

On the verbal width of finitely generated pro- p groups

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Abstract

Let p be a prime. It is proved that a non-trivial word w from a free group F has finite width in every finitely generated pro- p group if and only if $w \notin (F')^p F''$. Also it is shown that any word w has finite width in a compact p -adic group.

1 Introduction

Let F be a free group on k independent generators. We will call an element w from F a word. If G is a group, then we say that $g \in G$ is a w -**value** in G if there are $g_1, \dots, g_k \in G$ such that $g = w(g_1, \dots, g_k)^{\pm 1}$. We denote the set of the all w -values in G by $G^{\{w\}}$. A simple argument (see [6]) shows that if G is profinite, then $w(G)$ (the abstract subgroup generated by $G^{\{w\}}$) is closed if and only if there exists l such that any element from $w(G)$ is a product of at most l elements from $G^{\{w\}}$. The smallest such number l is called the **width** of w in G .

In this paper we consider a particular case of the following question: which words do have finite width in a finitely generated profinite group G ? The most important achievement in this subject is a recent work of N. Nikolov and D. Segal (see [13]), where they proved that if w is either d -locally finite or w is a simple commutator, then w has finite width in any d -generated profinite group G (we recall that a group word w is d -**locally finite** if every d -generator group H satisfying $w(H) = 1$ is finite).

The main result of this paper is as follows.

Theorem 1.1. *Let $w \neq 1$ be an element of a free group F . Then the following are equivalent:*

1. $w(H)$ is closed for every finitely generated pro- p group H ;
2. $w \notin (F')^p F''$.

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The proofs of two implications in Theorem 1.1 are quite different. In order to prove $2 \Rightarrow 1$ we characterize the words $w \in F$ satisfying $w \notin (F')^p F''$. We say that w is a \mathcal{N}_p -**word** if for every finitely generated pro- p group H , $H/\overline{w(H)}$ is nilpotent-by-finite, where $\overline{w(H)}$ denotes the closure of $w(H)$ in H . For example, x^{p^n} is a \mathcal{N}_p -word by Zelmanov's solution of the restricted Burnside problem. An Engel word is another example of \mathcal{N}_p -word (see [20]).

Theorem 1.2. *Let w be an element of a free group F . Then the following are equivalent:*

1. w is a \mathcal{N}_p -word;
2. if H is a free pro- p group on two generators then $H/w(H)$ is nilpotent-by-finite;
3. $w(C_p \wr \mathbb{Z}) \neq \{1\}$;
4. $w \notin (F')^p F''$.

This theorem reduces the proof of the implication $2 \Rightarrow 1$ from Theorem 1.1 to the case when H is virtually nilpotent. This case is solved using the following more general result, which also answers a question posed by L. Pyber:

Theorem 1.3. *Let G be a compact p -adic analytic group. Then any word w of a free group F has finite width in G .*

Note that in Theorem 1.3 we do not assume that G is a pro- p group.

In order to prove other implication from Theorem 1.1 we show that if $w \in (F')^p F''$ and H is a non-abelian free pro- p group, then $H^{\{w\}}$ is "very small" (more concretely we show that no power of $H^{\{w\}}$ can contain a non-trivial normal subgroup of H).

We use the following notation. If S is a set and m is a natural number then $S^{(m)}$ denotes the cartesian product of m copies of S . If S is a subset of a group H , then S^{*m} is the set of all products $s_1^{\pm 1} \cdots s_n^{\pm 1}$, where $n \leq m$ and $s_i \in S$. We will also use the same notation when the operation in H is additive, so in this case $S^{*m} = \{\pm s_1 \pm \cdots \pm s_n \mid n \leq m, s_i \in S\}$. We will say that S has **finite width** in H if there exists l such that the subgroup $\langle S \rangle$ generated by S is equal to S^{*l} . We use $[,]_L$ to denote the Lie bracket and simply $[,]$ for the group commutator.

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2 Proof of Theorem 1.3

In the following $|\cdot|_p$ is the standard p -adic valuation on \mathbb{Q}_p : if $a \in p^k \mathbb{Z}_p \setminus p^{k+1} \mathbb{Z}_p$, then $|a|_p = p^{-k}$. Let $X = (X_1, \dots, X_m)$ be m commuting indeterminates and let $\mathbb{Q}_p[[X]]$ denote the set of formal power series over \mathbb{Q}_p . The ring $\mathbb{Q}_p\{X\}$ of restricted power series in X over the p -adic field \mathbb{Q}_p consists of the formal power series $\sum_i a_i X^i$ in $\mathbb{Q}_p[[X]]$ such that $|a_i|_p \rightarrow 0$ as $|i| \rightarrow \infty$. Here $i = (i_1, \dots, i_m)$ ranges over $\mathbb{Z}_{\geq 0}^{(m)}$, $|i| = i_1 + \dots + i_m$, $X^i = X_1^{i_1} \dots X_m^{i_m}$. Let $f = \sum_i a_i X^i$ be in $\mathbb{Q}_p\{X\}$ and $x \in \mathbb{Z}_p^{(m)}$. Then the series $\sum_i a_i x^i$ converges to a limit in \mathbb{Q}_p which we denote by $f(x)$. The subring $\mathbb{Z}_p\{X\}$ of $\mathbb{Q}_p\{X\}$ consists of the series $\sum_i a_i X^i$ in $\mathbb{Q}_p\{X\}$ all of whose coefficients are in \mathbb{Z}_p .

Let $L = (\mathbb{Z}_p^{(m)}, +)$. We will consider L as a p -adic manifold. Hence for each points a of L we can define a local ring H_a at a with the maximal ideal \mathfrak{m}_a . The dual of $\mathfrak{m}_a/\mathfrak{m}_a^2$ is the tangent space of L at a and it is denoted by $T_a L$. If K is another p -adic manifold and $g: K \rightarrow L$ is an analytic map, then the induced map of the tangent spaces $T_x K \rightarrow T_y L$ ($y = g(x)$) is denoted by $T_x g$. For details, see [16, Part II. Chapter III]. We denote the element $(0, \dots, 0)$ by \mathbf{e} .

Lemma 2.1. *Let $Y = (Y_1, \dots, Y_n)$ and $f = (f_1, \dots, f_m)$ be a m -tuple consisting of m formal power series from $\mathbb{Z}_p\{Y\}$ such that $f(\mathbf{e}) = \mathbf{e}$. Put*

$$S = f(\mathbb{Z}_p^{(n)}) = \{(f_1(x), \dots, f_m(x)) \mid x \in \mathbb{Z}_p^{(n)}\} \subseteq \mathbb{Z}_p^{(m)}.$$

Then the width of S in $(\mathbb{Z}_p^{(m)}, +)$ is finite.

Proof. Put $K = \mathbb{Z}_p^{(n)}$ and $L = (\mathbb{Z}_p^{(m)}, +)$ and let A be the closed subgroup of L generated by S . If $L_1 = \{l \in L \mid p^k l \in A \text{ for some } k\}$, then we can find a subgroup L_2 of L such that $L = L_1 \oplus L_2$. We can choose new coordinates $\{x_1, \dots, x_m\}$ of L such that L_1 is given by equations $\{x_{s+1} = \dots = x_m = 0\}$. Then in new coordinates the map f looks like $(h_1, \dots, h_s, 0, \dots, 0)$. Hence, without loss of generality, we can suppose that S generates an open subgroup in L . We may clearly assume that $L \neq 0$.

For any $a \in K$ define $g_a(Y) = f(Y) - f(a)$. Then g_a is an analytic map from K to L and $g_a(a) = \mathbf{e}$. This analytic map induces a map between the tangent space $T_a g_a: T_a K \rightarrow T_{\mathbf{e}} L$. Let us calculate the image of this map.

Let ∂_i be the partial derivation with respect to the i th coordinate. Then the functionals $e_i: \mathfrak{m}_a/\mathfrak{m}_a^2 \rightarrow \mathbb{Q}_p$ defined by $e_i(q) = (\partial_i q)(a)$ form a basis of $T_a K$. In the same way the functionals $h_i: \mathfrak{m}_{\mathbf{e}}/\mathfrak{m}_{\mathbf{e}}^2 \rightarrow \mathbb{Q}_p$ defined by $h_i(q) = (\partial_i q)(\mathbf{e})$ form a basis of $T_{\mathbf{e}} L$.

Notice that $(T_a g_a)(e_i)(x_j) = e_i(f_j - f_j(a)) = \partial_i f_j(a)$. Thus,

$$(T_a g_a)(e_i) = \sum_{j=1}^m \partial_i f_j(a) h_j.$$

Now consider the subspace of $T_{\mathbf{e}} L$ generated by all images of $T_a g_a$ for all a . If this subspace is different from $T_{\mathbf{e}} L$, then there are constants $\alpha_1, \dots, \alpha_m$, not

all zero, such that for all $a \in K$ and $1 \leq i \leq N$,

$$0 = \sum_{j=1}^m \alpha_j \partial_i f_j(a) = \partial_i \left(\sum_{j=1}^m \alpha_j f_j \right)(a).$$

Thus $\partial_i(\sum_{j=1}^m \alpha_j f_j) = 0$ for all i . Hence $g = \sum_{j=1}^m \alpha_j f_j$ is a constant function. But since $g(\mathbf{e}) = \mathbf{e}$, g is the zero function. But this contradicts the assumption that S generates an open subgroup in L .

Hence the subspace of $T_{\mathbf{e}}L$ generated by all images of $T_a g_a$ is equal to $T_{\mathbf{e}}L$. Therefore we can find m elements a_1, \dots, a_m in K such that

$$T_{a_1} g_{a_1}(T_{a_1} K) + \dots + T_{a_m} g_{a_m}(T_{a_m} K) = T_{\mathbf{e}}L. \quad (2.1)$$

Define a map h from $K^{(m)}$ to L by $h(b_1, \dots, b_m) = g_{a_1}(b_1) + \dots + g_{a_m}(b_m)$. Put $b = (a_1, \dots, a_m)$. Then, by 2.1, $T_b h(T_b K^{(m)}) = T_{\mathbf{e}}L$. Hence from [16, Theorem 10.2, p.85] we obtain that h is a submersion and so $h(K^{(m)})$ contains an open in L subset. Thus S^{*m} contains an open in $A = \langle S \rangle$ subset.

Since A is a profinite group, there exists an open subgroup B of A and $a \in A$ such that $a + B \subseteq S^{*m}$. Since $A = S^{*l} + B$ for some l , $A = S^{*(m+l)}$. \square

Recall that a pro- p group G is called **powerful** if $[G, G] \leq G^p$ when $p > 2$ or $[G, G] \leq G^4$ when $p = 2$. We say that a finitely generated pro- p group G is **uniform** if G is powerful and without torsion. A **uniform \mathbb{Z}_p -Lie lattice** is a Lie ring L such that L is a finitely generated free \mathbb{Z}_p -module and $[L, L]_L \leq pL$ when $p > 2$ or $[L, L]_L \leq 4L$ when $p = 2$. According to Lazard (see, for example, [4]), there is an equivalence between the category of uniform pro- p groups and the category of uniform \mathbb{Z}_p -Lie lattices. The uniform \mathbb{Z}_p -Lie lattice \mathbf{H} corresponding to a uniform pro- p group H has H itself as its underlying set and the Lie ring operations are given in terms of the group operations as follows: for all $z \in \mathbb{Z}_p$ and all $x, y \in H$ we have

$$\begin{aligned} z \cdot x &= x^z, \\ g + h &= \lim_{n \rightarrow \infty} (g^{p^n} h^{p^n})^{p^{-n}}, \\ [g, h]_L &= \lim_{n \rightarrow \infty} [g^{p^n}, h^{p^n}]^{p^{-2n}}. \end{aligned} \quad (2.2)$$

Conversely, given a uniform \mathbb{Z}_p -Lie lattice \mathbf{H} of rank m , the uniform pro- p group H corresponding to \mathbf{H} can be constructed via the Baker-Campbell-Hausdorff formula. Its underlying set is again \mathbf{H} , and the group product of $x, y \in \mathbf{H}$ is given by $xy = \Phi(x, y)$. Recall that the Baker-Campbell-Hausdorff formula is $\Phi(x_1, x_2) = \log(e^{x_1} e^{x_2})$ regarded as a formal power series in two non-commuting variables (see [4, Section II.6.5]).

From now on we fix a system of free \mathbb{Z}_p -generators of \mathbf{H} . Thus to any element x from \mathbf{H} corresponds a m -tuple (x_1, \dots, x_m) from $\mathbb{Z}_p^{(m)}$. We will refer to this m -tuple as coordinates of x and we will regard \mathbf{H} (and so H) as $\mathbb{Z}_p^{(m)}$. Then, by [2, Proposition II.8.1]), the multiplication in H is given by a m -tuple (F_1, \dots, F_m) where $F_i \in \mathbb{Z}_p\{X\}$.

Corollary 2.2. Let $Y = (Y_1, \dots, Y_n)$ and $f = (f_1, \dots, f_m)$ be a m -tuple consisting of m formal power series from $\mathbb{Z}_p\{Y\}$. Put

$$S = f(\mathbb{Z}_p^{(n)}) = \{(f_1(x), \dots, f_m(x)) \mid x \in \mathbb{Z}_p^{(n)}\} \subseteq H.$$

Suppose that $f(\mathbf{e}) = \mathbf{e}$ and the group generated by S is abelian. Then the width of S in H is finite.

Proof. Since the group $\langle S \rangle$ is abelian, we can apply Lemma 2.1, because the width of S in H is the same as the width of S in $(H, +)$. \square

Lemma 2.3. Let $Y = (Y_1, \dots, Y_n)$ and $f = (f_1, \dots, f_m)$ be a m -tuple consisting of m formal power series from $\mathbb{Z}_p\{Y\}$. Put

$$S = f(\mathbb{Z}_p^{(n)}) = \{(f_1(x), \dots, f_m(x)) \mid x \in \mathbb{Z}_p^{(n)}\} \subseteq H.$$

Suppose that $f(\mathbf{e}) = \mathbf{e}$ and S is a normal set in H . Then the width of S in H is finite.

Proof. Let T be the closed subgroup generated by S . Since S is a normal set in H , T is a normal subgroup of H . Hence the set

$$R = \{x \in H \mid x^{p^k} \in [T, T] \text{ for some } k\}$$

is also normal subgroup in H . Put $\bar{H} = H/R$. Note that \bar{H} is a uniform pro- p group. Moreover, R is an ideal of H . Hence we can choose new coordinates $\{x_1, \dots, x_m\}$ of H in such way that R is defined by equations $\{x_1 = \dots = x_s = 0\}$. Suppose that in these new coordinates the map f looks like (g_1, \dots, g_m) . Note that the first s coordinates (x_1, \dots, x_s) of H determine uniquely an element \bar{x} from \bar{H} and they are coordinates of \bar{x} with respect to some system of \mathbb{Z}_p -generators of \bar{H} . Thus, the composition of f with the natural epimorphism to \bar{H} looks like (g_1, \dots, g_s) . Since the set $\bar{S} = SR/R$ generates an abelian subgroup in \bar{H} , we obtain from the previous corollary that there exists l_1 such that $T = S^{*l_1}R$. On the other hand $R/[T, T]$ is finite, whence we also obtain that $T = S^{*l_2}[T, T]$ for some l_2 .

Since H is uniform, T is finitely generated. Let $t_1, \dots, t_l \in S$ be the generators of T as a pro- p group. Then $[T, T] = [t_1, T] \cdots [t_l, T]$ (see [4, Proof of Proposition 1.19]) and so, since S is normal, $[T, T] \subseteq S^{*2l}$. Hence $T = S^{*(2l+l_2)}$. \square

Proof of Theorem 1.3. Since G is p -adic analytic, G has an open uniform normal pro- p subgroup H (see [4, Corollary 8.34]). Let $\{a_i \mid 1 \leq i \leq |G : H|\}$ be a transversal of G over H . For each $i = (i_1, \dots, i_k)$ define the function $g_i: H^{(2k)} \rightarrow H$ by means of

$$g_i(h_{1,i}, \dots, h_{2k,i}) = w(a_{i_1} h_{1,i}, \dots, a_{i_k} h_{k,i}) w(a_{i_1} h_{k+1,i}, \dots, a_{i_k} h_{2k,i})^{-1}.$$

Choose any order on k -tuples and put $f = \prod_i g_i$. Then f is a function from $H^{2k|G:H|^k}$ to H . Moreover, if we regard H as $\mathbb{Z}_p^{(m)}$ then f is an m -tuple of

functions from $\mathbb{Z}_p\{Y\}$, where $Y = (Y_1, \dots, Y_n)$ and $N = 2Mk|G:H|^k$. Put $S = f(\mathbb{Z}_p^{(n)})$ and let T be the closed subgroup generated by S .

If $h, h_1, \dots, h_k \in H$ and $1 \leq i_1, \dots, i_k \leq |G:H|$, then

$$w(a_{i_1} h_1, \dots, a_{i_k} h_k)^h = w(a_{i_1} [a_{i_1}, h] h_1^h, \dots, a_{i_k} [a_{i_k}, h] h_k^h).$$

Therefore S is a normal set in H , and so, T is a normal subgroup of G . By the previous lemma, there exists l such that $T = S^{*l}$. Since $S^{*l} \subseteq (G^{\{w\}})^{*2Mk|G:H|^k}$, we obtain that $T \subseteq (G^{\{w\}})^{*m_1}$ for some m_1 .

Consider the group $\bar{G} = G/T$. Note that the word w takes only finitely many different values in \bar{G} . Since \bar{G} is a p -adic analytic group, it is linear. Hence, by Merzlyakov's solution of Hall's problem for linear groups [10], $w(\bar{G}) = w(G)/T$ is finite. Thus, $w(G) = (G^{\{w\}})^{*m_2} T = (G^{\{w\}})^{*(m_1+m_2)}$ for some m_2 . \square

We recall that P. Hall's question is

Question. Let G be a group and w a word from a free group. Suppose that the word w takes only finitely many different values in G . Is it true that $w(G)$ is finite?

S. Ivanov [8] answered this question for arbitrary groups in the negative; he constructed a group H and a word $w(x, y)$ such that $w(H)$ is infinite cyclic but $w(x, y)$ has only one non-trivial value in H . Ivanov's example is not residually finite. So as far as I know, P. Hall's question for profinite groups is still open.

3 Verbal subgroups corresponding to \mathcal{N}_p -words

In this section we prove that if H is a finitely generated pro- p group then $w(H)$ is closed for any \mathcal{N}_p -word w . First we prove Theorem 1.2.

Proof of Theorem 1.2. The implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ are clear.

Suppose now that the third condition holds. If w is not a \mathcal{N}_p -word, then by [3] (see also [18]), there is a finitely generated non p -adic analytic pro- p group H such that $w(H) = 1$. From [19] it follows that $w(C_p \wr \mathbb{Z}) = 1$, a contradiction. Hence $3 \Rightarrow 1$.

Now $4 \Rightarrow 3$ follows, for example, from [11, Theorem 22.43]. \square

Theorem 3.1. *Let w be a \mathcal{N}_p -word and G a finitely generated pro- p group. Then $w(G)$ is closed.*

Proof. Let $d = d(G)$ and H be a free pro- p group on generators x_1, \dots, x_d, z . Since w is a \mathcal{N}_p -word, $\gamma_n(H^{p^t}) \leq \overline{w(H)}$ for some n and t .

Denote by y_1, \dots, y_s generators of $\langle x_1, \dots, x_d \rangle^{p^t}$. Note that y_i are pro- p words in x_i (and do not involve z). By Theorem 1.3, there exists k such that for any, $i_1, \dots, i_n \in \{1, \dots, s\}$,

$$[z, y_{i_1}, y_{i_2}, \dots, y_{i_n}] \equiv v_{i_1, \dots, i_n} \pmod{\gamma_{n+2}(H^{p^t})},$$

where v_{i_1, \dots, i_n} is a product of at most k w -values in H . Thus,

$$v_{i_1, \dots, i_n}(x_1, \dots, x_d, z) = [z, y_{i_1}, y_{i_2}, \dots, y_{i_n}]r_{i_1, \dots, i_n}(x_1, \dots, x_d, z)$$

with $r_{i_1, \dots, i_n}(x_1, \dots, x_d, z) \in \gamma_{n+2}(H^{p^t})$.

Let h_1, \dots, h_d be generators of G .

Claim 1 Let $a \in \gamma_m(G^{p^t})$, $g \in G$ and $r \in \gamma_l(H^{p^t})$. Then

$$r(h_1, \dots, h_d, g) \equiv r(h_1, \dots, h_d, ga) \pmod{\gamma_{m+l-1}(G^{p^t})}.$$

When $l = 1$, the claim is clear. The general case follows by induction on l .

Claim 2 $\gamma_{n+1}(G^{p^t}) = \prod_{i_1, \dots, i_n} v_{i_1, \dots, i_n}(h_1, \dots, h_d, G^{p^t})$.

By induction on m we prove that if $m \geq n + 1$ then

$$\gamma_{m+1}(G^{p^t}) = \prod_{i_1, \dots, i_n} v_{i_1, \dots, i_n}(h_1, \dots, h_d, H^{p^t})\gamma_m(G^{p^t}).$$

This implies the claim because the set

$$\prod_{i_1, \dots, i_n} v_{i_1, \dots, i_n}(h_1, \dots, h_d, H^{p^t})$$

is closed.

The base of induction $m = n + 1$ is clear. Suppose it holds for m . Let prove it for $m + 1$. Let $h \in \gamma_{m+1}(G^{p^t})$. By the inductive hypothesis, there are $g_{i_1, \dots, i_n} \in H^{p^t}$ and $u \in \gamma_m(G^{p^t})$ such that

$$h = \prod_{i_1, \dots, i_n} v_{i_1, \dots, i_n}(h_1, \dots, h_d, g_{i_1, \dots, i_n})u.$$

We can write

$$u \equiv \prod_{i_1, \dots, i_n} [t_{i_1, \dots, i_n}, \tilde{y}_{i_1}, \dots, \tilde{y}_{i_n}] \pmod{\gamma_{m+1}(G^{p^t})},$$

where $\tilde{y}_j = y_j(h_1, \dots, h_d)$ and $t_{i_1, \dots, i_n} \in \gamma_{m-n}(G^{p^t})$.

Thus,

$$h \equiv \prod_{i_1, \dots, i_n} [g_{i_1, \dots, i_n} t_{i_1, \dots, i_n}, \tilde{y}_{i_1}, \dots, \tilde{y}_{i_n}]r_{i_1, \dots, i_n}(h_1, \dots, h_d, g_{i_1, \dots, i_n}) \pmod{\gamma_{m+1}(G^{p^t})}.$$

By Claim 1,

$$r_{i_1, \dots, i_n}(h_1, \dots, h_d, g_{i_1, \dots, i_n}) \equiv r_{i_1, \dots, i_n}(h_1, \dots, h_d, g_{i_1, \dots, i_n} t_{i_1, \dots, i_n}) \pmod{\gamma_{m+1}(G^{p^t})}.$$

Hence we have

$$h \equiv \prod_{i_1, \dots, i_n} v_{i_1, \dots, i_n}(h_1, \dots, h_d, g_{i_1, \dots, i_n} t_{i_1, \dots, i_n}) \pmod{\gamma_{m+1}(G^{p^t})}.$$

This finishes the proof of the claim.

It follows that the closed subgroup $\gamma_{n+1}(G^{p^t})$ is contained in $w(G)$. We may therefore apply Theorem 1.3 and deduce that

$$w(G)/\gamma_{n+1}(G^{p^t}) = w(G/\gamma_{n+1}(G^{p^t}))$$

is closed. Hence $w(G)$ is closed. \square

4 Words of infinite width

In this section we prove that if F is a non-abelian free group and $1 \neq w \in (F')^p F''$, then $w(H)$ is not closed in a free finitely generated non-abelian free pro- p group H . These examples generalize the example of Roman'kov [15] who proved the same statement for $w = [[x, y], [z, u]]$.

Theorem 4.1. *Let F be a non-abelian free group, p a prime number and H a non-abelian finitely generated free pro- p group. Suppose that $1 \neq w \in (F')^p F''$. Then the verbal subgroup $w(H)$ is not closed.*

We explain first the strategy of the proof of this theorem. We suppose the contrary, that there exists k such that any element of $w(H)$ is a product of at most k w -values in H . Hence since $w \in (F')^p F''$ there exists a number l depending on w such that any element of $w(H)$ is a product of at most l values of the word $x^p[y, z]$ in H' . Now, note that H' is a free pro- p group of infinite rank and $w(H)$ is a normal subgroup of H' . Then the following proposition leads us to a contradiction.

Proposition 4.2.

1. *Let K be a free pro- p group of rank d and $\{1\} \neq N$ a closed normal subgroup of K . Then there exists an element $g \in N$ such that g cannot be represented as a product of less than $\lfloor d/3 \rfloor$ values of the word $x^p[y, z]$ in K .*

2. *Let K be a free pro- p group of infinite rank and $\{1\} \neq N$ a closed normal subgroup of K . Then for any $l \in \mathbb{N}$ there exists an element $g \in N$ such that g cannot be represented as a product of less than l values of the word $x^p[y, z]$ in K .*

Before the proof of the proposition we present an auxiliary result. For any pro- p group G , let $D_i(G)$ be the i th dimension subgroup of G (see e.g [4, Chapter 11]). Let K be a free pro- p group of rank d and $\{1\} \neq N$ a closed normal subgroup of K . Put $N_i = N \cap D_i(K)$ and define the following numbers:

$$a_i = \log_p |D_i(K) : D_{i+1}(K)|, \quad b_i = \log_p |K : D_{i+1}(K)|,$$

$$c_i = \log_p |N_i : N_{i+1}|, \quad d_i = \log_p |N : N_{i+1}| = \log_p |ND_{i+1}(K) : D_{i+1}(K)|.$$

Lemma 4.3. *When n tends to infinity the following holds*

1. $a_n = \frac{d^n}{n}(1 + o(1))$;
2. $b_n = \frac{d^{n+1}}{(d-1)n}(1 + o(1))$;
3. $c_n = \frac{d^n}{n}(1 + o(1))$;
4. $d_n = \frac{d^{n+1}}{(d-1)n}(1 + o(1))$.

Proof. Let L be a free Lie algebra generated by d elements. Then L can be graded in a standard way if we suppose that free generators are elements of degree 1. Then we can write $L = \oplus_i L_i$. The following formula

$$\dim L_i = M_d(n) = \frac{1}{n} \sum_{k|n} \mu(k) d^{n/k}$$

is well-known. Moreover, $M_d(n) = \frac{d^n}{n}(1 + o(1))$ (see [7, Chapter VIII]).

Recall that if G is a pro- p group, then $L_p(G) = \oplus D_i(G)/D_{i+1}(G)$ has the structure of a restricted Lie \mathbb{F}_p -algebra. Moreover, $L_p(K)$ is a free p -restricted Lie \mathbb{F}_p -algebra. The construction of a free Lie restricted \mathbb{F}_p -algebra from a free Lie \mathbb{F}_p -algebra is described, for example, in [2, Exercise 2.3.4]. It follows that if $n = p^s m$, where m and p are coprime, then

$$a_n = \log_p |D_n(K) : D_{n+1}(K)| = \sum_{k=0}^s M_d(mp^k).$$

This is an easy exercise to obtain from this that $a_n = \frac{d^n}{n}(1 + o(1))$.

Now, since

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \infty \text{ and } \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \frac{d-1}{d},$$

we obtain that $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{d-1}{d}$. Hence $b_n = \frac{d^{n+1}}{(d-1)n}(1 + o(1))$.

Put $G = K/N$. Then p -enveloping algebra of $L_p(G)$ is a proper quotient of a free \mathbb{F}_p -algebra on d -generators. Hence, by [5] (see also [12, Theorem 3.1] and [14, Theorem 15]), there exists $\alpha < d$ such that

$$a_i - c_i = \log_p |D_i(G)/D_{i+1}(G)| \leq \alpha^n$$

when n tends to infinity. This inequality with the previous estimation for a_n implies that $c_n = \frac{d^n}{n}(1 + o(1))$. As in the case b_n , we obtain that $d_n = \frac{d^{n+1}}{(d-1)n}(1 + o(1))$. \square

Now we are ready to prove Proposition 4.2.

Proof of Proposition 4.2. 1. Put $V_n = \{g_1^p[g_2, g_3] \in K/D_{n+1}(K) \mid g_i \in K/D_{n+1}(K)\}$. Then, by Lemma 4.3,

$$|V_n| \leq |K/D_n(K)|^3 = p^{3b_{n-1}} = p^{\frac{3d^n}{(d-1)(n-1)}(1+o(1))}.$$

On the other hand applying again Lemma 4.3, we obtain that

$$|ND_{n+1}(K)/D_{n+1}(K)| = p^{\frac{d^{n+1}}{(d-1)n}(1+o(1))}.$$

Comparing these two expressions we obtain 1.

2. In order to prove 2, it is enough to observe that for any $d > 1$, a free pro- p group of infinite rank is residually-free pro- p of rank d . Hence for any $d > 1$ there exists a homomorphism of K onto a free pro- p group of rank d such that the image of N is not trivial. Then we can apply 1. Since d is arbitrary, we obtain 2. \square

Remark 4.4. The first statement of Proposition 4.2 has the following interpretation. For any closed subset V of K define its Hausdorff dimension (see [1]):

$$\dim_H V = \liminf_{n \rightarrow \infty} \frac{\log |VD_n(K) : D_n(K)|}{\log |K : D_n(K)|}.$$

In the proof of Proposition 4.2 we have shown that $\dim_H N = 1$ for any non-trivial normal subgroup N of K and $\dim_H K^{\{x^p[y, z]\}} \leq \frac{3}{d}$.

5 Final remarks

5.1 The finite verbal width and the Restricted Burnside Problem

If H is a finitely generated pro- p group and $w = x^{p^n}$, then all known proofs that $w(H)$ is closed use the Zelmanov's solution of the Restricted Burnside Problem. For example, from Zelmanov's result it follows that w is a \mathcal{N}_p -word and then we can apply Theorem 3.1. In [17, page 53], Dan Segal suggested that it would be very interesting to prove that H^{p^n} is closed without appealing to Zelmanov's result. Using the ideas of the previous section we will show that it would give an alternative solution of the Restricted Burnside Problem.

Theorem 5.1. *Let F be a free group and $w \in F$. Put $t = w^p$. Then if $t(G)$ is closed for any finitely generated pro- p group G , then $w(G)$ is open for any finitely generated pro- p group G .*

Proof. Note that in the proof of Theorem 3.1 we have used the solution of the Restricted Burnside Problem (we needed H^{p^s} to be open). However, if we assume that w satisfies a stronger condition: $G/\overline{w(G)}$ is nilpotent for any finitely generated pro- p group G , then we obtain that w is of finite width without appealing to the solution of the Restricted Burnside Problem.

Let, now, H be a non abelian finitely generated free pro- p group and $\overline{w(H)}$ the closure of $w(H)$ in H . First we assume that $\overline{w(H)}$ is not open. Since H is a non-abelian free pro- p group, then $\overline{w(H)}$ is a free pro- p group of infinite rank. Now, using the argument of the proof of Theorem 4.1, we obtain that $t(H)$ is not closed.

Hence we can assume that $\overline{w(H)}$ is open. In particular, $H/\overline{w(H)}$ is nilpotent. Thus, from the first paragraph we obtain that $w(H)$ is also closed and so open. \square

5.2 Pronilpotent groups

In this subsection we show how our previous results on pro- p groups can be generalized on pronilpotent groups. The possibility of this generalization has been suggested to us by Dan Segal.

We say that w is a \mathcal{N} -word if for any finitely generated pronilpotent group H , $H/\overline{w(H)}$ is nilpotent-by-finite, where $\overline{w(H)}$ denotes the closure of $w(H)$ in H . The following characterization of \mathcal{N} -words is due to Dan Segal.

Theorem 5.2. *Let w be an element of a free group F . Then the following are equivalent*

1. w is a \mathcal{N} -word;
2. w is a \mathcal{N}_p -word for all primes p .

Proof. The implication $1 \Rightarrow 2$ is clear. Let us prove $2 \Rightarrow 1$.

Let d be a natural number, U a free group on d generators and T the maximal residually nilpotent quotient of $U/w(U)$. Since w is a \mathcal{N}_2 -word, the pro-2 completion T_2 of T is virtually nilpotent, and so of finite rank. Using [9, Lemma 9, page 386], we obtain that there exists a finite set π of primes such that T is embedded in $\prod_{p \in \pi} T_p$. Applying again that w is a \mathcal{N}_p -word for all primes $p \in \pi$, we conclude that T is virtually nilpotent.

Let now H be a finitely generated pronilpotent group and $\bar{H} = H/\overline{w(H)}$. Put $d = d(H)$. Let T_1 be a dense d -generated subgroup of \bar{H} . Then, T_1 is a quotient of T . Hence T_1 and \bar{H} are virtually nilpotent. \square

Now, we are ready to prove the main theorem of this subsection.

Theorem 5.3. *Let $1 \neq w$ be an element of a free group F . Then the following two statements are equivalent:*

1. $w(H)$ is closed for every finitely generated pronilpotent group H ;
2. $w \notin \bigcup_{p \text{ prime}} (F')^p F''$.

Proof. The implication $1 \Rightarrow 2$ follows from Theorem 1.1. Now assume that $w \notin \bigcup_p (F')^p F''$. Then by Theorems 1.2 and 5.2, w is a \mathcal{N} -word. We will argue as in the proof of Theorem 3.1.

Let G be a finitely generated pronilpotent group, $d = d(G)$ and H a free pronilpotent group on generators x_1, \dots, x_d, z . Since w is a \mathcal{N} -word, $\gamma_n(H^t) \leq$

$\overline{w(H)}$ for some n and t . We write G as $G = G_1 \times G_2$, where G_1 is the product of all the Sylow pro- p subgroups with $p \in \pi(t)$ and G_2 is the product of the rest of the Sylow pro- p subgroups. In the same way we write $H = H_1 \times H_2$. Note that $w(G)$ is closed if and only if $w(G_1)$ and $w(G_2)$ are closed. Since $w(G_1)$ is closed by Theorem 1.1, it is enough to prove only that $\overline{w(G_2)}$ is closed.

By our construction of H_2 we have that $\gamma_n(H_2) \leq \overline{w(H_2)}$. Repeating the argument from the proof of Theorem 3.1, we obtain that any element of $\gamma_n(G_2)$ is a product of a bounded number of w -values in G_2 . By Roman'kov's result [15], $w(G_2/\gamma_n(G_2))$ is closed, whence we conclude that $w(G_2)$ is also closed. \square

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