Chapter 10
Rotating Flows

Flows in rotating frames of reference include atmospheric flows and flows in turbomachinery. Fictitious inertia forces are added to the Navier-Stokes equations: the centrifugal force is easily included the conventional framework, but the Coriolis force introduces some new physics. Relevant dimensionless parameters are the Ekman and Rossby numbers.

The case of dominant Coriolis force corresponds to a small Rossby number. See Fig. 10.1.

10.1 Equations of motion

The equations of motion in a rotating frame of reference can be found in most dynamics textbooks. The derivation relies on the observation that, as the base vectors rotate at constant angular velocity \( \Omega \) (Fig. 10.2), we have

\[
\partial_t \xi_j = \Omega \times \xi_j
\]  

(10.1)

Note that \( \Omega \) is the rotation vector (Poisson vector) of the frame of reference as seen from an assumed inertial reference. Then, the second derivative of the vector position \( \mathbf{r} = x_i \xi_i \) is easily obtained

\[
\partial_{tt} \mathbf{r} = \partial_{tt}^2 x_i \xi_i + 2 \partial_t x_i \Omega \times \xi_i + x_i \Omega \times (\Omega \times \xi_i)
\]

(10.2)

In addition to the apparent acceleration, two new terms appear: the Coriolis acceleration which depend on velocity, and the centripetal acceleration which depends on position relative to the axis of rotation.
D. Fultz’s movie: Rotating flows

- Coriolis force: motion of vortex ring (self-propelled) or similar object
- Rossby number, geostrophic flow
  - linear equations
  - 2-D: Taylor-Proudman theorem
  - Taylor columns
  - non-local effects: $\nabla^2$
  - Rossby waves
- atmospheric motion: large scale only
- Ekman layers: secondary flows

Figure 10.1: Small Rossby number flows, movie by D. Fultz

Figure 10.2: Inertial and rotating frames of reference
10.2. SCALING

Moving these two accelerations to the other side of Newton’s second law, they become fictitious forces (with signs reversed), and the Navier-Stokes equations become

$$\partial_t \mathbf{u} + \mathbf{\omega} \times \mathbf{u} = -\nabla \left( \frac{p}{\rho} + \frac{u^2}{2} + gz \right) - \nu \nabla \times \mathbf{\omega} - \Omega \times (\Omega \times \mathbf{r}) - 2\Omega \times \mathbf{u} \tag{10.3}$$

The first added term is the centrifugal force, quadratic in $\Omega$. Because it can be rewritten in the form

$$\Omega \times (\Omega \times \mathbf{r}) = -\nabla \left( \frac{1}{2} \Omega^2 r'^2 \right) \tag{10.4}$$

where $r'$ denotes the distance from the axis of rotation, the centrifugal term is easily combined with the pressure term.

The term linear in $\Omega$ is the Coriolis force. It is essential in explaining our large-scale weather systems, among other features. This chapter revolves around it.

Questions for discussion: Is your lab an inertial lab? Rotor?

10.1.1 Coriolis force

Effects of the Coriolis force are illustrated in the movie shown in class. The trajectory of a vortex ring in the rotating tank is a good example (Fig. 10.3). Note that the direction of rotation is actually for tank rotation as seen from (inertial) lab.

A more dramatic effect is found in the earth’s atmosphere, with the dominant westerlies at mid-latitudes (Fig. 10.4).

Of course, many complicating dynamics are encountered in applications. But an understanding of what happens when rotation is dominant serves as a caution for what might happen in combination with other factors: simple first!

10.2 Scaling

As usual, with length and time as the only dimensions, we can take two scaling quantities: some velocity and length scales $U$ and $L$ are the usual default, but the imposed angular speed $\Omega$ will identify relevant ranges of $L$. Then, ignoring gravity and centrifugal forces, the various terms in the momentum equation can be estimated:
Figure 10.3: Sketch from movie: trajectory of a vortex ring in rotating tank seen from above

Figure 10.4: Dominant westerly winds at mid-latitudes as a consequence of buoyancy-induced equatorial-polar circulation and of the Coriolis force
10.3. GEOSTROPHIC APPROXIMATION

1. \( \mathbf{u} \cdot \nabla \mathbf{u} \sim \frac{U^2}{L} \)
2. \( 2\Omega \times \mathbf{u} \sim \Omega U \)
3. \( \nu \nabla^2 \mathbf{u} \sim \nu \frac{U}{L^2} \)

Since we are exploring the effects of rotation, the Coriolis term cannot be neglected, and we compare all other to it by dividing throughout by \( \Omega U \). Then, we have

1. Coriolis term: \( \sim 1 \)
2. Time derivative: \( \sim \frac{1}{\Omega} \partial_t \), which indicates that, in the absence of other forcing, the characteristic time scale will be of the order of the period of rotation.
3. Convective term: \( \sim \frac{U}{L} = Ro \), which defines the Rossby number
4. Viscous term: \( \sim \nu \frac{U}{L^2} = Ek \), which defines the Ekman number.

Note that we have no way to evaluate the pressure term directly.

Various choices can be made at this stage. The case of \( Ek \sim 1 \) is important near walls, or at the surface of the ocean: in the Ekman layer, the direction of motion is not the same as the direction of the applied stress, a fact known to sailors long ago and explained by Ekman. See the flow visualization of Ekman layers in the movie shown in class. Ignoring boundary layers for lack of time, we assume the Ekman number to be very small (large scale, small viscous effects). Then, away from solid surfaces and interfaces, the Rossby number tells us how important rotation is: the limiting case of interest here is of dominant Coriolis forces, i.e. \( Ro \ll 1 \). For given \( U \) and \( \Omega \) as scaling quantities, this corresponds to large scale flows: hurricanes rather than tornadoes! But note that the relevant scale depends on the apparent speed of motion and on the rotation speed.

This limit \( Ro \ll 1 \) is known as the geostrophic approximation, and is very important for large-scale meteorology.

10.3 Geostrophic approximation

Extreme case, the phenomena associated with rotation takes their most extreme form (and the equations are simpler!) Assumptions: steady flow, small
Rossby and Ekman numbers

\[ Ro = \frac{U}{\Omega L} \ll 1 \quad (10.5) \]

\[ Ek = \frac{\nu}{\Omega L^2} \ll 1 \quad (10.6) \]

Then, the viscous and convective terms drop out, and only the time derivative and the pressure terms remain to balance the Coriolis force. The quasi-static version (no external forcing to impose a time variation) is easiest:

\[ 2\Omega \times u = -\frac{1}{\rho} \nabla p \quad (10.7) \]

The time derivative can be restored for time-dependent problems, e.g. Rossby waves below.

The undergraduate understanding (from hydrostatics) of the pressure gradient as a force should be revisited here: the fluid motion is perpendicular to the pressure gradient.

The geostrophic equations are linear for velocity, which makes them easier to manipulate. Projecting them along streamlines, we get:

\[ u \cdot \nabla p = 0 \quad (10.8) \]

similar to Crocco’s equation, but for static pressure. There is no velocity in the direction of the pressure gradient, so pressure is constant along streamlines. Furthermore,

\[ \Omega \cdot \nabla p = 0 \quad (10.9) \]

so that pressure is also constant in the direction of the axis of rotation. Finally, taking the divergence of the geostrophic equation, we get

\[ \nabla^2 p = 2\rho \Omega \cdot \omega \quad (10.10) \]

so that the component of vorticity along the axis of rotations acts as a source for pressure variations throughout the field. This equation explains the direction of rotation around high and low pressures a mid-latitudes Fig. 10.5. A high pressure corresponds to maximum of \( p \), therefore to a negative value of the r.h.s.: with \( \Omega \) pointing toward the North Star, the dot product will be negative for clockwise rotation in the northern hemisphere. This explains the cyclonic and anticyclonic circulation at mid-latitudes (why not in tropical regions?)
10.3. GEOSTROPHIC APPROXIMATION

10.3.1 Taylor columns

One of the most intriguing aspects of small $Ro$ flows is the ‘vorticity’ equation. Taking the curl of the geostrophic (momentum) equation, we get for uniform $\Omega$ and divergence-free velocity

\[(\Omega \cdot \nabla)u = 0\]  \hspace{1cm} (10.11)

i.e. the directional derivative of velocity (each and every component) along the axis of rotation vanishes. This is known as the Taylor-Proudman theorem.

What does it say? We have learned to read directional derivatives, and this is a good application: the directional derivative of velocity (the vector, or any component of it) in the direction of the axis of rotation vanishes. Interpreted, this means that the flow is two-dimensional, restricted to the plane normal to the axis of rotation!

The movie seen in class shows many spectacular illustrations of this phenomenon, for example:

- in a rotating tank, a bump at the bottom effectively freezes the entire column of fluid above it, with the flow forced to avoid the column in order to stay in-plane.
• in a rotating tank, moving an obstacle up and down induces vorticities of opposite signs below and above the obstacle.

The two-dimensionality of flow in rapidly rotating systems as a great impact on turbulence, therefore on mixing. Keep this in mind when using turbulence models (generally assuming 3-D properties) in rotating systems. Read Tritton, Section 16.4, p.219-226, for additional information.

Summary of small-Ro approximation on Fig. 10.6.

10.4 Rossby waves

This section has some aspects in common with Ch. 11, except that no linearization is required here: geostrophic motion is already linear! Tritton, Section 16.7, p.232-238.

The configuration is shown on Fig. 10.7. The most significant element, beside rotation, is the layer of varying depth, and we denote the angle

$$\gamma = d_y h$$  \hspace{1cm} (10.12)

It can be shown that this simulates the variation of Coriolis force with latitude. We assume constant depth in the streamwise direction

$$d_x h = 0$$  \hspace{1cm} (10.13)

and homogeneity in the direction of rotation. We cannot impose a no-slip condition, since viscous effects scaled out of the problem.

Mass balance is given by

$$\partial_x u + \partial_y v + \partial_z w = 0$$  \hspace{1cm} (10.14)

Because $u$ and $v$ are independent of $z$, we have

$$\partial^2_{zz} w = 0$$  \hspace{1cm} (10.15)

therefore

$$\partial_z w = C = \frac{w_h}{h} = \frac{\gamma v}{h}$$  \hspace{1cm} (10.16)

Hence

$$\partial_x u + \partial_y v + \frac{\gamma v}{h} = 0$$  \hspace{1cm} (10.17)
10.4. Rossby Waves

**Simplifications**

\[ \text{Ro} \ll 1 \]
\[ \text{Ek} \ll 1 \]

Then

\[ \frac{\partial u}{\partial t} + 2 \Omega \times u = -\frac{1}{\rho} \nabla p \]

**Linear!**

\[ \nabla \cdot u = 0 \]
\[ \nabla \cdot \nabla p = 0 \]
\[ \nabla^2 p = \rho \Omega \cdot \omega \]

along streamline

along axis of rotation

Pressure sources

\[ \text{Taylor-Proudman} \rightarrow 2D \text{ motion} \]

Figure 10.6: Ideas related to small Ro flows: geostrophic motion
The dynamics are described by the unsteady geostrophic equations

\[ \partial_t u + 2\Omega \times u = -\frac{1}{\rho} \nabla p \]  \hspace{1cm} (10.18)

Breaking it into components:

\[ \partial_t u - 2\Omega v = -\frac{1}{\rho} \partial_x p \]
\[ \partial_t v + 2\Omega u = -\frac{1}{\rho} \partial_y p \]  \hspace{1cm} (10.19)

The equations are linear in \( u \) and \( v \). Eliminating pressure

\[ -\frac{1}{\rho} \partial_x \partial_y p = \partial^2_{ty} u - 2\Omega \partial_y v \]
\[ = \partial^2_{tx} v + 2\Omega \partial_x u \]  \hspace{1cm} (10.20)

or

\[ \partial^2_{ty} u = \partial^2_{tx} v + 2\Omega (\partial_x u + \partial_y v) = \partial^2_{tx} v - 2\Omega \frac{\gamma \gamma'}{h} \]  \hspace{1cm} (10.21)
We need one more simplification (not to imply that our simplifying assumptions must be true in all cases, but merely looking for the simplest solution at first). It is conceivable that there are solutions with $u$ independent of $y$ — still allowing for $v$ and $w$ to vary with $y$. This yields an equation for $v$ only:

$$\partial_{xx}^2 v - \frac{\gamma^2 \Omega}{h} v = 0 \quad (10.22)$$

This equation admits oscillatory solutions. Prompted by the experiments (movie), we restrict our search to traveling waves, of the form

$$v = v_0 e^{i(\omega t - \kappa x)} \quad (10.23)$$

For such waves, the partial operators $\partial_t$ and $\partial_x$ become algebraic operators $i\omega$ and $-i\kappa$, respectively. Substitution and rearrangement gives the dispersion relation

$$\kappa = \frac{2\Omega \gamma}{h \omega} \quad (10.24)$$

From this, we can calculate the phase velocity (i.e. the velocity of crest propagation) as

$$c_p = \frac{\omega}{\kappa} = \frac{h \omega^2}{2\Omega \gamma} \quad (10.25)$$

which depends on $\omega$, and the group velocity (i.e. the velocity of energy propagation)

$$c_g = \partial_{\kappa} \omega = -\frac{h \omega^2}{2\Omega \gamma} = -c_p \quad (10.26)$$

Rossby waves are important to us. Remember that the jet stream is one consequence of the Coriolis force. It is common for the jet stream to show 3 to 5 periods of oscillation around mid-latitudes: one dramatic example of Rossby waves Fig. 10.8.

10.5 Advanced topics and ideas for further reading

Zonal motion on larger planets.

Ekman layers, ocean surface motion and drift relative to wind speed, Langmuir circulation.
Problems

1. Imagine a turbine rotor spinning at 5000 rpm. Between rotor blades, the centerline air speed is of the order of 50 m/s, and the blade chord is 3 cm with an inter-blade distance of 1.5 cm. Evaluate the Rossby number characteristic of the mean flow, and of the boundary layer turbulence (mixing length of the order of the boundary layer thickness).

2. Identify nonlocal phenomena in flows with $Ro \ll 1$.

3. Can geostrophic flow be potential?

4. Does Bernoulli’s equation apply to geostrophic flow, and in which form?

5. Based on respective equations, discuss similarities and differences between Stokes flow and geostrophic flow.