(T) \equiv \ln T - \kappa T + 1 = 0.$$

The function $g(T)$ has a unique maximum at $T = \kappa^{-1}$ and $g(\kappa^{-1}) = \ln(\kappa^{-1})$, provided that κ is positive, so that $g(T)$ has roots if and only if $\kappa \le 1$. Hence, the solutions will form finite time singularities if $\kappa \le 1$ and will exist globally if $\kappa > 1$.

In Figs. 3 and 4, we represent $\theta(x, t)$ and $u(x, t)$ at $t = 0, 0.11, 0.22, 0.33$ and 0.45 when $\kappa = 0.5$. As we can see, the L^∞ norm of $u(x, t)$ tends to decrease but the solution forms a finite time singularity (later than in the case $\kappa = 0$). In Figs. 5 and 6, we represent $\theta(x, t)$ and $u(x, t)$ at $t = 0, 0.11, 0.22, 0.33$ and 0.45 when $\kappa = 1.5$. The solution exists globally and the L^∞ norms of $\theta(x, t)$ and $u(x, t)$ decay.

4. Global existence of solutions in the periodic case for small data

In this section we will consider the equation with critical viscosity and $\kappa > 0$

$$\theta_t + ((H\theta)\theta)_x = -\kappa H\theta_x. \tag{4.6}$$

Our aim is to prove the following:

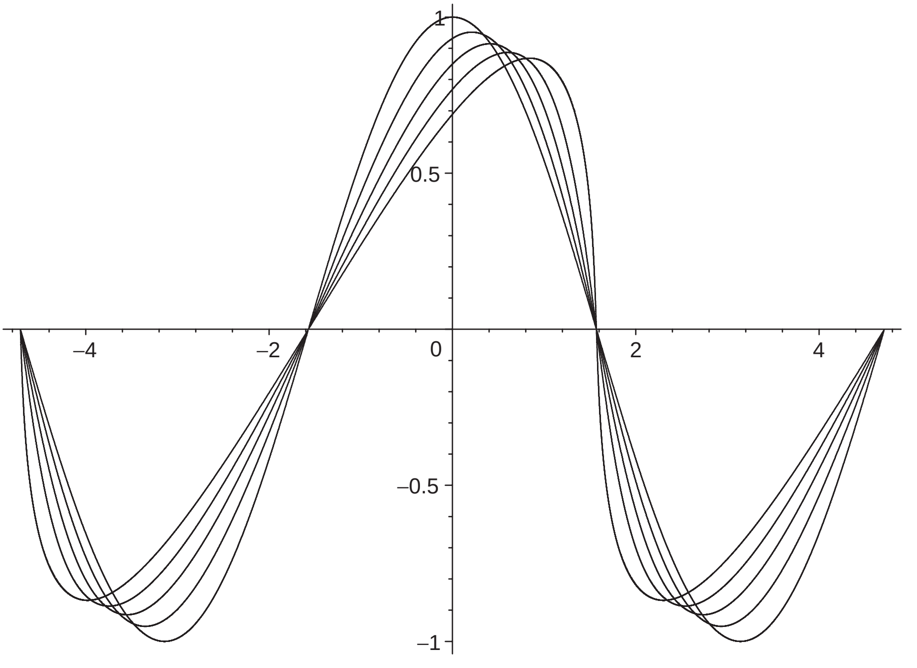

Fig. 4. u at $t = 0,\ 0.11,\ 0.22,\ 0.33,\ 0.45$ for $\kappa = 0.5$.

Theorem 4.1. *If the initial data* θ_0 *verifies* $\int_{-\pi}^{\pi} \theta_0(x)\, dx = 0$, $\|\theta_0\|_{L^\infty} < \kappa$ *and* $\|\Lambda^{\frac{3}{2}}\theta_0\|_{L^2} < \infty$, *then there is a classical solution of Eq.* (4.6) *that satisfies* $\theta \in C^1([0,\infty));\ W^{\frac{3}{2}}([-\pi,\pi])$ *and* $\|\theta(-,t)\|_{L^\infty} < \kappa$ *for every* $t \geq 0$.

Proof. This will be based in the following sequence of facts:

Fact 1. *If* $\theta \in C^1([-\pi,\pi] \times [0,T])$ *is a solution of* (4.6) *where the initial data* θ_0 *satisfies the hypothesis given in the theorem above, then we have*:

(i) $M(t) = \max_x \theta(x,t)$, *is a positive monotonically decreasing Lipschitz function.*
(ii) $m(t) = \min_x \theta(x,t)$, *is a negative monotonically increasing Lipschitz function.*

Proof. The proof follows the scheme introduced in Section 2. Let $x(t)$, $\overline{x}(t)$ be chosen in such a way that

$$M(t) = \theta(x(t), t),$$

$$m(t) = \theta(\overline{x}(t), t)$$

and assume that t is a point of differentiability of both Lipschitz functions $M(\cdot)$, $m(\cdot)$. Then we have

$$M'(t) \leq -(M(t) + \kappa)\Lambda\theta(x(t), t) \leq 0,$$

$$m'(t) \geq -(m(t) + \kappa)\Lambda\theta(\overline{x}(t), t) \geq 0. \qquad \square$$

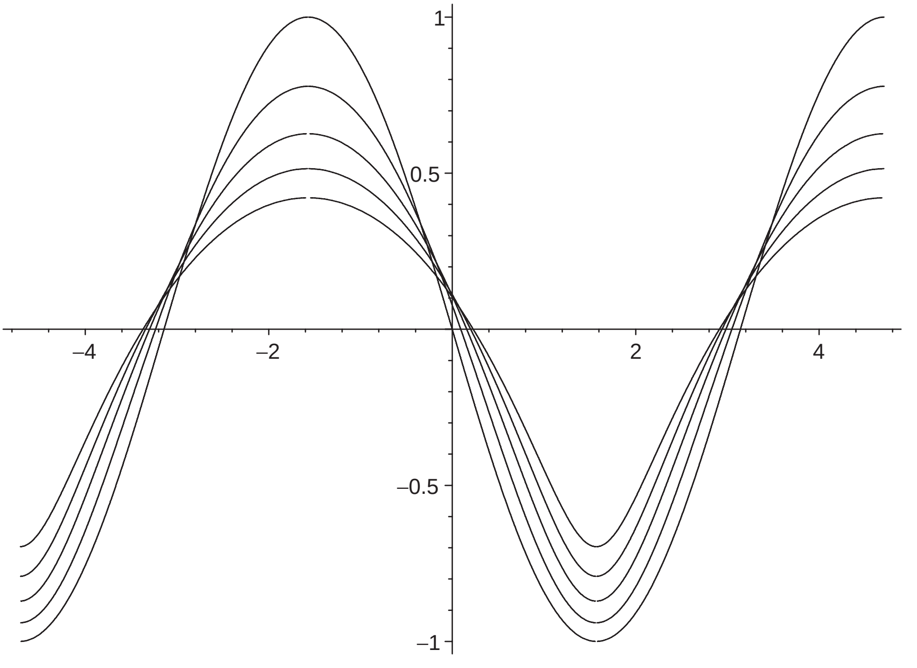

Fig. 5. θ at $t = 0,\ 0.11,\ 0.22,\ 0.33,\ 0.45$ for $\kappa = 1.5$.

Fact 2. $\|\theta(-,t)\|_{L^2}$ *is monotonically decreasing.*

Proof. From Eq. (4.6) we have

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 = \int \theta\theta_t = \int \theta_x \theta H\theta - \kappa \int \theta \Lambda\theta$$
$$= -\frac{1}{2}\int \theta^2 \Lambda\theta - \kappa \int \theta\Lambda\theta.$$

Since

$$\int \theta^2(x)\Lambda\theta(x)\,dx = \int \theta^2(x)\int \frac{\theta(x)-\theta(y)}{[\sin\frac{x-y}{2}]^2}\,dy\,dx = -\int \theta^2(y)\int \frac{\theta(x)-\theta(y)}{[\sin\frac{x-y}{2}]^2}\,dx\,dy$$
$$= \int\int \frac{\theta(x)+\theta(y)}{2}\frac{[\theta(x)-\theta(y)]^2}{[\sin\frac{x-y}{2}]^2}\,dy\,dx$$

and

$$\int \theta(x)\Lambda\theta(x)\,dx = \int\int \frac{[\theta(x)-\theta(y)]^2}{[\sin\frac{x-y}{2}]^2}\,dy\,dx,$$

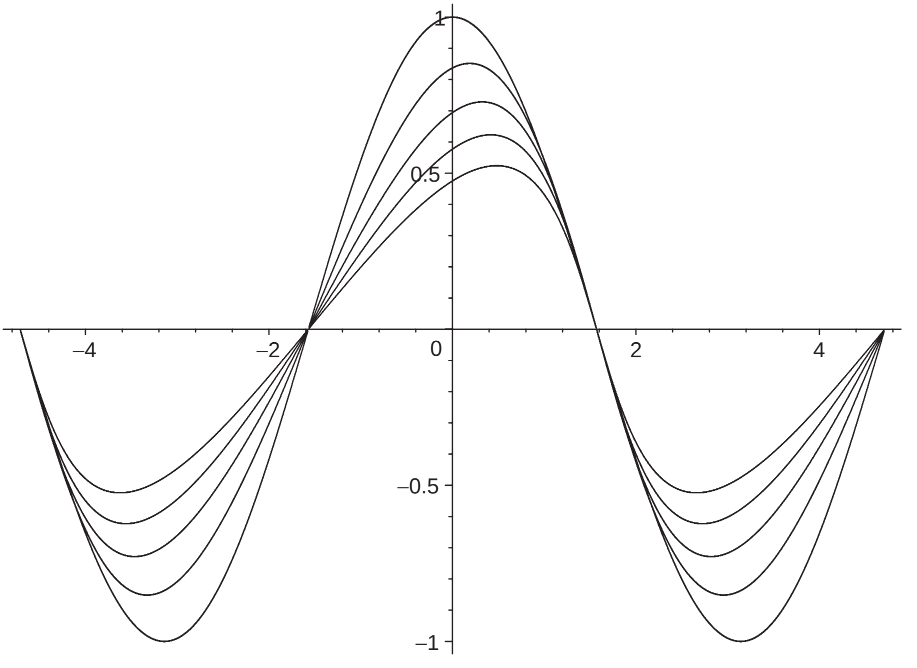

Fig. 6. u at $t = 0,\ 0.11,\ 0.22,\ 0.33,\ 0.45$ for $\kappa = 1.5$.

we obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 = -\frac{1}{2}\int\int\left[\frac{\theta(x)+\theta(y)}{2}+2\kappa\right]\frac{[\theta(x)-\theta(y)]^2}{[\sin\frac{x-y}{2}]^2}\,dy\,dx$$
$$\leq -\frac{\kappa}{2}\|\Lambda^{\frac{1}{2}}\theta\|_{L^2}^2 \leq -\frac{\kappa}{2}\|\theta\|_{L^2}^2, \tag{4.7}$$

which implies the result. Furthermore we get $\|\theta\|_{L^2}^2 \leq \|\theta_0\|_{L^2}^2 e^{-\kappa t}$. $\square$

Fact 3.

$$\int_0^T \|\Lambda^{\frac{1}{2}}\theta\|_{L^2}^2\,dt \leq \frac{1}{\kappa}\|\theta_0\|_{L^2}^2.$$

We obtain integrating inequality (4.7).

Next, let us consider

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{1}{2}}\theta\|_{L^2}^2 = \int \Lambda^{\frac{1}{2}}\theta\Lambda^{\frac{1}{2}}\theta_t = -\int \Lambda\theta(\theta H\theta)_x - \kappa\int|\Lambda\theta|^2\,dx$$

$$\begin{aligned}
&= -\int \Lambda\theta\theta_x H\theta - \int (\Lambda\theta)^2\theta - \kappa\|\Lambda\theta\|_{L^2}^2 \\
&= \int H(\Lambda\theta\theta_x)\theta - \int (\Lambda\theta)^2\theta - \kappa\|\Lambda\theta\|_{L^2}^2 \\
&= \frac{1}{2}\int \theta[(H\theta_x)^2 - (\theta_x)^2] - \int (\Lambda\theta)^2\theta - \kappa\|\Lambda\theta\|_{L^2}^2 \\
&= -\frac{1}{2}\int \theta[(H\theta_x)^2 + (\theta_x)^2] - \kappa\|\Lambda\theta\|_{L^2}^2 \\
&\leq (\|\theta\|_{L^\infty} - \kappa)\|\Lambda\theta\|_{L^2}^2.
\end{aligned}$$

Since $\|\theta_0\|_{L^\infty} < \kappa$, we have

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{1}{2}}\theta\|_{L^2}^2 \leq -c(\kappa)\|\Lambda\theta\|_{L^2}^2,$$

where $c(\kappa) = (\kappa - \|\theta\|_{L^\infty}) > 0$.

An integration of our last inequality yields

$$\int_0^T \|\Lambda\theta\|_{L^2}^2\, dt \leq C(\kappa)\|\Lambda^{\frac{1}{2}}\theta_0\|_{L^2}^2,$$

where $C(\kappa) = \frac{1}{2c(\kappa)}$.

Fact 4. *The evolution of the norm $\|\Lambda^{\frac{3}{2}}\theta\|_{L^2}$ exists.*

This follows from the following estimates:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{3}{2}}\theta\|_{L^2}^2 &= -\int \Lambda^{\frac{3}{2}}\theta\Lambda^{\frac{3}{2}}(\theta H\theta)_x - \kappa\|\Delta\theta\|_{L^2}^2 \\
&= \int \Lambda^{\frac{3}{2}}\theta\Lambda^{\frac{3}{2}}\Lambda H(\theta H\theta) - \kappa\|\Delta\theta\|_{L^2}^2 \\
&= \int \Delta\theta\Delta[\frac{1}{2}(H\theta)^2 - \frac{1}{2}\theta^2] - \kappa\|\Delta\theta\|_{L^2}^2 \\
&= \int \Delta\theta(\Delta(H\theta)H\theta + |\nabla H\theta|^2 - \theta\Delta\theta - |\nabla\theta|^2) - \kappa\|\Delta\theta\|_{L^2}^2.
\end{aligned}$$

Let us observe that

$$\begin{aligned}
|\int [\Delta\theta\Delta(H\theta)H\theta - \theta\Delta\theta]| &= |\int [\theta\frac{1}{2}((\Delta H\theta)^2 - (\Delta\theta)^2) + \theta(\Delta\theta)^2]| \\
&\leq \|\theta\|_{L^\infty}\|\Delta\theta\|_{L^2}^2
\end{aligned}$$

and since $\|\Lambda\theta\|_{L^4}^2 \leq C\|\Lambda\theta\|_{L^2}\|\Lambda^{\frac{3}{2}}\theta\|_{L^2}$ we have

$$\begin{aligned} |\int \Delta\theta[|\nabla H\theta|^2 - |\nabla\theta|^2]| &\leq C\|\Delta\theta\|_{L^2}\|\Lambda\theta\|_{L^4}^2 \\ &\leq C\|\Delta\theta\|_{L^2}\|\Lambda\theta\|_{L^2}\|\Lambda^{\frac{3}{2}}\theta\|_{L^2} \\ &\leq \delta\|\Delta\theta\|_{L^2}^2 + \frac{C}{\delta}\|\Lambda\theta\|_{L^2}^2\|\Lambda^{\frac{3}{2}}\theta\|_{L^2}^2, \end{aligned}$$

where C is a constant and we choose $\delta = \frac{\kappa - \|\theta_0\|_{L^\infty}}{2}$. By Fact 1 follows that

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{3}{2}}\theta\|_{L^2}^2 \leq \frac{C}{\delta}\|\Lambda\theta\|_{L^2}^2\|\Lambda^{\frac{3}{2}}\theta\|_{L^2}^2 + \frac{1}{2}(\|\theta_0\|_{L^\infty} - \kappa)\|\Delta\theta\|_{L^2}^2.$$

Therefore

$$\|\Lambda^{\frac{3}{2}}\theta\|_{L^2}^2(t) \leq \|\Lambda^{\frac{3}{2}}\theta_0\|_{L^2}^2 e^{\frac{C}{\delta}\int_0^t \|\Lambda\theta\|_{L^2}^2(s)\,ds} \leq \|\Lambda^{\frac{3}{2}}\theta_0\|_{L^2}^2 e^{C(\kappa)\|\Lambda^{\frac{1}{2}}\theta_0\|_{L^2}^2}$$

by Fact 3.

Fact 5. *Facts 1–4 continue to hold for the equation*

$$\begin{aligned} \theta_t + (\theta H\theta)_x &= -\kappa\Lambda\theta + \varepsilon\theta_{xx}, \\ \theta(x,0) &= \theta_0, \end{aligned} \tag{4.8}$$

uniformly on $\varepsilon > 0$, *under the hypothesis that* $\|\Lambda^3\theta_0\|_{L^2} < \infty$.

(a) *Given* $\varepsilon > 0$ *and initial data* $\theta_0 \in W^3(-\pi,\pi)$ *such that*:

$$\int_{-\pi}^{\pi} \theta_0(x)\,dx = 0 \quad \textit{and} \quad \|\theta_0\|_{L^\infty} < \kappa,$$

there exists

$$T = T(\varepsilon, \|\theta_0\|_{L^\infty}, \|(\theta_0)_{xxx}\|_{L^2}) > 0,$$

with $\|(\theta)_{xxx}\|_{L^2} < \infty$ *for* $0 \leq t \leq T$.

The part (a) follows from the following estimates:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta_{xxx}\|_{L^2}^2 &= -\int \theta_{xxx}(\theta H\theta)_{xxxx} - \kappa\|\Lambda^{\frac{7}{2}}\theta\|_{L^2}^2 - \varepsilon\|\Lambda^4\theta\|_{L^2}^2 \\ &= \int \theta_{xxxx}(\theta H\theta)_{xxx} - \kappa\|\Lambda^{\frac{7}{2}}\theta\|_{L^2}^2 - \varepsilon\|\Lambda^4\theta\|_{L^2}^2. \end{aligned}$$

Let us observe that

$$\begin{aligned} |\int \theta_{xxxx}(\theta H\theta)_{xxx}| &\leq \|\theta_{xxxx}\|_{L^2}\|(\theta H\theta)_{xxx}\|_{L^2} \\ &\leq \|\theta_{xxxx}\|_{L^2}[\|\theta_{xxx}\|_{L^2}(\|H\theta\|_{L^\infty}+\|\theta\|_{L^\infty}) \\ &\quad +(\|\theta_{xx}\|_{L^\infty}+\|H\theta_{xx}\|_{L^\infty})\|\theta_x\|_{L^2}] \\ &\leq C\|\Lambda^4\theta\|_{L^2}\|\theta_{xxx}\|_{L^2}\|\Lambda\theta\|_{L^2} \\ &\leq \frac{\varepsilon}{4}\|\Lambda^4\theta\|^2_{L^2}+\frac{C}{\varepsilon}\|\theta_{xxx}\|^2_{L^2}\|\Lambda\theta\|^2_{L^2}. \end{aligned}$$

That is

$$\frac{1}{2}\frac{d}{dt}\|\theta_{xxx}\|^2_{L^2} \leq \frac{C}{\varepsilon}\|\theta_{xxx}\|^2_{L^2}\|\Lambda\theta\|^2_{L^2} \tag{4.9}$$

and since $\|\Lambda\theta\|_{L^2} \leq C\|\Lambda^3\theta\|_{L^2}$, we obtain local existence for $\|\theta_{xxx}\|_{L^2}$.

Therefore $\theta \in C^2$ for $0 \leq t \leq T$ and Facts 1–3 follows for $\varepsilon > 0$.

In particular from (4.9) and using the following inequality

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta\|^2_{L^2} &= -\frac{1}{2}\int \theta^2\Lambda\theta - \kappa\int \theta\Lambda\theta - \varepsilon\|\Lambda\theta\|^2_{L^2} \\ &\leq -\varepsilon\|\Lambda\theta\|^2_{L^2}. \end{aligned}$$

we get

$$\|\theta_{xxx}\|^2_{L^2} \leq \|(\theta_0)_{xxx}\|^2_{L^2} e^{C_\varepsilon \int_0^T \|\Lambda\theta\|^2_{L^2} ds} \leq \|(\theta_0)_{xxx}\|^2_{L^2} e^{C_\varepsilon\|\theta_0\|^2_{L^2}}$$

where $C_\varepsilon = C_\varepsilon(\varepsilon, \theta_0)$ is a constant, allowing us to conclude that $\|\theta_{xxx}\|_{L^2} < C$ for all time and that θ_{xx} is a continuous function, which gives us the maximum principle for (4.8).

Proof of Theorem 4.1 (*Conclusion*). Therefore, for fixed $\varepsilon > 0$, one obtains a solution $\theta^\varepsilon \in C^2([-\pi, \pi] \times [0, \infty))$ of the problem

$$\begin{aligned} \theta^\varepsilon_t + (\theta^\varepsilon H\theta^\varepsilon)_x &= -\kappa\Lambda\theta^\varepsilon + \varepsilon\theta^\varepsilon_{xx}, \\ \theta^\varepsilon(x, 0) &= \theta^\varepsilon_0, \end{aligned}$$

where θ^ε_0 is the convolution of θ_0 with a smooth approximation of the identity, so that, uniformly on $\varepsilon > 0$, we have

$$\|\theta^\varepsilon(-, t)\|_{L^\infty} < \kappa \quad \text{for every } t \geq 0,$$

$$\|\theta^\varepsilon(-,t)\|_{L^2}^2 \le \|\theta_0\|_{L^2}^2 e^{-\kappa t},$$

$$\int_0^\infty \|\Lambda\theta^\varepsilon\|_{L^2}^2\,dt \le C(\kappa)\|\Lambda^{\frac{1}{2}}\theta_0\|_{L^2}^2,$$

$$\|\Lambda^{\frac{3}{2}}\theta^\varepsilon\|_{L^2} \le C(\kappa,\theta_0)\|\Lambda^{\frac{3}{2}}\theta_0\|_{L^2}.$$

We are now in position to use compacity to select a converging subsequence θ^ε to obtain a solution θ of Eq. (4.6) satisfying the requirements of Theorem 4.1. □

Appendix

Adding the second-order viscosity term $\mu\theta_{xx}$ to the 1D inviscid QG model equation, we obtain

$$\theta_t + (H(\theta)\theta)_x = \mu\theta_{xx}, \qquad \text{(A.1)}$$

$$\theta(x,0) = \theta_0(x).$$

In [1,18] they show the existence of singularities with a specific initial data for Eq. (A.1).

Introducing the complex valued function, $z(x,t) = H\theta(x,t) + i\theta(x,t)$ as previously, we find that (A.1) is the imaginary part of the complex viscous Burgers equation,

$$z_t + zz_x = \mu z_{xx}, \qquad \text{(A.2)}$$

$$z(x,0) = z_0(x).$$

One can solve (A.2) explicitly by the (complex) Hopf–Cole transform as follows. We consider the change of variable $z \mapsto w$, defined by

$$z(x,t) = -2\mu\frac{w_x(x,t)}{w(x,t)}.$$

By elementary computations we find that $w(x,t)$ satisfies the complex heat equation,

$$w_t = \mu w_{xx},$$

$$w(x,0) = \exp\left(\frac{1}{2\mu}\int_{-\infty}^x z_0(s)\,ds\right).$$

We first consider the case of the whole domain of $\mathbb{R}$. Using the well-known heat kernel representation of the solution $w(x,t)$, we obtain the explicit solution of the complex Burgers equation as

$$z(x,t) = \frac{\int_{-\infty}^{\infty}\frac{x-y}{t}\exp\left[-\frac{|x-y|^2}{2\mu t} - \frac{1}{2\mu}\int_{-\infty}^{y} z_0(s)\,ds\right]dy}{\int_{-\infty}^{\infty}\exp\left[-\frac{|x-y|^2}{2\mu t} - \frac{1}{2\mu}\int_{-\infty}^{y} z_0(s)\,ds\right]dy}. \qquad \text{(A.3)}$$

Substituting $z_0(x) = (H\theta_0)(x) + i\theta_0(x)$, and taking the imaginary part of (A.3), we find explicitly the solution of (A.1) given by

$$\theta(x,t) = \frac{-\tilde{B}(x,t)\int_{-\infty}^{\infty} \frac{x-y}{t} A(x,y,t)\,dy + \tilde{A}(x,t)\int_{-\infty}^{\infty} \frac{x-y}{t} B(x,y,t)\,dy}{\tilde{A}^2(x,t) + \tilde{B}^2(x,t)}, \qquad \text{(A.4)}$$

where we denoted

$$A(x,y,t) = \exp\left[-\frac{|x-y|^2}{2\mu t} - \frac{1}{2\mu}\int_{-\infty}^{y} H\theta_0(s)\,ds\right]\cos\left(\frac{1}{2\mu}\int_{-\infty}^{y} \theta_0(s)\,ds\right),$$

$$B(x,y,t) = \exp\left[-\frac{|x-y|^2}{2\mu t} - \frac{1}{2\mu}\int_{-\infty}^{y} H\theta_0(s)\,ds\right]\sin\left(\frac{1}{2\mu}\int_{-\infty}^{y} \theta_0(s)\,ds\right)$$

and

$$\tilde{A}(x,t) = \int_{-\infty}^{\infty} A(x,y,t)\,dy, \quad \tilde{B}(x,t) = \int_{-\infty}^{\infty} B(x,y,t)\,dy.$$

Next, in the periodic case, we can solve (A.2) explicitly, using the Fourier series combined with the Hopf–Cole transform. We first solve the complex heat equation by the standard Fourier series method as

$$w(x,t) = \sum_{k\in\mathbb{Z}} \hat{w}_0(k) e^{-\mu k^2 t + ikx},$$

where

$$\begin{aligned}\hat{w}_0(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} w_0(x)e^{-ikx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left[\frac{1}{2\mu}\int_0^x z_0(y)\,dy - ikx\right]dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left[\frac{1}{2\mu}\int_0^x (H\theta_0)(y)\,dy + i\left(\frac{1}{2\mu}\int_0^x \theta_0(y)\,dy - kx\right)\right]dx.\end{aligned}$$

Hence,

$$\theta(x,t) = -2\mu\,\mathrm{Im}\left\{\frac{w_x}{w}\right\} = -2\mu\,\mathrm{Re}\left\{\frac{\sum_{k\in\mathbb{Z}} k\hat{w}_0(k)e^{-\mu k^2 t + ikx}}{\sum_{k\in\mathbb{Z}} \hat{w}_0(k)e^{-\mu k^2 t + ikx}}\right\}$$

with $\hat{w}_0(k)$ given above.

References

[1] G. Baker, X. Li, A. Morlet, Analytic structure of two 1D-transport equations with nonlocal fluxes, Physica D 91 (1996) 349–375.

[2] D. Chae, The quasi-geostrophic equation in the Triebel–Lizorkin spaces, Nonlinearity 16 (2) (2003) 479–495.
[3] D. Chae, J. Lee, Global Well-Posedness in the super critical dissipative Quasi-geostrophic equations, Commun. Math. Phys. 233 (2003) 297–311.
[4] P. Constantin, Energy spectrum of quasi-geostrophic turbulence, Phys. Rev. Lett. 89 (18) (2002) 1804501-4.
[5] P. Constantin, D. Córdoba, J. Wu, On the critical dissipative Quasi-geostrophic equation, Indiana Univ. Math. J. 50 (2001) 97–107.
[6] P. Constantin, P. Lax, A. Majda, A simple one-dimensional model for the three dimensional vorticity, Comm. Pure Appl. Math. 38 (1985) 715–724.
[7] P. Constantin, A. Majda, E. Tabak, Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar, Nonlinearity 7 (1994) 1495–1533.
[8] P. Constantin, J. Wu, Behavior of solutions of 2D quasi-geostrophic equations, SIAM J. Math. Anal. 30 (1999) 937–948.
[9] A. Córdoba, D. Córdoba, A pointwise estimate for fractionary derivatives with applications to P.D.E., Proc. Natl. Acad. Sci. 100 (26) (2003) 15316–15317.
[10] A. Córdoba, D. Córdoba, A maximum principle applied to Quasi-geostrophic equations, Comm. Math. Phys. 249 (2004) 511–528.
[11] A. Córdoba, D. Córdoba, M.A. Fontelos, Formation of singularities for a transport equation with nonlocal velocity, preprint.
[12] D. Córdoba, Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation, Ann. of Math. 148 (1998) 1135–1152.
[13] D. Córdoba, C. Fefferman, Growth of solutions for QG and 2D Euler equations, J. Amer. Math. Soc. 15 (3) (2002) 665–670.
[14] M.A. Fontelos, J.J.L. Velázquez, On some breakup and singularity formation mechanisms for inviscid liquid jets. SIAM J. Appl. Math. 59 (6) (1999) 2274–2300.
[15] M.A. Fontelos, J.J.L. Velázquez, Fractal-like singularities for an inviscid one-dimensional model of fluid jets, European J. Appl. Math. 11 (1) (2000) 29–60.
[16] I. Held, R. Pierrehumbert, S. Garner, K. Swanson, Surface quasi-geostrophic dynamics, J. Fluid Mech. 282 (1995) 1–20.
[17] A.S. Vasudeva Murthy, The Constantin–Lax–Majda model for the vorticity equation revisited, J. Indian Inst. Sci. 78 (1998) 109–117.
[18] A. Morlet, Further properties of a continuum of model equations with globally defined flux, J. Math. Anal. Appl. 221 (1998) 132–160.
[19] K. Ohkitani, M. Yamada, Inviscid and inviscid-limit behavior of a surface quasi-geostrophic flow, Phys. Fluids 9 (1997) 876–882.
[20] J.N. Pandey, The Hilbert Transform of Schwartz Distributions and Applications, Wiley, New York, 1996.
[21] J. Pedlosky, Geophysical Fluid Dynamics, Springer, New York, 1987.
[22] S. Resnick, Dynamical problems in nonlinear advective partial differential equations, Ph.D. Thesis, University of Chicago, Chicago, 1995.
[23] T. Sakajo, On global solutions for the Constantin–Lax–Majda equation with a generalized viscosity term, Nonlinearity 16 (4) (2003) 1319–1328.
[24] S. Schochet, Explicit solutions of the viscous model vorticity equation, Commun. Pure Appl. Math. 39 (1986) 531–537.
[25] M.E. Schonbek, T.P. Schonbek, Asymptotic behavior to dissipative quasi-geostrophic flows, SIAM J. Math. Anal. 35 (2) (2003) 357–375.
[26] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, 1970.
[27] J. Wu, Dissipative quasi-geostrophic equations with L^p data, Electron. J. Differential Equations 56 (2001) 1–13.
[28] J. Wu, Inviscid limits and regularity estimates for the solutions of the 2-D dissipative Quasi-geostrophic equations, Indiana Univ. Math. J. 46 (4) (1997) 1113–1124.