Integral inequalities for the Hilbert transform applied to a nonlocal transport equation

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Abstract

We prove several weighted inequalities involving the Hilbert transform of a function \( f(x) \) and its derivative. One of those inequalities,

\[-\int \frac{f_x(x)(Hf(x) - Hf(0))}{|x|^\alpha} \, dx \geq C_\alpha \int \frac{(f(x) - f(0))^2}{|x|^{1+\alpha}} \, dx,
\]
is used to show finite time blow-up for a transport equation with nonlocal velocity.

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1. Introduction

In this paper we show the existence of finite-time singularities for a Burgers type equation with nonlocal velocity in one space variable,

\[ f_t - (Hf)f_x = 0, \]

where \( H(\cdot) \) denotes the Hilbert transform (see (2) below), improving the results of our previous paper [4] in the sense that we can obtain here blow-up for a much wider class of initial data. One motivation for the study of that equation is its analogy with both, the 2D quasi-geostrophic equation and the 3D Euler in vorticity form (see [2,3,5,6,8–10] and [11] for a variety of 1D models involving nonlocal operators, the majority of them have their origin in the seminal paper of Constantin, Lax and Majda [3]). For instance the 2D quasi-geostrophic equation belongs to the class of transport equations with nonlocal velocities:

\[ \theta_t - R_2(\theta) \frac{\partial \theta}{\partial x_1} + R_1(\theta) \frac{\partial \theta}{\partial x_2} = 0, \]

where the velocity is \( u = (-R_2(\theta), R_1(\theta)) \) and \( R(\theta) = (R_1(\theta), R_2(\theta)) \) are the Riesz transforms of \( \theta \). This is a well-known model for the dynamic of mixtures of cold and hot air.

Another motivation for (1) is its similarity in structure with the Birkhoff–Rott equation for the evolution of vortex sheets (see [1,4] and [7]).

In this paper we present examples showing the existence of such singularities, our proofs below will take advantage of certain weighted norm inequalities satisfied by the Hilbert transform. There is a vast literature about such topic, but, in our case, we need very precise estimates, separately for both even and odd functions, which do not follow directly from the general theorems. Since the weights involved are powers of the independent variable, the Mellin transform is a very adequate instrument to produce those sharp estimates.

In the following we shall consider the Hilbert transform defined by the formulas:

\[ Hf(x) = \frac{1}{\pi} \text{PV} \int \frac{f(y)}{x-y} \, dy, \]

or \( \hat{H}f(\xi) = -i \text{sign}(\xi) \hat{f}(\xi) \), where \( \hat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x) \, dx \) denotes the Fourier transform.

We shall make use also of the identity,

\[ \int_0^\infty t^{-\beta} \frac{dt}{t-1} = \pi \cot(\pi \beta), \]

valid for \( 0 < \text{Re}(\beta) < 1 \) and the explicit formula:

\[ H(|x|^\alpha) = -\frac{1}{\pi} \tan \left( \frac{\alpha \pi}{2} \right) \text{sign}(x)|x|^\alpha, \]

with \(-3 < \alpha < 1 \) and \( \alpha \neq 0, -1, -2 \). The distribution \( 1/|x|^\beta \) is defined by,

\[ \frac{1}{|x|^\beta}(\varphi) = \text{PV} \int \frac{\varphi(x) - \varphi(0)}{|x|^\beta} \, dx \quad \text{if } 1 < \beta < 2, \]

and by,

\[ \frac{1}{|x|^\beta}(\varphi) = \text{PV} \int \frac{\varphi(x) - \varphi(0) - x\varphi_x(0)}{|x|^\beta} \, dx \quad \text{if } 2 < \beta < 3. \]

To present our results we find it convenient to introduce the following functional spaces: For \( 0 < \alpha < 2 \), \( H_\alpha^1(\mathbb{R}) \) is the closure of \( C_0^1(\mathbb{R}) \) under the norm:

\[ \|f\|^2 = \|f\|_{L^\infty}^2 + \int \frac{(f(x) - f(0))^2}{|x|^{1+\alpha}} \, dx + \int \frac{(f_x(x))^2}{|x|^\alpha-1} \, dx, \]

and for \( \alpha \leq 0 \) we define \( H_\alpha^1(\mathbb{R}) \) as before taking out the correction term \( f(0) \).

We will prove the following inequalities involving Hilbert transforms:
Theorem 1.1. (I) For every $\alpha$, $0 < \alpha < 2$, there exists a strictly positive constant $C_\alpha$ so that the inequality,

$$- \int \frac{f_x(x)Hf(x)}{|x|^\alpha} \, dx \geq C_\alpha \int \frac{(f(x) - f(0))^2}{|x|^{1+\alpha}} \, dx,$$

holds for every even function in $C^1(\mathbb{R}) \cap H_\alpha^1(\mathbb{R})$.

(II) Let $f \in C^1(\mathbb{R}) \cap H_\alpha^1(\mathbb{R})$ be a nonnegative (or nonpositive) function. Then for every $\alpha$, $1 < \alpha < 2$, there exists a strictly positive constant $C_\alpha$ so that

$$- \int f_x(x)[Hf(x) - Hf(0)] |x|^{\alpha} \, dx \geq C_\alpha \int \frac{(f(x) - f(0))^2}{|x|^{1+\alpha}} \, dx.$$

(III) The inequality,

$$- \int f_x(x)Hf(x) |x|^\alpha \, dx \geq C_\alpha \int f^2(x) |x|^{\alpha-1} \, dx,$$

with $C_\alpha > 0$, is satisfied when $0 < \alpha < 2$ and $f \in C^1(\mathbb{R}) \cap H_\alpha^1(\mathbb{R})$ is an even function or when $1 < \alpha < 2$ and $f \in C^1(\mathbb{R}) \cap H_{1-\alpha}(\mathbb{R})$ is either nonnegative or nonpositive.

As it was mentioned before, we will use these inequalities to prove the formation of finite time singularities for the Burgers-type equation (1) with nonlocal velocity field.

The Cauchy problem is well posed for Eq. (1) giving us classical solutions $f(x,t)$ for a short period of time $0 \leq t < T$, for smooth enough initial data. Concerning blow-up solutions of Eq. (1) we have the following theorem:

Theorem 1.2. Let $f(x,0) \in C^1_{0+}(\mathbb{R})$ be positive and compactly supported. Then there is no global in time, locally bounded (in space) solution to Eq. (1).

The analysis involved in Theorem 1.1 leads to other interesting inequalities about the Hilbert transform, namely:

Theorem 1.3. Let $f \in H_\alpha^1(\mathbb{R})$. Then

(I) $0 < \alpha < 2$ we have:

$$\int_{-\infty}^{\infty} \frac{|(Hf)(x) - (Hf)(0)|^2}{|x|^{\alpha+1}} \, dx \geq \min \left\{ \tan^2 \left( \frac{1}{4} \pi \alpha \right), \cot^2 \left( \frac{1}{4} \pi \alpha \right) \right\} \int_{-\infty}^{\infty} \frac{|(f(x) - f(0))^2}{|x|^{\alpha+1}} \, dx.$$

(II) $-2 < \alpha < 0$ we have:

$$\int_{-\infty}^{\infty} \frac{|(Hf)(x)|^2}{|x|^{\alpha+1}} \, dx \geq \min \left\{ \tan^2 \left( \frac{1}{4} \pi \alpha \right), \cot^2 \left( \frac{1}{4} \pi \alpha \right) \right\} \int_{-\infty}^{\infty} \frac{|f(x)|^2}{|x|^{\alpha+1}} \, dx.$$

Theorem 1.4. (I) Let $f$ be an even function. Then for every $\alpha$, $0 < \alpha < 2$, we have:

$$\int_{-\infty}^{\infty} \frac{|f(x) - f(0)|Hf(x)}{|x|^{\alpha+1}} \, dx \leq 0 \quad \text{for } f \in H_\alpha^1(\mathbb{R}),$$

$$\int_{-\infty}^{\infty} \frac{f(x)Hf(x)}{|x|^{\alpha+1}} \, dx \geq 0 \quad \text{for } f \in H_{1-\alpha}(\mathbb{R}).$$
Let $f$ be an odd function. Then for every $\alpha$, $0 < \alpha < 2$, we have:

\[
\int_{-\infty}^{\infty} \frac{f(x) [Hf(x) - Hf(0)]}{|x|^\alpha} \, dx \geq 0 \quad \text{for } f \in H^1_\alpha(\mathbb{R}),
\]

(10)

\[
\int_{-\infty}^{\infty} \frac{f(x)Hf(x)}{|x|^{-\alpha}x} \, dx \leq 0 \quad \text{for } f \in H^{-1}_\alpha(\mathbb{R}).
\]

(11)

In Section 2 we present the proof of Theorem 1.1. Section 3 will be devoted to prove blow-up for Eq. (1) (Theorem 1.2). Finally Section 4 contains the proof of the inequalities for the Hilbert transform presented in Theorems 1.3 and 1.4.

2. Proof of Theorem 1.1

An important ingredient in the proofs of the above inequalities will be the use of Mellin transforms of functions $f(x)$ in $\mathbb{R}^+$. Such a transform is defined by the formula:

\[
Mf(\lambda) = \int_0^{\infty} x^{i\lambda} f(x) \frac{dx}{x}.
\]

It will be crucial in our analysis the fact that the restrictions of Hilbert transforms to the positive and negative real axis are Mellin multipliers as we will explicitly show below. A useful property of Mellin transforms is the following Plancherel’s identity for functions supported in $\mathbb{R}^+$:

\[
\int_0^{\infty} f(x)g(x) \frac{dx}{x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda)Mg(\lambda) \, d\lambda.
\]

We can consider two different cases of $f(x)$, even and odd, and then discuss the general case.

Notice that $f_x(x)Hf(x) = (f(x) - f(0))_x H(f(x) - f(0)) = g_x(x)Hg(x)$ for $g(x) = f(x) - f(0)$ so we may, without loss of generality, assume that $f(0) = 0$ for $f(x)$ even. Then,

\[
\int_{-\infty}^{\infty} \frac{f_x(x)Hf(x)}{|x|^\alpha} \, dx = 2 \int_0^{\infty} \frac{f_x(x)Hf(x)}{x^\alpha} \, dx,
\]

(12)

and using Plancherel’s identity it can be written in the form:

\[
- \int_0^{\infty} \frac{f_x(x)Hf(x)}{x^\alpha} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{F(\lambda)}m_x(\lambda)F(\lambda) \, d\lambda,
\]

(13)

where

\[
F(\lambda) = M\left(\frac{f}{x^{\alpha/2}}\right)(\lambda) = \int_0^{\infty} x^{i\lambda - \alpha/2 - 1} f(x) \, dx,
\]

and

\[
m_x(\lambda) = -\overline{A(\lambda)}B(\lambda),
\]

with $A(\lambda)$ and $B(\lambda)$ being Mellin multipliers that can be deduced from the Mellin transform of $f_x(x)$ and $Hf(x)$ respectively. In order to find them, let us compute:

\[
\int_0^{\infty} x^{i\lambda - \alpha/2} f_x(x) \, dx = -\left(i\lambda - \frac{\alpha}{2}\right)F(\lambda) \equiv A(\lambda)F(\lambda),
\]
\[ \int_{0}^{\infty} x^{i\lambda/2 - 1} Hf(x) \, dx = \frac{1}{\pi} \int_{0}^{\infty} x^{i\lambda/2 - 1} \left[ \int_{-\infty}^{\infty} \frac{f(y)}{x - y} \, dy \right] \, dx \]

\[ = \frac{1}{\pi} \int_{0}^{\infty} x^{i\lambda/2 - 1} \left[ \int_{0}^{\infty} \frac{f(y)}{x - y} \, dy + \int_{0}^{\infty} \frac{f(y)}{x + y} \, dy \right] \, dx \]

\[ = \frac{2}{\pi} \int_{0}^{\infty} x^{i\lambda/2} \left[ \int_{0}^{\infty} \frac{f(y)}{x^2 - y^2} \, dy \right] \, dx = \frac{2}{\pi} \int_{0}^{\infty} \left[ \int_{0}^{\infty} x^{i\lambda/2} \, dx \right] f(y) \, dy. \]

Then we evaluate the last integral in \( x \) performing the change of variables \( x = yt \):

\[ \int_{0}^{\infty} \frac{x^{i\lambda/2}}{x^2 - y^2} \, dx = \int_{0}^{\infty} \frac{x^{i\lambda/2}}{x^2 - y^2} \, dx = y^{i\lambda/2 - 1} \int_{0}^{\infty} \frac{x^{i\lambda/2}}{t^2 - 1} \, dt, \]

and the change \( t^2 = s \) together with (3) to evaluate:

\[ \int_{0}^{\infty} \frac{t^{i\lambda/2}}{t^2 - 1} \, dt = \frac{1}{2} \int_{0}^{\infty} \frac{s^{i\lambda/2 - 1/2}}{s - 1} \, ds = \frac{\pi}{2} \cot \left( \frac{-i\lambda + \alpha/2 + 1}{2} \pi \right). \]

Hence

\[ \frac{2}{\pi} \int_{0}^{\infty} \left[ \int_{0}^{\infty} x^{i\lambda/2} \, dx \right] f(y) \, dy = \cot \left( \frac{-i\lambda + \alpha/2 + 1}{2} \pi \right) \int_{0}^{\infty} y^{i\lambda/2 - 1} f(y) \, dy = -\tan \left( \frac{-i\lambda + \alpha/2}{2} \pi \right) F(\lambda). \]

Therefore

\[ B(\lambda) = -\tan \left( \frac{-i\lambda + \alpha/2}{2} \pi \right), \]

and

\[ m_s(\lambda) = \left( i\lambda + \alpha \right) \frac{\tan \left( \frac{-i\lambda + \alpha/2}{2} \pi \right)}{2}, \]

\[ \text{Re}[m_s(\lambda)] = \frac{\lambda \sin \pi \lambda + \frac{\alpha}{2} \sin \frac{1}{2} \pi \alpha}{\cosh \pi \lambda + \cos \frac{1}{2} \pi \alpha}. \] (14)

If \(-2 < \alpha < 2, \alpha \neq 0\), then

\[ \frac{\lambda \sin \pi \lambda + \frac{\alpha}{2} \sin \frac{1}{2} \pi \alpha}{\cosh \pi \lambda + \cos \frac{1}{2} \pi \alpha} > C_\alpha > 0, \]

and we may conclude that

\[ -\int_{0}^{\infty} \frac{f_s(x)Hf(x)}{x^\alpha} \, dx \geq C_\alpha \int_{-\infty}^{\infty} \frac{|f(x)|^2}{|x|^\alpha} \, dx, \]

or in general, if \( f(0) \neq 0 \), we obtain inequality (4).

In the case of \( f \) being odd, we evaluate:

\[ -\int_{0}^{\infty} \frac{f_s(x)(Hf(x) - Hf(0))}{x^\alpha} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \, d\lambda \, F(\lambda)m_a(\lambda)F(\lambda), \]

where

\[ m_a(\lambda) = -A(\lambda)B(\lambda), \]
with $A(\lambda)$ and $B(\lambda)$ being the Mellin multipliers coming from the Mellin transform of $f_\lambda(x)$ and $Hf(x) - Hf(0)$ respectively. To find them, let us compute:

$$
\int_0^\infty x^{i\lambda - \alpha/2} f_\lambda(x) \, dx = -(i\lambda - \alpha/2) F(\lambda) \equiv A(\lambda) F(\lambda),
$$

$$
\int_0^\infty x^{i\lambda - \alpha/2 - 1} ((Hf)(x) - (Hf)(0)) \, dx
$$

$$
\begin{align*}
&= \frac{1}{\pi} \int_0^\infty x^{i\lambda - \alpha/2 - 1} \left[ \int_0^\infty \frac{f(y)}{x-y} \, dy + \int_0^\infty \frac{f(y)}{y} \, dy - \int_0^\infty \frac{f(y)}{x+y} \, dy + \int_0^\infty \frac{f(y)}{y} \, dy \right] \, dx \\
&= \frac{2}{\pi} \int_0^\infty x^{i\lambda - \alpha/2 - 1} 2x^2 \left[ \int_0^\infty \frac{f(y)/y}{x^2 - y^2} \, dy \right] \, dx \\
&= \cot \left( \frac{-i\lambda + \alpha/2}{2} \right) \int_0^\infty x^{i\lambda - \alpha/2 - 1} f(y) \, dy = \cot \left( \frac{-i\lambda + \alpha/2}{2} \right) F(\lambda).
\end{align*}
$$

Hence

$$
m_a(\lambda) = -(i\lambda + \alpha/2) \cot \left( \frac{-i\lambda + \alpha/2}{2} \right),
$$

so that

$$
\text{Re} \left( -(i\lambda + \alpha/2) \cot \left( \frac{-i\lambda + \alpha/2}{2} \right) \right) = -\frac{1}{2} \alpha \sin \frac{1}{2} \pi \alpha + \lambda \sinh \pi \lambda - \sin \frac{1}{2} \pi \alpha,
$$

and

$$
-\int_0^\infty \frac{f_\lambda(x)(Hf(x) - Hf(0))}{x^\alpha} \, dx = \frac{1}{2\pi} \int_0^\infty \frac{\lambda \sinh \pi \lambda - \alpha}{\cos \pi \lambda - \cos \frac{1}{2} \pi \alpha} |F(\lambda)|^2.
$$

If $f$ is an arbitrary function (without loss of generality we can again assume that $f(0) = 0$), then we can write:

$$
f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \equiv f_s(x) + f_a(x),
$$

where $f_s(x)$ and $f_a(x)$ are even and odd respectively.

Notice that

$$
-\int_{-\infty}^\infty \frac{f_\lambda(x)(Hf(x) - Hf(0))}{|x|^{\alpha}} \, dx = -\int_{-\infty}^\infty \frac{|f_{s,\lambda}(x) + f_{a,\lambda}(x)|}{|x|^{\alpha}} (Hf_s(x) + Hf_a(x) - Hf_a(0)) \, dx
$$

$$
= -\int_{-\infty}^\infty \frac{f_{s,\lambda}(x) Hf_s(x)}{|x|^{\alpha}} \, dx - \int_{-\infty}^\infty \frac{f_{a,\lambda}(x) Hf_a(x)}{|x|^{\alpha}} \, dx
$$

$$
= -2 \int_0^\infty \frac{f_{s,\lambda}(x) Hf_s(x)}{x^\alpha} \, dx - 2 \int_0^\infty \frac{f_{a,\lambda}(x) Hf_a(x)}{x^\alpha} \, dx.
$$

Therefore:
\[- \int_{-\infty}^{\infty} \frac{f_s(x)(Hf(x) - Hf(0))}{|x|^\alpha} \, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda \, \text{Re}[m_s(\lambda)] |F_s(\lambda)|^2 + \frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda \, \text{Re}[m_a(\lambda)] |F_a(\lambda)|^2 \]

\[\geq \frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda \, \text{inf}_\lambda (\text{Re}[m_s(\lambda)]) |F_s(\lambda)|^2 + \frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda \, \text{inf}_\lambda (\text{Re}[m_a(\lambda)]) |F_a(\lambda)|^2 \]

\[= \frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda \, \text{inf}_\lambda (\text{Re}[m_s(\lambda)]) + \text{inf}_\lambda (\text{Re}[m_a(\lambda)]) \left( |F_s(\lambda)|^2 + |F_a(\lambda)|^2 \right) \]

\[+ \frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda \, \text{inf}_\lambda (\text{Re}[m_s(\lambda)]) - \text{inf}_\lambda (\text{Re}[m_a(\lambda)]) \left( |F_s(\lambda)|^2 - |F_a(\lambda)|^2 \right) \]

\[= \text{inf}_\lambda (\text{Re}[m_s(\lambda)]) + \text{inf}_\lambda (\text{Re}[m_a(\lambda)]) \int_{0}^{\infty} \frac{|f_s(x)|^2 + |f_a(x)|^2}{|x|^\alpha + 1} \, dx \]

\[+ \text{inf}_\lambda (\text{Re}[m_s(\lambda)]) - \text{inf}_\lambda (\text{Re}[m_a(\lambda)]) \int_{0}^{\infty} \frac{|f_s(x)|^2 - |f_a(x)|^2}{|x|^\alpha + 1} \, dx, \]

where we have used the notation,

\[F_a(\lambda) = M\left(-\frac{f_a}{x^{\alpha/2}}\right), \quad F_s(\lambda) = M\left(-\frac{f_s}{x^{\alpha/2}}\right). \]

It can be verified (see the lemma at the end of this section) that in the range \(1 < \alpha < 2\), \(\text{inf}_\lambda (\text{Re}[m_s(\lambda)]) > 0\), \(\text{inf}_\lambda (\text{Re}[m_a(\lambda)]) < 0\) and \(\text{inf}_\lambda (\text{Re}[m_s(\lambda)]) + \text{inf}_\lambda (\text{Re}[m_a(\lambda)]) > 0\). If \(f(x)\) is everywhere nonnegative, then \(|f_s(x)|^2 - |f_a(x)|^2 \geq 0\), and therefore:

\[- \int_{0}^{\infty} \frac{f_s(x)(Hf(x) - Hf(0))}{|x|^\alpha} \, dx \geq \text{inf}_\lambda (\text{Re}[m_s(\lambda)]) + \text{inf}_\lambda (\text{Re}[m_a(\lambda)]) \int_{0}^{\infty} \frac{|f_s(x)|^2 + |f_a(x)|^2}{|x|^\alpha + 1} \, dx \]

\[= \text{inf}_\lambda (\text{Re}[m_s(\lambda)]) + \text{inf}_\lambda (\text{Re}[m_a(\lambda)]) \int_{0}^{\infty} \frac{f^2(x) + f^2(-x)}{|x|^\alpha + 1} \, dx = C_\alpha \int_{-\infty}^{\infty} \frac{f^2(x)}{|x|^\alpha + 1} \, dx. \]

**Lemma 2.1.** For \(1 < \alpha < 2\), we have:

\[\text{inf}_\lambda (\text{Re}[m_s(\lambda)]) + \text{inf}_\lambda (\text{Re}[m_a(\lambda)]) > 0. \]

The proof of the lemma is based in two facts:

**Fact 1:**

\[\min_\lambda (\pi \text{Re}(m_a(\lambda))) = \frac{-s}{1 + t}. \]

**Fact 2:** For every \(\lambda\)

\[\frac{\lambda \sinh \lambda + s}{\cosh \lambda - t} > \frac{s}{1 + t}. \]
where \( s = \frac{\pi \alpha}{2} \sin \frac{\pi \alpha}{2} \) and \( t = -\cos \frac{\pi \alpha}{2} \) (\( 1 < \alpha < 2 \)). The first fact is trivial and the second is a consequence of the following observations:

\[
\lambda \sinh \lambda > 2(\cosh \lambda - 1), \quad \frac{s}{1 + t} = 2 \frac{\pi \alpha}{\tan \frac{\pi \alpha}{4}} < 2.
\]

3. Applications to a nonlocal transport equation

This section contains the proof of the application described in the introduction dealing with a nonlocal partial differential equation, where the inequalities obtained in the previous section will be crucial in proving the formation of finite time singularities.

The equation is:

\[
f_t - (Hf)_x = 0.
\]

In order to prove Theorem 1.2 we shall consider a general smooth initial data \( f_0(x) = f(x, 0) \), positive and compactly supported, and we will show that \( f_x(x, t) \) blows up in finite time.

It is a straightforward exercise to obtain a priori inequality:

\[
\frac{d}{dt} \left( \left\| f(\cdot, t) \right\|_{L^2}^2 + \left\| f_{xx}(\cdot, t) \right\|_{L^2}^2 \right) \leq C \left( \left\| f(\cdot, t) \right\|_{L^2}^2 + \left\| f_{xx}(\cdot, t) \right\|_{L^2}^2 \right)^{3/2},
\]

which implies local (in time) existence for the Cauchy problem with initial data in Sobolev’s space \( H^2(\mathbb{R}) \). We may thus assume a \( C^{1,\alpha} \) solution \( f(x, t) \). We have a particle dynamics given by the ordinary differential equation

\[
X'(t) = -Hf(X(t), t)
\]

and the equation implies that \( f \) is constant along trajectories. Let \( \text{supp}(f_0) \subset [a_0, b_0] \) then the positivity of \( f \) implies that \( Hf(a_0) \leq 0, Hf(b_0) \geq 0 \) which yields \( \text{supp}(f(x, t)) \subset [a_0, b_0] \) for \( t > 0 \).

If we change coordinates to a system of reference in which the maximum is stationary, i.e. we define \( x_M(t) \) to be the trajectory where \( f \) reaches its maximum, and

\[
x' = x - x_M(t), \quad t' = t,
\]

we obtain from (1) the equation:

\[
\tilde{f}_t' - x'_M(t)\tilde{f}_x' - H\tilde{f}(x', t')\tilde{f}_x' = 0,
\]

(15)

where \( \tilde{f}(x', t') = f(x' + x_M(t), t) \). In order to simplify notation let us omit from now on the explicit dependence of the function on the second variable \( t = t' \).

But

\[
\frac{dx_M(t)}{dt} = -Hf(x_M(t)),
\]

\[
Hf(x) = \frac{1}{\pi} \text{PV} \int \frac{f(y)}{x - y} dy = \frac{1}{\pi} \text{PV} \int \frac{f(y)}{(x - x_M(t)) - (y - x_M(t))} dy = H\tilde{f}(x'),
\]

and

\[
Hf(x(x_M(t))) = H\tilde{f}(x' = 0),
\]

so that (15) becomes

\[
\tilde{f}_t' - (H\tilde{f}(x') - H\tilde{f}(0))\tilde{f}_x' = 0.
\]

also we have \( \text{supp}_{x'}(\tilde{f}(\cdot, t)) \subset [-L, L] \) for a suitable finite constant \( L \).

Hence, denoting \( g = \tilde{f}_{\text{max}} - \tilde{f} \),

\[
g_{t'} = -(Hg(x') - Hg(0))g_{x'}.
\]

(16)

Dividing (16) by \( |x'|^\alpha \), \( 1 < \alpha < 2 \), and using Theorem 1.1 we get:
\[
\frac{d}{dr'} \int_{-L}^{L} \frac{g(x')}{|x'|^\alpha} \, dx' = - \int_{-L}^{L} \frac{(Hg(x') - Hg(0))g_{x'}(x')}{|x'|^\alpha} \, dx' \\
= - \int_{-\infty}^{\infty} \frac{(Hg(x') - Hg(0))g_{x'}(x')}{|x'|^\alpha} \, dx' \geq C_{\alpha} \int_{-\infty}^{\infty} \frac{g^2(x')}{|x'|^{\alpha+1}} \, dx' \\
\geq C_{\alpha} \int_{-L}^{L} \frac{g^2(x')}{|x'|^{\alpha+1}} \, dx' \geq C_{L,\alpha} \left( \int_{-L}^{L} \frac{g(x')}{|x'|^\alpha} \, dx' \right)^2,
\]
where \(C_{L,\alpha}\) is a positive constant. Therefore \(\int_{-L}^{L} \frac{g(x')}{|x'|^\alpha} \, dx'\) blows up in finite time and consequently \(f_x\) also blows up in finite time since

\[
\int_{-\infty}^{\infty} \frac{(Hf)(x)}{|x'|^\alpha} \, dx' \leq \sup_{x'} \left( \frac{g(x')}{|x'|^\alpha} \right) \int_{-\infty}^{\infty} \frac{dx'}{|x'|^{\alpha+1}} \leq \frac{2L^{2-\alpha}}{2-\alpha} \sup_x |g_x|.
\]

4. Proof of Theorems 1.3 and 1.4

In the proof of Theorem 1.3 we will consider two different cases of \(f(x)\), even and odd, and then discuss the general case.

If \(f(x)\) is even, i.e. \(f(x) = f(-x)\), then

\[
\int_{-\infty}^{\infty} \frac{|(Hf)(x)|^2}{|x|^{\alpha+1}} \, dx = \int_{-\infty}^{\infty} \frac{|H[f(x) - f(0)]|^2}{|x|^{\alpha+1}} \, dx = 2 \int_{0}^{\infty} \frac{|(Hf)(x)|^2}{|x|^{\alpha+1}} \, dx,
\]

since \(Hf\) is an odd function. In order to simplify the notation we can assume, without loss of generality, that \(f(0) = 0\).

To estimate the right-hand side of (17) we make use of the Mellin transform and write,

\[
\int_{0}^{\infty} \frac{|(Hf)(x)|^2}{|x|^{\alpha+1}} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \, d\lambda \, m(\lambda) |F(\lambda)|^2,
\]

where

\[
F(\lambda) = \int_{0}^{\infty} x^{i\lambda-\alpha/2-1} f(x) \, dx,
\]

and \(m(\lambda) = |G(\lambda)|^2\) with \(G(\lambda)\) being the Mellin multiplier for \(Hf\) with \(f\) an even function, that is:

\[
\int_{0}^{\infty} x^{i\lambda-\alpha/2-1} Hf(x) \, dx = G(\lambda) F(\lambda),
\]

where

\[
G(\lambda) = -\tan\left(\frac{-i\lambda + \alpha/2}{2\pi}\right).
\]

Then

\[
m(\lambda) = \left| \tan\left(\frac{-i\lambda + \alpha/2}{2\pi}\right) \right|^2 = \frac{\cos^2\left(\frac{1}{4} \pi \alpha\right) \sinh^2\left(\frac{1}{2} \pi \lambda\right) + \sin^2\left(\frac{1}{4} \pi \alpha\right) \cosh^2\left(\frac{1}{2} \pi \lambda\right)}{\cos^2\left(\frac{1}{4} \pi \alpha\right) \cosh^2\left(\frac{1}{2} \pi \lambda\right) + \sin^2\left(\frac{1}{4} \pi \alpha\right) \sinh^2\left(\frac{1}{2} \pi \lambda\right)} \geq \tan^2\left(\frac{1}{4} \pi \alpha\right),
\]
as one can easily verify.

Therefore, inequality (7) follows immediately for the even case since
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \, m(\lambda) \left| F(\lambda) \right|^2 \geq \frac{1}{2\pi} \tan^2 \left( \frac{1}{4} \pi \alpha \right) \int_{-\infty}^{\infty} d\lambda \, \left| F(\lambda) \right|^2 = \tan^2 \left( \frac{1}{4} \pi \alpha \right) \int_{-\infty}^{\infty} \frac{|f(x)|^2}{|x|^{\alpha+1}} \, dx,
\]
in the range \(0 < \alpha < 2\). In fact, one can show
\[
\int_{-\infty}^{\infty} \frac{|F(\lambda)|^2}{|x|^{\alpha+1}} \, dx \geq \min \left\{ \tan^2 \left( \frac{1}{4} \pi \alpha \right), 1 \right\} \int_{-\infty}^{\infty} \frac{|f(x)|^2}{|x|^{\alpha+1}} \, dx,
\]
for \(-2 < \alpha < 2, \alpha \neq 0\).

Next we consider \(f(x)\) to be an odd function and \(-2 < \alpha < 0\). Then (17) and (18) still hold, and we have to compute:
\[
\int_{0}^{\infty} x^{|i\lambda-\alpha/2|-1} Hf(x) \, dx = \cot \left( \frac{-i\lambda + \alpha/2}{\pi} \right) \int_{0}^{\infty} \left| f(y) \right| \, dy = \cot \left( \frac{-i\lambda + \alpha/2}{\pi} \right) F(\lambda).
\]
Therefore,
\[
m(\lambda) = \left| \cot \left( \frac{-i\lambda + \alpha/2}{\pi} \right) \right|^2 = \frac{\cos^2 \left( \frac{1}{4} \pi \alpha \right) \cosh^2 \left( \frac{1}{4} \pi \lambda \right) + \sin^2 \left( \frac{1}{4} \pi \alpha \right) \sinh^2 \left( \frac{1}{4} \pi \lambda \right)}{\cos^2 \left( \frac{1}{4} \pi \alpha \right) \sinh^2 \left( \frac{1}{4} \pi \lambda \right) + \sin^2 \left( \frac{1}{4} \pi \alpha \right) \cosh^2 \left( \frac{1}{4} \pi \lambda \right)} \geq \min \left\{ \cot^2 \left( \frac{1}{4} \pi \alpha \right), 1 \right\},
\]
which yields the inequality
\[
\int_{-\infty}^{\infty} \frac{|F(\lambda)|^2}{|x|^{\alpha+1}} \, dx \geq \min \left\{ \cot^2 \left( \frac{1}{4} \pi \alpha \right), 1 \right\} \int_{-\infty}^{\infty} \frac{|f(x)|^2}{|x|^{\alpha+1}} \, dx,
\]
for \(-2 < \alpha < 0\). When \(0 < \alpha < 2\), we estimate:
\[
\int_{-\infty}^{\infty} \frac{|(Hf)(x) - (Hf)(0)|^2}{|x|^{\alpha+1}} \, dx.
\]
As before let us now evaluate:
\[
\int_{0}^{\infty} x^{|i\lambda-\alpha/2|-1} \left( (Hf)(x) - (Hf)(0) \right) \, dx = \cot \left( \frac{-i\lambda + \alpha/2}{\pi} \right) \int_{0}^{\infty} \left| f(y) \right| \, dy,
\]
so that
\[
\int_{-\infty}^{\infty} \frac{|(Hf)(x) - (Hf)(0)|^2}{|x|^{\alpha+1}} \, dx \geq \min \left\{ \cot^2 \left( \frac{1}{4} \pi \alpha \right), 1 \right\} \int_{-\infty}^{\infty} \frac{|f(x)|^2}{|x|^{\alpha+1}} \, dx.
\]
Finally, given an arbitrary function \(f\), we write:
\[
f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \equiv f_s(x) + f_a(x),
\]
where \(f_s(x)\) and \(f_a(x)\) are even and odd respectively. Therefore we have:
\[
\int_{-\infty}^{\infty} \frac{|(Hf)(x)|^2}{|x|^{\alpha+1}} \, dx = \int_{-\infty}^{\infty} \frac{|H[f_s(x) - f_s(0)]|^2}{|x|^{\alpha+1}} \, dx + \int_{-\infty}^{\infty} \frac{|(Hf_a)(x)|^2}{|x|^{\alpha+1}} \, dx
\]
\[
+ 2 \int_{-\infty}^{\infty} \frac{H[f_s(x) - f_s(0)][(Hf_a)(x)]}{|x|^{\alpha+1}} \, dx.
\]
\[
\int_{-\infty}^{\infty} \frac{|H[f_s(x) - f_s(0)]|^2}{|x|^\alpha + 1} \, dx + \int_{-\infty}^{\infty} \frac{|(Hf_a)(x)|^2}{|x|^\alpha + 1} \, dx
\]
\[
\geq \min \left\{ \tan^2 \left( \frac{1}{4} \pi \alpha \right), 1 \right\} \int_{-\infty}^{\infty} \frac{|f_s(x) - f_s(0)|^2}{|x|^\alpha + 1} \, dx + \min \left\{ \cot^2 \left( \frac{1}{4} \pi \alpha \right), 1 \right\} \int_{-\infty}^{\infty} \frac{|f_a(x)|^2}{|x|^\alpha + 1} \, dx
\]

for \(-2 < \alpha < 0\).

Analogously, for \(0 < \alpha < 2\) we get the estimate:
\[
\int_{-\infty}^{\infty} \frac{|(Hf_a)(x) - (Hf_a)(0)|^2}{|x|^\alpha + 1} \, dx \geq \min \left\{ \tan^2 \left( \frac{1}{4} \pi \alpha \right), \cot^2 \left( \frac{1}{4} \pi \alpha \right) \right\} \int_{-\infty}^{\infty} \frac{|f_a(x)|^2}{|x|^\alpha + 1} \, dx.
\]

This completes the proof of Theorem 1.3.

Next, in order to prove Theorem 1.4 we will consider two different cases of \(f(x)\), even and odd.

If \(f(x)\) is even and \(0 < \alpha < 2\), then
\[
\int_{-\infty}^{\infty} \frac{|f(x) - f(0)|Hf(x)}{|x|^\alpha + 1} \, dx = 2 \int_{0}^{\infty} \frac{|f(x) - f(0)|Hf(x)}{|x|^\alpha + 1} \, dx,
\]
and again we can assume, without loss of generality, that \(f(0) = 0\). As in the previous section we write:
\[
\int_{0}^{\infty} \frac{f(x)Hf(x)}{|x|^\alpha + 1} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\lambda x} - e^{-i\lambda x}}{\cosh \pi \lambda + \cos \frac{1}{2} \pi \alpha} |F(\lambda)|^2 \, d\lambda,
\]
where \(m(\lambda)F(\lambda)\) is the Mellin transform of \(Hf(x)\), that is,
\[
m(\lambda) = -\tan \left( \frac{-i\lambda + \alpha/2}{2} \frac{\pi}{\pi} \right)
\]
and therefore,
\[
\int_{0}^{\infty} \frac{f(x)Hf(x)}{|x|^\alpha + 1} \, dx = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin \frac{1}{2} \pi \alpha}{\cosh \pi \lambda + \cos \frac{1}{2} \pi \alpha} |F(\lambda)|^2 \, d\lambda.
\]

This last identity implies:
\[
\int_{0}^{\infty} \frac{f(x)Hf(x)}{|x|^\alpha + 1} \, dx < 0,
\]
if \(0 < \alpha < 2\), and
\[
\int_{0}^{\infty} \frac{f(x)Hf(x)}{|x|^\alpha + 1} \, dx > 0,
\]
if \(-2 < \alpha < 0\).

If \(f(x)\) is odd then \(Hf(x)\) is even and by the properties of the Hilbert transform we can write:
\[
\int_{-\infty}^{\infty} \frac{f(x)Hf(x)}{|x|^\alpha + 1} \, dx = -\int_{-\infty}^{\infty} \frac{H(Hf(x))Hf(x)}{|x|^\alpha + 1} \, dx,
\]
and applying previous inequalities to the even function $Hf(x)$ yields

$$\int_0^\infty \frac{f(x)Hf(x)}{x^{\alpha+1}} \, dx > 0,$$

if $0 < \alpha < 2$, and

$$\int_0^\infty \frac{f(x)Hf(x)}{x^{\alpha+1}} \, dx < 0,$$

if $-2 < \alpha < 0$.

References