Commun. Math. Phys. 249, 511–528 (2004)
Digital Object Identifier (DOI) 10.1007/s00220-004-1055-1

Communications in
Mathematical
Physics

A Maximum Principle Applied to Quasi-Geostrophic Equations

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Received: 8 July 2003 / Accepted: 18 September 2003
Published online: 16 March 2004 –

Abstract: We study the initial value problem for dissipative 2D Quasi-geostrophic equations proving local existence, global results for small initial data in the super-critical case, decay of L^p-norms and asymptotic behavior of viscosity solution in the critical case. Our proofs are based on a maximum principle valid for more general flows.

1. Introduction

The two dimensional quasi-geostrophic equation (QG) is an important character of Geophysical Fluid Dynamics, see [9, 17 and 15]. It has the following form

$$(\partial_t + u \cdot \nabla)\,\theta = -\kappa(-\triangle)^{\frac{\alpha}{2}}\theta, \tag{1.1}$$

$$u = \nabla^{\perp}\psi, \quad \theta = -(-\triangle)^{\frac{1}{2}}\psi,$$

where ψ is the stream function. Here θ represents the potential temperature, u the velocity and κ is the viscosity. In this paper we examine existence, regularity and decay for solutions of the initial value problem. We will consider initial data $\theta(x, 0) = \theta_0(x)$, $x \in R^2$ or T^2. The parameters α, $0 \leq \alpha \leq 2$, and $\kappa \geq 0$ will be fixed real numbers.

The inviscid equation ($\kappa = 0$) was studied analytically and numerically by Constantin, Majda and Tabak [9]. They showed that there is a physical and mathematical analogy between the inviscid QG and 3D incompressible Euler equations. For both equations it is still an open problem to know if there are solutions that blow-up in finite time. For further analysis see [11, 13 and 3].

If $\kappa > 0$, Constantin and Wu [10] showed that viscous solutions remain smooth for all time when $\alpha \in (1, 2]$. In the critical case $\alpha = 1$, under the assumption of small L^∞

⋆ Partially supported by BFM2002-02269 grant.
⋆⋆ Partially supported by BFM2002-02042 grant.

norm, the global regularity was proven in [8]. Chae and Lee [4] studied the super-critical case $0 \leq \alpha \leq 1$ proving global existence for small initial data in the scale invariant Besov spaces. Many other results on the dissipative 2D Quasi-geostrophic equation can be found in [18, 2, 22–24, 19 and 14].

Ref. [18] contains a proof of a maximum principle for (1.1):

$$\|\theta(\cdot,t)\|_{L^p} \leq \|\theta_0\|_{L^p} \quad \text{for } 1 < p \leq \infty \quad \text{for all } t \geq 0.$$

For $\kappa = 0$, the L^p norms ($1 \leq p \leq \infty$) of θ are conserved for all time. In particular, that implies that energy is also conserved, because the velocity can be written in the following form

$$u = \left(-\partial_{x_2}\Lambda^{-1}\theta,\ \partial_{x_1}\Lambda^{-1}\theta\right) = (-R_2\theta,\ R_1\theta),$$

where Λ represents the operator $(-\Delta)^{\frac{1}{2}}$ and R_j are the Riesz transforms (see [20]).

In Sect. 2 we give a different proof of Resnick's maximum principle (see ref. [8]), showing a decay of the L^p norms. In Sect. 3 we present several estimates leading to local existence results. Section 4 contains one of the main results, namely the decay of the L^∞-norm.

The case $\alpha = 1$ is specially relevant because the viscous term $\kappa\Lambda\theta$ models the so-called Eckmann's pumping (see ref. [1] and [7]) which has been observed in quasi-geostrophic flows. On the other hand, several authors (see ref. [18] and [10]), have emphasized the deep analogy existing between Eq. (1.1) with $\alpha = 1$ and the 3D incompressible Navier-Stokes equations. In Sect. 5 of this paper we consider the notion of viscosity solution for the Eq. (1.1) adding an artificial viscosity term $\epsilon\Delta\theta$ to the right-hand side, and taking the limit, as $\epsilon \to 0$, of the corresponding solutions with the same initial data. We prove that for the critical case (α=1) there exist two times $T_1 \leq T_2$ (depending only upon the initial data θ_0 and $\kappa > 0$), so that viscosity solutions are smooth on the time intervals $t \leq T_1$ or $t \geq T_2$. Furthermore for $t \geq T_2$ we have a decay of the Sobolev norm $\|\theta\|_{H^s} = O(t^{-\frac{1}{2}})$.

Now we list some notations that will be used in the subsequent sections. As usual, $\widehat{f}$ is the Fourier transform of f, i.e.,

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^2}\int f(x)e^{-i\xi\cdot x}dx.$$

And $I^\alpha = \Lambda^{-\alpha}$, J^α denote the Riesz and Bessel potentials, given respectively by

$$\widehat{I^\alpha f}(\xi) = |\xi|^{-\alpha}\widehat{f}(\xi) \qquad \widehat{J^\alpha f}(\xi) = (1+|\xi|^2)^{-\frac{\alpha}{2}}\widehat{f}(\xi).$$

Throughout the paper we will make use of Sobolev's norms $\|f\|_{H^s}$ and of the duality of B.M.O. (bounded mean oscillation) with Hardy's space $\mathcal{H}^1$. We refer again to [20] for the corresponding definitions and properties. Besides the "$\leq$" symbol which has a very precise meaning, we will make use of the following standard notation: "$a << b$" if there exists a constant $C > 0$ (independent of all relevant parameters) so that $a \leq Cb$.

Finally, it is a pleasure to thank C.Fefferman for his helpful comments and his strong influence in our work.

2. Maximum Principle

In this section we present a proof, using fractional integral operators, of the maximum principle and decay of the L^p norms for the following scalar equation:

$$(\partial_t + u \cdot \nabla)\, \theta = -\kappa \Lambda^\alpha \theta.$$

Throughout this paper it will be assumed that the vector u satisfies either $\nabla \cdot u = 0$ or $u_i = G_i(\theta)$, together with the appropriate hypothesis about regularity and decay at infinity, which will be specified each time, in order to allow the integration by parts needed in our proofs.

Proposition 2.1. *Let* $0 < \alpha < 2$, $x \in R^2$ *and* $\theta \in S$, *the Schwartz class, then*

$$\Lambda^\alpha \theta(x) = C_\alpha PV \int \frac{[\theta(x) - \theta(y)]}{\mid x - y \mid^{2+\alpha}} dy, \tag{2.2}$$

where $C_\alpha > 0$.

Proof. We write Λ^α as an integral operator (see [20])

$$\begin{aligned}
\Lambda^\alpha \theta(x) = \Lambda^{\alpha-2}(-\Delta\theta) &= c_\alpha \int \frac{-\Delta_y \theta(y)}{\mid x - y \mid^\alpha} dy \\
&= c_\alpha \int \frac{\Delta_y [\theta(x) - \theta(y)]}{\mid x - y \mid^\alpha} dy \\
&= lim_{\epsilon \to 0} c_\alpha \int_{|x-y| \ge \epsilon} \frac{\Delta_y [\theta(x) - \theta(y)]}{\mid x - y \mid^\alpha} dy \\
&\equiv lim_{\epsilon \to 0} c_\alpha \Lambda^\alpha_\epsilon \theta,
\end{aligned}$$

where $c_\alpha = \frac{\Gamma(1-\frac{\alpha}{2})}{\pi 2^\alpha \Gamma(\frac{\alpha}{2})}$.

An application of Green's formula gives us

$$\begin{aligned}
\Lambda^\alpha_\epsilon \theta(x) = \tilde{c}_\alpha & \int_{|x-y| \ge \epsilon} \frac{[\theta(x) - \theta(y)]}{\mid x - y \mid^{2+\alpha}} dy \\
&+ \int_{|x-y| = \epsilon} [\theta(x) - \theta(y)] \frac{\partial \frac{1}{|x-y|^\alpha}}{\partial \eta} d\sigma(y) \\
&- \int_{|x-y| = \epsilon} \frac{1}{\mid x - y \mid^\alpha} \frac{\partial [\theta(x) - \theta(y)]}{\partial \eta} d\sigma(y) \\
&\equiv I_1 + I_2 + I_3,
\end{aligned}$$

where $\frac{\partial}{\partial \eta}$ is the normal derivative and $\tilde{c}_\alpha > 0$. Furthermore

$$I_2 = \frac{1}{\epsilon^{\alpha+1}} \int_{|x-y| = \epsilon} [\theta(x) - \theta(y)] d\sigma(y) = O(\epsilon^{2-\alpha}),$$

$$I_3 = \frac{1}{\epsilon^\alpha} \int_{|x-y| = \epsilon} \frac{\partial [\theta(x) - \theta(y)]}{\partial \eta} d\sigma(y) = O(\epsilon^{2-\alpha}),$$

therefore

$$lim_{\epsilon \to 0} I_2 = lim_{\epsilon \to 0} I_3 = 0$$

which yields (2.2).

Proposition 2.2. *Let* $0 < \alpha < 2$, $x \in T^2$ *and* $\theta \in S$, *the Schwartz class, then*

$$\Lambda^\alpha \theta(x) = C_\alpha \sum_{\nu \in Z^2} PV \int_{T^2} \frac{[\theta(x) - \theta(y)]}{\mid x - y - \nu \mid^{2+\alpha}} dy \tag{2.3}$$

with $C_\alpha > 0$

Proof.

$$\Lambda^\alpha \theta(x) = \sum_{|\nu|>0} |\nu|^\alpha \widehat{\theta}(\nu) e^{i\nu\cdot x} = -\sum_{|\nu|>0} |\nu|^{\alpha-2} \widehat{\Delta\theta}(\nu) e^{i\nu\cdot x}.$$

Let $\Phi_\epsilon(x) = (|x|^{\alpha-2})_\epsilon * \varphi_\epsilon(x)$, where $(|x|^{\alpha-2})_\epsilon = \left[|x|^{\alpha-2} \cdot \chi\left(\frac{|x|}{\epsilon}\right)\right]$ with $\chi \in C^\infty(0,\infty)$,

$$\chi(x) = \begin{cases} 0 \; if & |x| \leq 1 \\ 1 \; if & |x| \geq 2 \end{cases}$$

and $\varphi_\epsilon(x) = \epsilon^{-2}\varphi(\frac{x}{\epsilon})$ is a standard approximation of the identity: $0 \leq \varphi \in C^\infty$, $sop\varphi \subset B_1$ and $\int \varphi = 1$. Now we can write

$$\begin{aligned} \Lambda^\alpha \theta(x) &= -lim_{\epsilon\to 0} \sum \Phi_\epsilon(\nu) \widehat{\Delta\theta}(\nu) e^{i\nu\cdot x} \\ &= -lim_{\epsilon\to 0} \left(\sum \Phi_\epsilon(\nu) e^{i\nu\cdot x}\right) * \left(\sum \widehat{\Delta\theta}(\nu) e^{i\nu\cdot x}\right). \end{aligned}$$

Poisson's summation yields:

$$\begin{aligned} \Lambda^\alpha \theta(x) &= -lim_{\epsilon\to 0} \left(\sum \widehat{\Phi}_\epsilon(x-\nu)\right) * \Delta\theta(x) \\ &= lim_{\epsilon\to 0} \sum \int_{T^2} \widehat{\Phi}_\epsilon(x-y-\nu) \Delta(\theta(x) - \theta(y)) dy \\ &= lim_{\epsilon\to 0} \sum \int_{T^2} \Delta(\widehat{\Phi}_\epsilon)(x-y-\nu)(\theta(x) - \theta(y)) dy. \end{aligned}$$

Since

$$\begin{aligned} \widehat{\Phi}_\epsilon(\eta) &= (\widehat{|x|^{\alpha-2}})_\epsilon(\eta) \cdot \widehat{\varphi}_\epsilon(\eta) = (\widehat{|x|^{\alpha-2}})_\epsilon(\eta) \cdot \widehat{\varphi}(\epsilon\eta) \\ \Delta\widehat{\Phi}_\epsilon(\eta) &= \Delta((\widehat{|x|^{\alpha-2}})_\epsilon)(\eta) \cdot \widehat{\varphi}(\epsilon\eta) + O(\epsilon), \\ (\widehat{|x|^{\alpha-2}})_\epsilon(y) &= \frac{c_\alpha}{|y|^\alpha} - \int e^{-iyx} |x|^{\alpha-2}(1 - \chi(\frac{|x|}{\epsilon})) dx, \\ \Delta((\widehat{|x|^{\alpha-2}})_\epsilon)(y) &= \frac{\tilde{c}_\alpha}{|y|^{\alpha+2}} - \int e^{-iyx} |x|^\alpha (1 - \chi(\frac{|x|}{\epsilon})) dx, \end{aligned}$$

We get easily

$$\sum_\nu \Delta(\widehat{\Phi}_\epsilon)(y-\nu) = \tilde{c}_\alpha \sum_\nu \frac{1}{\mid y-\nu \mid^{2+\alpha}} + O\left(\sum_\nu \frac{1}{\mid y-\nu \mid^{2+\delta}} O(\epsilon^\delta)\right)$$

for some $\delta > 0$.

Therefore:

$$\Lambda^{\alpha}\theta(x) = lim_{\epsilon\to 0}\sum\int_{T^2}\Delta(\widehat{\Phi}_{\epsilon})(x-y-\nu)(\theta(x)-\theta(y))dy$$
$$= C_{\alpha}\sum_{\nu} PV\int_{T^2}\frac{[\theta(x)-\theta(y)]}{|\,x-y-\nu\,|^{2+\alpha}}dy.$$

Proposition 2.3. *Let* $0 \le \alpha \le 2$*,* $x \in R^2$ *and* $\theta \in S$ *(the Schwartz class). We have the pointwise inequality*

$$2\theta\Lambda^{\alpha}\theta(x) \ge \Lambda^{\alpha}\theta^2(x). \tag{2.4}$$

Proof. When $\alpha = 0, \alpha = 2$ the result is well known. For the remainder cases Proposition 2.1 (for the periodic case we use Proposition 2.2) gives us:

$$\Lambda^{\alpha}\theta(x) = PV\int\frac{[\theta(x)-\theta(y)]}{|\,x-y\,|^{2+\alpha}}dy.$$

Therefore,

$$\theta\Lambda^{\alpha}\theta(x) = PV\int\frac{[\theta(x)^2-\theta(y)\theta(x)]}{|\,x-y\,|^{2+\alpha}}dy$$
$$= \frac{1}{2}PV\int\frac{[\theta(y)-\theta(x)]^2}{|\,x-y\,|^{2+\alpha}}dy + \frac{1}{2}PV\int\frac{[\theta^2(x)-\theta^2(y)]}{|\,x-y\,|^{2+\alpha}}dy$$
$$\ge \frac{1}{2}\Lambda^{\alpha}\theta^2(x).$$

For a more general statement of Proposition 2.3 see [12]. The inequality (2.4) also holds in the periodic case.

Lemma 2.4. *With* $0 \le \alpha \le 2$*,* $x \in R^2, T^2$ *and* $\theta, \Lambda^{\alpha}\theta \in L^p$ *with* $p = 2^n$ *we get:*

$$\int|\theta|^{p-2}\theta\Lambda^{\alpha}\theta dx \ge \frac{1}{p}\int|\,\Lambda^{\frac{\alpha}{2}}\theta^{\frac{p}{2}}\,|^2\,dx. \tag{2.5}$$

Proof. The cases $\alpha = 0$ and $\alpha = 2$ are easy to check. For $0 < \alpha < 2$ we apply inequality (2.4) k times

$$\int|\theta|^{p-2}\theta\Lambda^{\alpha}\theta dx \ge \frac{1}{2}\int|\theta|^{p-2}\Lambda^{\alpha}\theta^2 dx = \int|\theta|^{p-4}\theta^2\Lambda^{\alpha}\theta^2 dx$$
$$\ge \frac{1}{4}\int|\theta|^{p-4}\Lambda^{\alpha}\theta^4 dx \ge \frac{1}{2^k}\int|\theta|^{p-2^k}\Lambda^{\alpha}\theta^{2^k}dx.$$

Taking $k = n - 1$ and using Parseval's identity with the Fourier transform we obtain inequality (2.5).

Lemma 2.5 (*Positivity Lemma*). *For $0 \leq \alpha \leq 2$, $x \in R^2, T^2$ and $\theta, \Lambda^\alpha\theta \in L^p$ with $1 \leq p < \infty$ we have:*

$$\int |\theta|^{p-2}\theta\Lambda^\alpha\theta dx \geq 0. \tag{2.6}$$

Proof. Again the cases $\alpha = 0$ and $\alpha = 2$ are easy to check directly. For $0 < \alpha < 2$ we have

$$\int |\theta|^{p-2}\theta\Lambda^\alpha\theta dx = lim_{\epsilon\to 0}\int |\theta|^{p-2}\theta\Lambda^\alpha_\epsilon\theta dx = lim_{\epsilon\to 0}\int |\theta|^{p-2}\theta I_1 dx,$$

where I_1 was defined above in (2.4). Then a change of variables yields

$$\begin{aligned}\int |\theta|^{p-2}\theta I_1 dx &= c_\alpha \int\int_{|x-y|\geq\epsilon} |\theta|^{p-2}(x)\theta(x)\frac{[\theta(x)-\theta(y)]}{|x-y|^{2+\alpha}}dydx\\ &= -c_\alpha \int\int_{|x-y|\geq\epsilon} |\theta|^{p-2}(y)\theta(y)\frac{[\theta(x)-\theta(y)]}{|x-y|^{2+\alpha}}dydx.\end{aligned}$$

And we get

$$\begin{aligned}&\int |\theta|^{p-2}\theta I_1 dx\\ &= \frac{1}{2}c_\alpha \int\int_{|x-y|\geq\epsilon} (|\theta|^{p-2}(x)\theta(x) - |\theta|^{p-2}(y)\theta(y))\frac{[\theta(x)-\theta(y)]}{|x-y|^{2+\alpha}}dydx.\\ &\geq 0\end{aligned}$$

Corollary 2.6 (*Maximum principle*). *Let θ and u be smooth functions on either R^2 or T^2 satisfying $\theta_t + u\cdot\nabla\theta + \kappa\Lambda^\alpha\theta = 0$ with $\kappa \geq 0$, $0 \leq \alpha \leq 2$ and $\nabla\cdot u = 0$ (or $u_i = G_i(\theta)$). Then for $1 \leq p \leq \infty$ we have:*

$$\|\theta(t)\|_{L^p} \leq \|\theta(0)\|_{L^p}.$$

Proof.

$$\begin{aligned}\frac{d}{dt}\int |\theta|^p dx &= p\int |\theta|^{p-2}\theta[-u\cdot\nabla\theta - \kappa\Lambda^\alpha\theta]dx\\ &= -\kappa p\int |\theta|^{p-2}\theta\Lambda^\alpha\theta dx \leq 0,\end{aligned}$$

where we have use the fact that $\nabla\cdot u = 0$ (or $u_i = G_i(\theta)$) and the positivity lemma.

Remark 2.7. When $p = 2^n$ $(n \geq 1)$ we have by Lemma 2.4 the following improved estimate:

$$\begin{aligned}\frac{d}{dt}\|\theta\|^p_{L^p} &= -\kappa p\int |\theta|^{p-2}\theta\Lambda^\alpha\theta dx\\ &\leq -\kappa\int |\Lambda^{\frac{\alpha}{2}}\theta^{\frac{p}{2}}|^2 dx.\end{aligned}$$

In the periodic case this inequality yields an exponential decay of $\|\theta\|_{L^p}$, $1 \leq p < \infty$. For the non-periodic case Sobolev's embedding and interpolation will give us the following

$$\begin{aligned}\frac{d}{dt}\|\theta\|_{L^p}^p &\leq -\kappa\left(\int \theta^{\frac{2p}{2-\alpha}}dx\right)^{\frac{2-\alpha}{2}}\\ &\leq -C\left(\|\theta\|_{L^p}^p\right)^{\frac{p-1+\frac{\alpha}{2}}{p-1}},\end{aligned}$$

where $C = C(\kappa, \alpha, p, \|\theta_0\|_1)$ is a positive constant. It then follows

$$||\theta(\cdot,t)||_{L^p}^p \leq \frac{||\theta_0||_{L^p}^p}{\left(1+\epsilon Ct||\theta_0||_{L^p}^{p\epsilon}\right)^{\frac{1}{\epsilon}}}$$

with $\epsilon = \frac{\alpha}{2(p-1)}$.

Remark 2.8. The decay for other L^p, $1 < p < \infty$, follows easily by interpolation. However, the L^∞ decay needs further arguments that will be presented in Sect. 4.

3. Local Existence and Small Data

The local (in time) existence theorem has been known (see refs. [9 and 3]) for the inviscid quasi-geostrophic equation when the initial data belong to the Sobolev space H^s, $s > 2$. Here we will improve slightly those results making use of well known properties of the space of functions of bounded mean oscillation (B.M.O.), namely the following:

a) J^α, $\alpha > 0$, maps B.M.O. continuously into $\Lambda_\alpha(R^2)$. Let us recall that when $0 < \alpha \leq 1$ we have (see [21])

$$\Lambda_\alpha(R^2) : ||f||_{\Lambda_\alpha} = ||f||_{L^\infty} + sup_{xy}\frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

b) If R is a Calderon-Zygmund Singular Integral and $b \in B.M.O.$, then we have the "commutator estimate":

$$||R(bf) - bR(f)||_{L^2} \ll ||f||_{L^2}||b||_{BMO}.$$

It then follows that if R has an odd kernel and $f \in L^2$, then $fR(f)$ belong to the Hardy space $\mathcal{H}^1$ and satisfies (see [5]):

$$||fR_jf||_{\mathcal{H}} \ll ||f||_{L^2}^2.$$

We shall also make use of the following, calculus inequality (see [16]): If $s < 0$ and $1 < p < \infty$, then:

$$||J^s(f\cdot g) - fJ^s(g)||_{L^p} \ll ||\nabla f||_{L^\infty}||J^{s-1}g||_{L^p} + ||g||_{L^\infty}||J^sf||_{L^p}.$$

This inequality follows from the estimate for the bilineal operators considered by R. Coifman and Y. Meyer [6] (Operateurs multilinearies (Ondelettets et Operateurs III), Theorem 1, p. 427): Define

$$T(b,f) = \int\int e^{ix(\xi+\eta)}p(\xi,\eta)\widehat{b}(\xi)\widehat{f}(\eta)d\xi d\eta,$$

where the symbol p satisfies

$$|D_\xi^\alpha D_\eta^\beta p(\xi,\eta)| \ll (1+|\xi|+|\eta|)^{-|\alpha|-|\beta|}$$

for $|\alpha|+|\beta| \le 2n+1$, $\xi,\eta \in R^n$. Then we have the estimate:

$$||T(b,f)||_{L^2} \ll ||b||_{L^\infty}||f||_{L^2}.$$

In our case, where n=2 (2n + 1 = 5), it implies the following inequality:

$$||\Lambda^s(R(\theta)\cdot\nabla^\perp\theta) - R(\theta)\cdot\nabla^\perp\Lambda^s\theta||_{L^2} \ll ||\Lambda^s\theta||_{L^2} sup_{|\alpha|\le 5}||R^\alpha\Lambda\theta||_{L^\infty},$$

where $\widehat{R^\alpha f}(\xi) = \frac{\xi^\alpha}{|\xi|^{|\alpha|}}\widehat{f}(\xi)$ are higher Riesz transforms. Therefore

$$||\Lambda^s(R(\theta)\cdot\nabla^\perp\theta) - R(\theta)\cdot\nabla^\perp\Lambda^s\theta||_{L^2} \ll ||\Lambda^s\theta||_{L^2}\left(||\theta||_{L^2} + ||\Lambda^{2+\epsilon}\theta||_{L^2}\right)$$

for every $\epsilon > 0$.

Theorem 3.1 (*Local existence*)**.** *Let $\alpha \ge 0$ and $\kappa > 0$ be given and assume that $\theta_0 \in H^m$, $m + \frac{\alpha}{2} > 2$. Then there exists a time $T = T(\kappa, ||\Lambda^m\theta_0||_{L^2}) > 0$ so that there is a unique solution to (1.1) in $C^1([0,T), H^m)$. Furthermore, when $\kappa = 0$ the same conclusion holds for $m > 2$, and in the critical case $\alpha = 1$ ($\kappa > 0$), we have local existence for all initial data θ_0 such that $||\Lambda\theta_0||_{L^4} < \infty$.*

Proof. If $\kappa > 0$ we have:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\Lambda^m\theta||^2_{L^2} &\ll \left|\int \Lambda^m\theta\{\Lambda^m(R(\theta)\cdot\nabla^\perp\theta) - R(\theta)\cdot\nabla^\perp\Lambda^m\theta\}\right| - \kappa||\Lambda^{m+\frac{\alpha}{2}}\theta_0||^2_{L^2}\\ &\ll ||\Lambda^m\theta||^2_{L^2}\left(||\theta||_{L^2} + ||\Lambda^{2+\epsilon}\theta||_{L^2}\right) - \kappa||\Lambda^{m+\frac{\alpha}{2}}\theta||^2_{L^2}\end{aligned}$$

for every $\epsilon > 0$. Taking $\epsilon = m + \frac{\alpha}{2} - 2$ we get

$$\frac{1}{2}\frac{d}{dt}||\Lambda^m\theta||^2_{L^2} \ll \frac{1}{\kappa}||\Lambda^m\theta||^4_{L^2} + ||\theta||_{L^2}||\Lambda^m\theta||^2_{L^2}$$

which yields the desired results.

In the case $\kappa = 0$, $m > 2$, we proceed in a similar manner:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\Lambda^m\theta||^2_{L^2} &= \int \Lambda^m\theta\{\Lambda^m(R(\theta)\cdot\nabla^\perp\theta) - R(\theta)\cdot\nabla^\perp\Lambda^m\theta\}\\ &\ll ||\Lambda^m\theta||^2_{L^2}\left(||\theta||_{L^2} + ||\Lambda^{2+\epsilon}\theta||_{L^2}\right).\end{aligned}$$

Therefore taking $\epsilon = m - 2 > 0$ one obtains:

$$\frac{1}{2}\frac{d}{dt}||\Lambda^m\theta||^2_{L^2} \ll ||\Lambda^m\theta||^3_{L^2} + ||\Lambda^m\theta||^2_{L^2}||\theta||_{L^2}.$$

Finally if $\alpha = 1$, $\kappa > 0$, let us consider:

$$\begin{aligned}\frac{d}{dt}\sum_j ||\frac{\partial\theta}{\partial x_j}||_{L^4}^4 &= 4\sum_{j=1,2}\int\left(\frac{\partial\theta}{\partial x_j}\right)^3\frac{\partial}{\partial x_j}(R(\theta)\cdot\nabla^\perp\theta) - 4\kappa\sum_{j=1,2}||\Lambda^{\frac{1}{2}}(\frac{\partial\theta}{\partial x_j})^2||_{L^2}^2\\ &\leq 4\sum_{j=1,2}\int\left(\frac{\partial\theta}{\partial x_j}\right)^3(R(\frac{\partial\theta}{\partial x_j})\cdot\nabla^\perp\theta) - C_1\kappa\sum_{j=1,2}||\frac{\partial\theta}{\partial x_j}||_{L^8}^4\\ &\leq C_2\sum_{j=1,2}||\frac{\partial\theta}{\partial x_j}||_{L^5}^5 - C_1\kappa\sum_{j=1,2}||\frac{\partial\theta}{\partial x_j}||_{L^8}^4,\end{aligned}$$

where C_1, C_2 are some universal positive constants.

Since

$$||\frac{\partial\theta}{\partial x_j}||_{L^5} \leq ||\frac{\partial\theta}{\partial x_j}||_{L^4}^{\frac{3}{5}}||\frac{\partial\theta}{\partial x_j}||_{L^8}^{\frac{2}{5}},$$

one obtains:

$$\begin{aligned}\frac{d}{dt}\sum_j ||\frac{\partial\theta}{\partial x_j}||_{L^4}^4 &\leq C_2\sum_j ||\frac{\partial\theta}{\partial x_j}||_{L^4}^3||\frac{\partial\theta}{\partial x_j}||_{L^8}^2 - C_1\kappa\sum_{j=1,2}||\frac{\partial\theta}{\partial x_j}||_{L^8}^4\\ &\leq \frac{C_3}{\kappa}\left(||\frac{\partial\theta}{\partial x_j}||_{L^4}^4\right)^{\frac{3}{2}},\end{aligned}$$

for some positive constant C_3. And from this estimate the results follow easily.

In the supercritical cases, $0 \leq \alpha \leq 1$, we have the following global existence results for small data.

Theorem 3.2. *Let $\kappa > 0$, $0 \leq \alpha \leq 1$, and assume that the initial data satisfies $||\theta_0||_{H^m} \leq \frac{\kappa}{C}$ (where $m > 2$ and $C = C(m) < \infty$ is a fixed constant). Then there exists a unique solution to (1.1) which belongs to H^m for all time $t > 0$.*

Proof. We have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(||\theta||_{L^2}^2 + ||\Lambda^m\theta||_{L^2}^2) &\leq -\kappa||\Lambda^{\frac{\alpha}{2}}\theta||_{L^2}^2 + C(||\theta||_{L^2}||\Lambda^m\theta||_{L^2}^2 + ||\Lambda^m\theta||_{L^2}^3)\\ &\quad - \kappa||\Lambda^{m+\frac{\alpha}{2}}\theta||_{L^2}^2.\end{aligned}$$

Since

$$||\Lambda^m\theta||_{L^2}^2 \leq ||\Lambda^{\frac{\alpha}{2}}\theta||_{L^2}^2 + ||\Lambda^{m+\frac{\alpha}{2}}\theta||_{L^2}^2$$

we obtain the inequality:

$$\frac{1}{2}\frac{d}{dt}(||\theta||_{L^2}^2 + ||\Lambda^m\theta||_{L^2}^2) \ll ||\Lambda^m\theta||_{L^2}^2(C(||\theta||_{L^2}^2 + ||\Lambda^m\theta||_{L^2}^2)^{\frac{1}{2}} - \kappa)$$

for some fixed constant $C < \infty$, and the theorem follows.

In the critical case $\alpha = 1$, $\kappa > 0$, we have the following:

Theorem 3.3 (*Global existence for small data*). *Let θ be a weak solution of (1.1) with an initial data $\theta_0 \in H^{\frac{3}{2}}$ satisfying $||\theta_0||_{L^\infty} \leq \frac{\kappa}{C}$ (where $C < \infty$ is a fixed constant). Then $\theta \in C^1([0,\infty); H^{\frac{3}{2}})$ is a classical solution.*

Proof. Using Eq. (1.1) we have

$$\frac{1}{2}\frac{d}{dt}||\Lambda^{\frac{3}{2}}\theta||_{L^2}^2 = \int \Lambda^{\frac{3}{2}}\theta\Lambda^{\frac{3}{2}}(R(\theta)\cdot\nabla^{\perp}\theta) - \kappa||\Delta\theta||_{L^2}^2.$$

Integration by parts gives us the following:

$$\begin{aligned}\Lambda^{-1}(R(\theta)\cdot\nabla^{\perp}\theta)(x) &= \tilde{c}\int\frac{R(\theta)\cdot\nabla^{\perp}\theta(y)}{|x-y|}dy\\ &= C[R_1(\theta\cdot R_2(\theta)) - R_2(\theta\cdot R_1(\theta))]\end{aligned}$$

for a suitable constant C. Therefore:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\Lambda^{\frac{3}{2}}\theta||_{L^2}^2 &= \int \Lambda^{\frac{3}{2}}\theta\Lambda^{\frac{3}{2}}(R(\theta)\cdot\nabla^{\perp}\theta)dx - \kappa||\Delta\theta||_{L^2}^2\\ &= C\int\Delta\theta\Delta(R_1(\theta\cdot R_2(\theta)) - R_2(\theta\cdot R_1(\theta)))dx - \kappa||\Delta\theta||_{L^2}^2\\ &= C\int\Delta\theta(R_1(\Delta\theta\cdot R_2(\theta)) - R_2(\Delta\theta\cdot R_1(\theta)))dx\\ &\quad + C\int\Delta\theta(R_1(\theta\cdot R_2(\Delta\theta)) - R_2(\theta\cdot R_1(\Delta\theta)))dx\\ &\quad + 2C\int\Delta\theta[R_1(\nabla\theta\cdot R_2(\nabla\theta)) - R_2(\nabla\theta\cdot R_1(\nabla\theta))]dx - \kappa||\Delta\theta||_{L^2}^2\\ &= C[I_1 + I_2 + 2I_3] - \kappa||\Delta\theta||_{L^2}^2.\end{aligned}$$

Our estimate will follow from the following observations:

$$I_2 = -\int\theta[R_1(\Delta\theta)R_2(\Delta\theta) - R_2(\Delta\theta)R_1(\Delta\theta)] = 0$$

$$\begin{aligned}|I_1| &\leq \left|\int R_1(\Delta\theta)\Delta\theta R_2(\theta)\right| + \left|\int R_2(\Delta\theta)\Delta\theta R_1(\theta)\right|\\ &\ll \sum_j ||R_j(\Delta\theta)\Delta\theta||_{\mathcal{H}}||\theta||_{BMO} \ll ||\Delta\theta||_{L^2}^2||\theta||_{L^\infty}.\end{aligned}$$

This is because for each Riesz transform R_j and a given L^2-function f, the product fR_jf is in Hardy's space $\mathcal{H}^1$ and satisfies $||fR_jf||_{\mathcal{H}^1} \ll ||f||_{L^2}^2$. Therefore

$$\left|\int\Delta\theta\cdot R_j(\Delta\theta)\cdot R_m(\theta)dx\right| \ll ||\Delta\theta||_{L^2}^2||R_m(\theta)||_{BMO} \ll ||\Delta\theta||_{L^2}^2||\theta_0||_{L^\infty}.$$

Finally I_3 is a sum of terms of the following form:

$$\int R_j(\Delta\theta)\frac{\partial\theta}{\partial x_k}\frac{\partial}{\partial x_l}R_m(\theta)dx, \qquad j,k,l,m=1,2.$$

Therefore we have the estimates:

$$\left|\int R_j(\Delta\theta)\frac{\partial\theta}{\partial x_k}\frac{\partial}{\partial x_l}R_m(\theta)dx\right| \ll ||\Delta\theta||_{L^2}||\Lambda\theta||_{L^4}^2.$$

Integration by parts yields

$$\begin{aligned}
||\Lambda\theta||_{L^4}^4 &= \sum_j\int\left(\frac{\partial\theta}{\partial x_j}\right)^4 dx\\
&= \left|\sum_j\int\theta\frac{\partial}{\partial x_j}\left(\left(\frac{\partial\theta}{\partial x_j}\right)^3\right)dx\right|\\
&= 3\left|\sum_j\int\theta\left(\frac{\partial\theta}{\partial x_j}\right)^2\frac{\partial^2\theta}{\partial x_j^2}dx\right|\\
&\ll ||\theta_0||_{L^\infty}||\Lambda\theta||_{L^4}^2||\Delta\theta||_{L^2}.
\end{aligned}$$

Thus,

$$||\Lambda\theta||_{L^4}^2 \ll ||\theta_0||_{L^\infty}||\Delta\theta||_{L^2},$$

that is

$$\frac{d}{dt}||\Lambda^{\frac{3}{2}}\theta||_{L^2}^2 \le (c||\theta_0||_{L^\infty}-\kappa)||\Delta\theta||_{L^2}^2 \tag{3.7}$$

for some universal constant c.

A well known approximation argument allows us to conclude the result: Let θ^n be the sequence of solutions to the following problems:

$$\theta_t^n + R(\theta^n)\cdot\nabla^{\perp}\theta^n = -\kappa\Lambda\theta^n + \frac{1}{n}\Delta\theta^n,$$

$$\theta_0^n \in C_0^\infty(R^2), \qquad ||\theta_0-\theta_0^n||_{L^\infty} \le \frac{\kappa}{2^n}, \qquad ||\theta_0-\theta_0^n||_{H^{\frac{3}{2}}} \le \frac{\kappa}{2^n}.$$

Then $||\Lambda^{\frac{3}{2}}\theta^n(\cdot,t)||_{L^2}^2$ is a decreasing sequence on t, uniformly on n. A compacity argument, taking limits as $n\to\infty$, will give us the desired estimate for θ.

4. Decay of the L^∞ Norm

Theorem 4.1. *If θ and u are smooth functions on $R^2\times[0,T)$ (or $T^2\times[0,T)$) satisfying $\theta_t+u\cdot\nabla\theta+\kappa\Lambda^\alpha\theta=0$ with $\kappa>0$, $0<\alpha\le 2$, $\theta(\cdot,t)\in H^s(R^2)$, $0\le t<T$, (or $H^s(T^2)$) ($s>1$) and $\nabla\cdot u=0$, then*

$$||\theta(\cdot,t)||_{L^\infty} \leq \frac{||\theta_0||_{L^\infty}}{\left(1+\alpha C t||\theta_0||^{\alpha}_{L^\infty}\right)^{\frac{1}{\alpha}}} \qquad 0 \leq t < T, \tag{4.8}$$

where $\theta_0 = \theta(\cdot, 0)$ *and* $C = C(\kappa, \theta_0) > 0$*. Furthermore, when* $\alpha = 0$ *we have the exponential decay* $||\theta(\cdot,t)||_{L^\infty} \leq ||\theta_0||_{L^\infty} e^{-\kappa t}$.

Proof. The case $\alpha = 0$ is straightforward. When $0 < \alpha \leq 2$ let $g(t) = |\theta(\cdot,t)|_{L^\infty}$ for $0 \leq t < T$. By the maximum principle $g(t)$ is bounded, and since $\theta(\cdot,t) \in H^s$, $s > 1$, it follows from the Riemann-Lebesgue lemma that $\theta(x,t)$ tends to 0 when $|x| \to \infty$. Therefore there always exists a point $x_t \in R^2$ where $|\theta|$ reaches its maximum, that is

$$g(t) = |\theta(x_t, t)|.$$

Assume that $\theta(x_t, t) \geq 0$ (for $\theta(x_t, t) \leq 0$ a similar argument will work), and let $h \geq 0$, then by the maximum principle

$$0 \leq g(t) - g(t+h) = \theta(x_t, t) - \theta(x_{t+h}, t+h) \leq \theta(x_t, t) - \theta(x_t, t+h) \leq c \cdot h,$$

where $c = sup_{0 \leq t < T}|\frac{\partial \theta}{\partial t}|$. Therefore $g(t)$ is a decreasing Lipschitz function and by H. Rademacher's theorem it is differentiable almost everywhere.

Let us consider t such that $g'(t)$ exists. For each $h > 0$ we take $x_{t+h} \in R^2$ such that

$$g(t+h) = \theta(x_{t+h}, t+h).$$

Then we can find a sequence $h_n \to 0$ such that $x_{t+h_n} \to \tilde{x}$ with $g(t) = \theta(\tilde{x}, t)$. (This follows by a compacity argument: let R be so that $|\theta(x,t)| \leq \frac{1}{2} g(t)$ if $|x| \geq R$ (observe that when $g(t) = 0$ everything trivializes), then for h small enough it happens that $|x_{t+h}| \leq 2R$).

We have:

$$\begin{aligned}
g'(t) &= lim_{h_n \to 0} \frac{\theta(x_{t+h_n}, t+h_n) - \theta(\tilde{x}, t)}{h_n} \\
&= lim_{h_n \to 0} \left[\frac{\theta(x_{t+h_n}, t+h_n) - \theta(x_{t+h_n}, t)}{h_n} + \frac{\theta(x_{t+h_n}, t) - \theta(\tilde{x}, t)}{h_n}\right] \\
&\leq lim_{h_n \to 0} \frac{\partial \theta}{\partial t}(x_{t+h_n}, \tilde{t})
\end{aligned}$$

with $t \leq \tilde{t} \leq t + h_n$. Therefore, we get the following inequality:

$$\frac{d\|\theta(\cdot,t)\|_{L^\infty}}{dt} = g'(t) \leq lim_{h_n \to 0} \frac{\partial \theta}{\partial t}(x_{t+h_n}, \tilde{t}) = \frac{\partial \theta}{\partial t}(\tilde{x}, t).$$

Equation (1.1) together with the fact that $\theta(\cdot, t)$ reaches its maximum at the point $\tilde{x}$ implies the equality:

$$\begin{aligned}
\frac{\partial \theta}{\partial t}(\tilde{x}, t) = -u \cdot \nabla\theta(\tilde{x}, t) - \kappa(-\Delta)^{\frac{\alpha}{2}}\theta(\tilde{x}, t) &= -\kappa(-\Delta)^{\frac{\alpha}{2}}\theta(\tilde{x}, t) \\
&= -\kappa \cdot PV \int \frac{[\theta(\tilde{x}, t) - \theta(y, t)]}{| x - y |^{2+\alpha}} dy.
\end{aligned}$$

Thus,

$$\frac{d\|\theta(\cdot,t)\|_{L^\infty}}{dt} \le -\kappa PV \int \frac{[\theta(\tilde{x},t)-\theta(y,t)]}{|\tilde{x}-y|^{2+\alpha}} dy \le 0.$$

We know that $\theta(\tilde{x},t)-\theta(y,t)\ge 0$ for all $y\in R^2$. So

$$I \equiv PV \int \frac{[\theta(\tilde{x},t)-\theta(y,t)]}{|\tilde{x}-y|^{2+\alpha}} dy = \int_\Omega + \int_{R^2/\Omega} \ge \int_\Omega,$$

where $\Omega \equiv \{y : |\tilde{x}-y| \le \delta\}$. We split $\Omega = \Omega_1 \cup \Omega_2$

$$y \in \Omega_1 \qquad if \qquad \theta(\tilde{x},t)-\theta(y,t) \ge \frac{\theta(\tilde{x},t)}{2},$$

and $y \in \Omega_2$ otherwise. Now

$$I \ge \int_\Omega \ge \int_{\Omega_1} = \frac{\theta(\tilde{x},t)}{2\delta^{2+\alpha}} Area(\Omega_1).$$

On the other hand we have the energy estimate

$$\begin{aligned} E(0) &= \int_{R^2} \theta^2(x,0)dx \ge \int_{R^2} \theta^2(x,t)dx \ge \int_{\Omega_2} \theta^2(x,t)dx \\ &\ge \frac{\theta^2(\tilde{x},t)}{4} Area(\Omega_2), \end{aligned}$$

therefore

$$I \ge \frac{\theta(\tilde{x},t)}{2\delta^{2+\alpha}}(Area(\Omega) - Area(\Omega_2)) \ge \frac{\theta(\tilde{x},t)}{2\delta^{2+\alpha}}(\pi\delta^2 - \frac{4E(0)}{\theta^2(\tilde{x},t)}).$$

To finish let us take $\delta = \sqrt{\frac{4E(0)}{\theta^2(\tilde{x},t)}}$, to get

$$\frac{d\|\theta(\cdot,t)\|_{L^\infty}}{dt} \le -C^2(\kappa, E(0))\cdot\theta^{1+\alpha}(\tilde{x},t) = -C^2(\kappa,E(0))\cdot\|\theta(\cdot,t)\|_{L^\infty}^{1+\alpha}$$

which yields inequality (4.8).

Corollary 4.2. *For solutions of the equation*

$$\theta_t + R(\theta)\cdot\nabla^\perp\theta = -\kappa\Lambda\theta + \epsilon\Delta\theta,$$

$\kappa > 0$, $\epsilon > 0$, *where either* $\theta_0 \in H^s(R^2)(or H^s(T^2))$, $s > \frac{3}{2}$, *or* $||\Lambda\theta_0||_{L^4} < \infty$, *we have:*

$$||\theta(\cdot,t)||_{L^\infty} \le \frac{||\theta_0||_{L^\infty}}{\left(1 + C\kappa t\frac{||\theta_0||_{L^\infty}}{||\theta_0||_{L^2}}\right)}$$

for some universal constant $C > 0$.

Proof. It follows from the argument of Theorem 4.1 and the observation that $\Delta\theta(x_t,t) \le 0$ at the points x_t where $\theta(\cdot,t)$ reaches its maximum value.

5. Viscosity Solutions

A weak solution of

$$\theta_t + R(\theta) \cdot \nabla^{\perp}\theta = -\kappa\Lambda\theta$$

will be called a viscosity solution with initial data $\theta_0 \in H^s(R^2)(H^s(T^2))$, $s > 1$, if it is the weak limit of a sequence of solutions, as $\epsilon \to 0$, of the problems

$$\theta_t^{\epsilon} + R(\theta^{\epsilon}) \cdot \nabla^{\perp}\theta^{\epsilon} = -\kappa\Lambda\theta^{\epsilon} + \epsilon\Delta\theta^{\epsilon}$$

with $\theta^{\epsilon}(x, 0) = \theta_0$. We know that each θ^{ϵ}, $\epsilon > 0$, is classical and $\theta^{\epsilon}(\cdot, t) \in H^s$ for each $t > 0$ satisfying

$$||\theta^{\epsilon}(\cdot, t)||_{L^{\infty}} \leq \frac{||\theta_0||_{L^{\infty}}}{1 + Ct\frac{\kappa||\theta_0||_{L^{\infty}}}{||\theta_0||_{L^2}}},$$

uniformly on $\epsilon > 0$, for all time $t \geq 0$. Furthermore, for $s > \frac{3}{2}$ there is a time $T_1 = T_1(\kappa, ||\theta_0||_{H^s})$ such that $||\Lambda^s\theta^{\epsilon}(t)||_{L^2} \leq 2||\Lambda^s\theta_0||_{L^2}$ for $0 \leq t < T_1$.

Lemma 5.1. *Let θ be a viscosity solution of QG with critical viscosity, i.e. $\alpha = 1$, $\kappa > 0$, then*

$$\int_0^{\infty} ||\Lambda^{\frac{1}{2}}\theta(\cdot, t)||_{L^2}^2 dt < \infty.$$

Proof. For each $\epsilon > 0$ we have

$$\begin{aligned}\frac{d}{dt}||\theta^{\epsilon}||_{L^2}^2 &= 2\int \theta^{\epsilon}R(\theta^{\epsilon}) \cdot \nabla^{\perp}\theta^{\epsilon} - 2\kappa\int \theta^{\epsilon}\Lambda\theta^{\epsilon} - 2\epsilon\int |\Lambda\theta^{\epsilon}|^2 \\ &= -2\kappa||\Lambda^{\frac{1}{2}}\theta^{\epsilon}||_{L^2}^2 - 2\epsilon||\Lambda\theta^{\epsilon}||_{L^2}^2 \leq -2\kappa||\Lambda^{\frac{1}{2}}\theta^{\epsilon}||_{L^2}^2,\end{aligned}$$

therefore

$$||\theta_0||_{L^2}^2 - ||\theta_0^{\epsilon}(\cdot, t)||_{L^2}^2 \geq 2\kappa\int_0^t ||\Lambda^{\frac{1}{2}}\theta^{\epsilon}(\cdot, t)||_{L^2}^2 dt,$$

i.e.

$$\int_0^{\infty} ||\Lambda^{\frac{1}{2}}\theta^{\epsilon}(\cdot, t)||_{L^2}^2 dt \leq \frac{1}{2\kappa}||\theta_0||_{L^2}^2$$

uniformly on $\epsilon > 0$. Taking the limit we get our result.

We also have the following:

Corollary 5.2. *For each $\delta > 0$, $\epsilon \geq 0$ and $n = 0, 1, 2, \ldots$ there exists a time $t_n^{\epsilon} \in [n\delta^{-1}, (n+1)\delta^{-1})$ such that $||\Lambda^{\frac{1}{2}}\theta^{\epsilon}(\cdot, t_n^{\epsilon})||_{L^2}^2 \leq \frac{\delta}{2\kappa}||\theta_0||_{L^2}^2$.*

Next we assume that $\theta_0 \in H^{\frac{3}{2}}$ and let us consider

$$\begin{aligned}
\frac{d}{dt}||\Lambda^{\frac{1}{2}}\theta^\epsilon||^2_{L^2} &= 2\int \Lambda^{\frac{1}{2}}\theta^\epsilon \Lambda^{\frac{1}{2}}(R(\theta^\epsilon)\cdot\nabla^\perp\theta^\epsilon) - 2\kappa||\Lambda\theta^\epsilon||^2_{L^2} - 2\epsilon||\Lambda^{\frac{3}{2}}\theta^\epsilon||^2_{L^2} \\
&\le \left|\int \Lambda\theta^\epsilon R(\theta^\epsilon)\cdot\nabla^\perp\theta^\epsilon\right| - 2\kappa||\Lambda\theta^\epsilon||^2_{L^2} \\
&\le C\sum_j ||\Lambda\theta^\epsilon||^2_{L^2}||R_j\theta^\epsilon||_{BMO} - 2\kappa||\Lambda\theta^\epsilon||^2_{L^2} \\
&\le C||\Lambda\theta^\epsilon||^2_{L^2}||\theta(\cdot,t)||_{L^\infty} - 2\kappa||\Lambda\theta^\epsilon||^2_{L^2} \\
&= (C||\theta^\epsilon(\cdot,t)||_{L^\infty} - 2\kappa)||\Lambda\theta^\epsilon||^2_{L^2}
\end{aligned}$$

for some universal constant C.

Because of the L^∞-decay we can find a time $T = T(\kappa, \theta_0)$ so that if $t \ge T$ then $C||\theta^\epsilon(\cdot,t)||_{L^\infty} < \kappa$ uniformly on $\epsilon > 0$.

Choosing t_n^ϵ to be the smallest element of the time sequence in Corollary 5.2 which is bigger than T, we obtain:

$$||\Lambda^{\frac{1}{2}}\theta^\epsilon(\cdot,t_n^\epsilon)||^2_{L^2} \ge \kappa\int_{t_n^\epsilon}^\infty ||\Lambda\theta^\epsilon(\cdot,t)||^2_{L^2}dt \ge \kappa\int_{(n+1)\delta^{-1}}^\infty ||\Lambda\theta^\epsilon(\cdot,t)||^2_{L^2}dt.$$

Therefore we have proved the following:

Lemma 5.3. *For each $\delta > 0$ there exists a time $T = T(\kappa, \theta_0)$ so that*

a) $\int_T^\infty ||\Lambda\theta^\epsilon(\cdot,t)||^2_{L^2}dt \le \frac{\delta}{\kappa^2}||\theta_0||^2_{L^2}$.

b) $||\Lambda^{\frac{1}{2}}\theta^\epsilon(\cdot,t)||^2_{L^2}$ *is a decreasing function of t, for* $t \ge T$ *and* $||\Lambda^{\frac{1}{2}}\theta^\epsilon(\cdot,T)||^2_{L^2} \le \frac{\delta}{2\kappa}||\theta_0||^2_{L^2}$.

c) There exists a time t_n^ϵ on each interval $[T+cn, T+c(n+1))$ so that (for an adequate c to be fixed later) $||\Lambda\theta^\epsilon(\cdot,t_n^\epsilon)||^2_{L^2} \le \frac{\kappa}{c}$.

For $t \ge T$ we may consider

$$\frac{d}{dt}||\Lambda\theta^\epsilon||^2_{L^2} = 2\int \Lambda\theta^\epsilon\Lambda(R(\theta^\epsilon)\cdot\nabla^\perp\theta^\epsilon) - 2\kappa||\Lambda^{\frac{3}{2}}\theta^\epsilon||^2_{L^2} - 2\epsilon||\Delta\theta^\epsilon||^2_{L^2}$$

and observe that

$$\begin{aligned}
\left|\int \Lambda\theta^\epsilon\Lambda(R(\theta^\epsilon)\cdot\nabla^\perp\theta^\epsilon)\right| &= \left|\int \Delta\theta^\epsilon(R(\theta^\epsilon)\cdot\nabla^\perp\theta^\epsilon)\right| \\
&= \left|\sum_j\int \frac{\partial\theta^\epsilon}{\partial x_j}R(\frac{\partial\theta^\epsilon}{\partial x_j})\cdot\nabla^\perp\theta^\epsilon\right| \\
&\ll ||\Lambda\theta^\epsilon||^3_{L^3} \le ||\Lambda\theta^\epsilon||_{L^2}||\Lambda\theta^\epsilon||^2_{L^4} \le ||\Lambda\theta^\epsilon||_{L^2}||\Lambda^{\frac{3}{2}}\theta^\epsilon||^2_{L^4}.
\end{aligned}$$

Therefore:

$$\frac{d}{dt}||\Lambda\theta^\epsilon||^2_{L^2} \le (C||\Lambda\theta^\epsilon||_{L^2} - \kappa)||\Lambda^{\frac{3}{2}}\theta^\epsilon||^2_{L^2}.$$

Let us observe now that our previous choice of T was made in such a way that $C||\Lambda\theta^\epsilon||_{L^2} \le \frac{\kappa}{2}$. Then for $t \ge T$ we obtain the decrease of $||\Lambda\theta^\epsilon||_{L^2}$, together with the sequence of "uniformly spaced" times t_n^ϵ, where $||\Lambda\theta^\epsilon(\cdot, t_n^\epsilon)||_{L^2} \le \frac{\kappa}{2C}$.

We conclude the existence of other time $\tilde{T} = \tilde{T}(\kappa, \theta_0)$ so that

$$\int_{\tilde{T}}^{\infty} ||\Lambda^{\frac{3}{2}}\theta^\epsilon||_{L^2}^2 dt \le C(\kappa)$$

uniformly on $\epsilon > 0$.

Assuming now that $\theta_0 \in H^2$ we get:

$$\frac{d}{dt}||\Lambda^{\frac{3}{2}}\theta^\epsilon||_{L^2}^2 = 2\int \Lambda^{\frac{3}{2}}\theta^\epsilon\Lambda^{\frac{3}{2}}(R(\theta^\epsilon)\cdot\nabla^\perp\theta^\epsilon) - 2\kappa||\Lambda^2\theta^\epsilon||_{L^2}^2 - 2\epsilon||\Lambda^{\frac{5}{2}}\theta^\epsilon||_{L^2}^2.$$

We have:

$$\begin{aligned}
&\int \Lambda^{\frac{3}{2}}\theta^\epsilon\Lambda^{\frac{3}{2}}(R(\theta^\epsilon)\cdot\nabla^\perp\theta^\epsilon)dx \\
&\quad = C\int \Delta\theta^\epsilon\Delta(R_1(\theta^\epsilon\cdot R_2(\theta^\epsilon)) - R_2(\theta^\epsilon\cdot R_1(\theta^\epsilon)))dx \\
&\quad = C\int \Delta\theta^\epsilon(R_1(\theta^\epsilon\cdot R_2(\Delta\theta^\epsilon)) - R_2(\theta^\epsilon\cdot R_1(\Delta\theta^\epsilon)))dx \\
&\qquad + C\int \Delta\theta^\epsilon(R_1(\Delta\theta^\epsilon\cdot R_2(\theta^\epsilon)) - R_2(\Delta\theta^\epsilon\cdot R_1(\theta^\epsilon)))dx \\
&\qquad + 2C\int \Delta\theta^\epsilon[R_1(\nabla\theta^\epsilon\cdot R_2(\nabla\theta^\epsilon)) - R_2(\nabla\theta^\epsilon\cdot R_1(\nabla\theta^\epsilon))]dx \\
&\quad = C[I_1 + I_2 + 2I_3].
\end{aligned}$$

We have that $I_1 = 0$, and

$$\begin{aligned}
|I_2| &= \left|\int R_2(\Delta\theta^\epsilon)\cdot\Delta\theta^\epsilon\cdot R_1(\theta^\epsilon) - \int R_1(\Delta\theta^\epsilon)\cdot\Delta\theta^\epsilon\cdot R_2(\theta^\epsilon)\right| \\
&\ll ||\Delta\theta^\epsilon||_{L^2}^2||\theta^\epsilon||_{BMO} \le ||\Delta\theta^\epsilon||_{L^2}^2||\theta^\epsilon||_{L^\infty}.
\end{aligned}$$

Again this is true because $fR_j(f)$ is in Hardy's space $\mathcal{H}^1$ for each L^2-function f.

To estimate I_3 let us observe the following:

$$|I_3| \ll ||\Delta\theta^\epsilon||_{L^2}||\Lambda\theta^\epsilon||_{L^4}^2.$$

And we have

$$\begin{aligned}
||\Lambda\theta^\epsilon||_{L^4}^4 &\cong \sum_j\int\left(\frac{\partial\theta^\epsilon}{\partial x_j}\right)^4 \le 3\sum_j\int\theta^\epsilon\left(\frac{\partial\theta^\epsilon}{\partial x_j}\right)^2\left|\frac{\partial^2\theta^\epsilon}{\partial x_j^2}\right| \\
&\ll ||\theta^\epsilon||_{L^\infty}||\Lambda\theta^\epsilon||_{L^4}^2||\Delta\theta^\epsilon||_{L^2},
\end{aligned}$$

which implies

$$|I_3| \ll ||\Delta\theta^\epsilon||_{L^2}^2||\theta^\epsilon||_{L^\infty}.$$

Therefore we obtain

$$\frac{d}{dt}||\Lambda^{\frac{3}{2}}\theta^{\epsilon}||_{L^2}^2 \leq (C||\theta^{\epsilon}||_{L^\infty} - \kappa)||\Delta\theta^{\epsilon}||_{L^2}^2.$$

In particular one can find a time $\overline{T} = \overline{T}(\kappa, \theta_0)$ so that for $t \geq \overline{T}$, $||\Lambda^{\frac{3}{2}}\theta^{\epsilon}(\cdot, t)||_{L^2}$ is bounded by $||\Lambda^{\frac{3}{2}}\theta_0^{\epsilon}||_{L^2}$ and decreasing ($||\theta^{\epsilon}(\cdot, t)||_{L^\infty} \leq \frac{\kappa}{2C}$). We get

$$\int_{\overline{T}}^{\infty} ||\Delta\theta^{\epsilon}||_{L^2}^2 dt < \infty$$

uniformly on $\epsilon > 0$. Then one can repeat this process now with Λ^2 and $\Lambda^{\frac{5}{2}}$ and so on. Therefore we have completed the proof of the following:

Theorem 5.4. *Let θ be a viscosity solution with initial data $\theta_0 \in H^s$, $s > \frac{3}{2}$, of the equation $\theta_t + R(\theta) \cdot \nabla^{\perp}\theta = -\kappa\Lambda\theta$ ($\kappa > 0$). Then there exist two times $T_1 \leq T_2$ depending only upon κ and the initial data θ_0 so that:*

1) If $t \leq T_1$ then $\theta(\cdot, t) \in C^1([0, T_1); H^s)$ is a classical solution of the equation satisfying

$$||\theta(\cdot, t)||_{H^s} \ll ||\theta_0||_{H^s}.$$

2) If $t \geq T_2$ then $\theta(\cdot, t) \in C^1([T_2, \infty); H^s)$, is also a classical solution and $||\theta(\cdot, t)||_{H^s}$ is monotonically decreasing in t, bounded by $||\theta_0||_{H^s}$, and satisfying

$$\int_{T_2}^{\infty} ||\theta||_{H^s}^2 dt < \infty.$$

In particular this implies that

$$||\theta(\cdot, t)||_{H^s} = O(t^{-\frac{1}{2}}) \qquad t \to \infty.$$

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Communicated by P. Constantin