Exercises

4.1. (i) Show that in a simply connected region of irrotational fluid motion the integral (4.2) is independent of the path between O and P.
(ii) Show that in a simply connected region of two-dimensional, incompressible fluid motion the integral

\[ \psi = \int_O^P u \, dy - v \, dx \]

is independent of the path between O and P, and hence serves as a definition of the stream function.

4.2. The velocity field

\[ u_r = \frac{Q}{2\pi r}, \quad u_\theta = 0, \]

where Q is a constant, is called a line source flow if Q > 0 and a line sink if Q < 0. Show that it is irrotational and that it satisfies \( \nabla \cdot \mathbf{u} = 0 \), save at \( r = 0 \), where it is not defined. Find the velocity potential and the stream function, and show that the complex potential is

\[ w = \frac{Q}{2\pi} \log z. \]
Observe that the stream function is a multivalued function of position. Why does this not contradict part (ii) of Exercise 4.1?

Fluid occupies the region \( x > 0 \), and there is a plane rigid boundary at \( x = 0 \). Find the complex potential for the flow due to a line source at \( z = d > 0 \), and show that the pressure at \( x = 0 \) decreases to a minimum at \( \left| y \right| = d \) and thereafter increases with \( \left| y \right| \).

[Any attempt to reproduce the flow of Fig. 4.13 at high Reynolds number would be fraught with difficulties. A viscous boundary layer would be present, to satisfy the no-slip condition, but for \( \left| y \right| > d \) the substantial adverse pressure gradient along the boundary would make separation inevitable (see §2.1). More fundamentally still, there are considerable practical difficulties in producing a line source, as opposed to a line sink, at high Reynolds number. These are more easily seen by considering the corresponding 3-D problem; a point sink can be simulated quite well by sucking at a small tube inserted in the fluid, but blowing down such a tube produces not a point source but a highly directional and usually turbulent jet (see, e.g. Lighthill 1986, pp. 100–103). The streamline pattern in Fig. 4.13 may nevertheless be observed in a Hele–Shaw cell (§7.7), although viscous effects are then paramount throughout the whole flow, so the pressure distribution is not given by Bernoulli’s equation.]

4.3. An irrotational 2-D flow has stream function \( \psi = A(x - c)y \), where \( A \) and \( c \) are constants. A circular cylinder of radius \( a \) is introduced, its centre being at the origin. Find the complex potential, and hence the stream function, of the resulting flow. Use Blasius’s theorem (4.62) to calculate the force exerted on the cylinder.
4.4. Show that the problem of irrotational flow past a circular cylinder may be formulated in terms of the velocity potential \( \phi(r, \theta) \) as follows:

\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0,
\]

with

\[
\phi \sim U r \cos \theta \text{ as } r \to \infty, \quad \frac{\partial \phi}{\partial r} = 0 \text{ on } r = a,
\]

and obtain the solution (4.33) by using the method of separation of variables.

When there is circulation round the cylinder, derive eqn (4.40), and confirm that the stagnation points vary in position with the parameter \( B \) in the manner of Fig. 4.4.

4.5. Establish the expression (4.63) for the moment, \( N \), of forces on a body in irrotational flow, using an argument similar to that for Blasius’s theorem.

4.6. By writing \( z = a + \varepsilon \) in eqn (4.57) and taking the limit \( \varepsilon \to 0 \) check that the choice of circulation (4.58) does indeed lead to a finite velocity at the trailing edge.

4.7. According to eqns (4.1) and (4.66), the force on a thin symmetric aerofoil with a sharp trailing edge is

\[
L = \pi \rho U^2 L \sin \alpha
\]

in a direction perpendicular to the uniform stream. This amounts to a component \( L \cos \alpha \) perpendicular to the aerofoil and a component \( L \sin \alpha \) parallel to the aerofoil, directed towards the leading edge. This latter component is, at first sight, rather curious; it might be thought that the net effect of a pressure distribution on a thin symmetric aerofoil should be almost normal to the aerofoil. That it is not is due to leading edge suction, i.e. a severe drop in pressure in the immediate vicinity of the rounded leading edge, this pressure drop being sufficient to make itself felt despite the small thickness of the wing on which it acts.

To see evidence of this, consider the extreme case of flow past a flat plate with circulation, as in Fig. 4.6(b) or Fig. 4.15. First, use eqns (4.56) and (4.57), on \( z = a e^{i\alpha} \), with \( \Gamma \) chosen according to eqn (4.58), to show that the flow speed on the plate is

\[
U \left| \cos \alpha \pm \left( \frac{1 - s}{1 + s} \right)^{1/4} \sin \alpha \right|,
\]

where the upper/lower sign corresponds to the upper/lower side of the plate, and \( s \) denotes \( X/2a \), which therefore runs between \(-1\) at the leading edge and \(+1\) at the trailing edge.
Show that the corresponding pressure distributions are
\[ p(s) = p(1) - \frac{1}{2} \rho U^2 \left[ \frac{(1-s)}{1+s} \sin^2 \alpha \pm 2 \left( \frac{1-s}{1+s} \right)^{\frac{1}{2}} \sin \alpha \cos \alpha \right], \]
(see Fig. 4.14). Note that there is a (negative) pressure singularity at the leading edge, whereas if the leading edge were rounded this pressure drop would be finite.

As far as the force component normal to the plate is concerned, note that the pressure difference across the plate is
\[ p_D = 2 \rho U^2 \left( \frac{1-s}{1+s} \right)^{\frac{1}{2}} \sin \alpha \cos \alpha. \]
This too has a singularity at the leading edge, but it is integrable. Show that
\[ \int_{-2a}^{2a} p_D \, dX = \mathcal{L} \cos \alpha, \]
in keeping with the Kutta–Joukowski Lift Theorem.

Finally, show that eqn (4.65) holds even if there is circulation \( \Gamma \) round the ellipse, and then take the case \( c = a \) to show that the torque on a flat plate about the origin is \( -\mathcal{L}a \cos \alpha \), i.e. as if the whole lift force \( \mathcal{L} \) were

\[ \frac{p(s) - p(1)}{\frac{1}{2} \rho U^2 \sin \alpha} \]

\[ \alpha = 10^\circ \]

**Fig. 4.14.** Theoretical pressure distribution on a flat plate at a 10° angle of attack.
Fig. 4.15. The torque on a flat plate in uniform flow is as if the lift \( \mathcal{L} \) were concentrated at a point one-quarter of the way along the plate from the leading edge.

applied at a point one-quarter of the way along the plate, as indicated in Fig. 4.15.

[The fact that this point is independent of \( \alpha \) is of practical value, and makes for smooth control of an aircraft.]

4.8. Show that the Joukowsky transformation \( Z = z + a^2/z \) can be written in the form

\[
\frac{Z - 2a}{Z + 2a} = \left( \frac{z - a}{z + a} \right)^2,
\]

so that, in particular,

\[
\text{arg}(Z - 2a) - \text{arg}(Z + 2a) = 2[\text{arg}(z - a) - \text{arg}(z + a)].
\]

Consider the circle in the \( z \)-plane which passes through \( z = -a \) and \( z = a \) and has centre \( ia \cot \beta \). Show that the above transformation takes it into a circular arc between \( Z = -2a \) and \( Z = 2a \), with subtended angle \( 2\beta \) (Fig. 4.16). Obtain an expression for the complex potential in the \( Z \)-plane, when the flow is uniform, speed \( U \), and parallel to the real axis. Show that the velocity will be finite at both the leading and trailing edges if

\[
\Gamma = -4\pi U a \cot \beta.
\]

[This exceptional circumstance arises only when the undisturbed flow is parallel to the chord line of the arc.]

4.9. Provided that \( f'(z_0) \neq 0 \), points in the neighbourhood of \( z = z_0 \) are mapped by \( Z = f(z) \), according to Taylor's theorem, in such a way that

\[
Z - Z_0 = f'(z_0)(z - z_0) + O((z - z_0)^2),
\]
where $Z_0 = f(z_0)$. Use this to show that a line source of strength $Q$ at $z = z_0$ is mapped into a line source of strength $Q$ at $Z = Z_0$, provided that $f'(z_0) \neq 0$.

Fluid occupies the region between two plane rigid boundaries at $y = \pm b$, and there is a line source of strength $Q$ at $z = 0$. Find the complex potential $w(z)$ for the flow

(i) by the method of images,

(ii) by using the mapping $Z = e^{\alpha z}$ with a suitably chosen $\alpha > 0$.

4.10. Use the momentum equation in its integral form (4.70) to show that there is a non-zero drag

$$F_c = \rho \Gamma^2 / 2d$$

on each of the aerofoils in Fig. 4.10.

Is this at odds with the Kutta–Joukowski Lift Theorem (4.66)?