Evaluating $\zeta(2)$

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I list several proofs of the celebrated identity:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{1}$$

As it is clear that

$$\frac{3}{4}\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{m=1}^{\infty} \frac{1}{(2m)^2} = \sum_{r=0}^{\infty} \frac{1}{(2r+1)^2},$$

(1) is equivalent to

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} = \frac{\pi^2}{8}. \tag{2}$$

Many of the proofs establish this latter identity first.

None of these proofs is original; most are well known, but some are not as familiar as they might be. I shall try to assign credit the best I can, and I would be grateful to anyone who could shed light on the origin of any of these methods. I would like to thank Tony Lezard, José Carlos Santos and Ralph Krause, who spotted errors in earlier versions, and Richard Carr for pointing out an egregious solecism.

Proof 1: Note that

$$\frac{1}{n^2} = \int_0^1 \int_0^1 x^{n-1} y^{n-1} \, dx \, dy$$

and by the monotone convergence theorem we get

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2} &= \int_0^1 \int_0^1 \left(\sum_{n=1}^{\infty} (xy)^{n-1} \right) dx\, dy \\ &= \int_0^1 \int_0^1 \frac{dx\, dy}{1 - xy}.\end{aligned}$$

We change variables in this by putting $(u, v) = ((x+y)/2, (y-x)/2)$, so that $(x, y) = (u - v, u + v)$. Hence

$$\zeta(2) = 2 \iint_S \frac{du\, dv}{1 - u^2 + v^2}$$

where S is the square with vertices $(0, 0)$, $(1/2, -1/2)$, $(1, 0)$ and $(1/2, 1/2)$. Exploiting the symmetry of the square we get

$$\begin{aligned}\zeta(2) &= 4 \int_0^{1/2} \int_0^u \frac{dv\, du}{1 - u^2 + v^2} + 4 \int_{1/2}^1 \int_0^{1-u} \frac{dv\, du}{1 - u^2 + v^2} \\ &= 4 \int_0^{1/2} \frac{1}{\sqrt{1 - u^2}} \tan^{-1} \left(\frac{u}{\sqrt{1 - u^2}} \right) du \\ &\quad +4 \int_{1/2}^1 \frac{1}{\sqrt{1 - u^2}} \tan^{-1} \left(\frac{1 - u}{\sqrt{1 - u^2}} \right) du.\end{aligned}$$

Now $\tan^{-1}(u/(\sqrt{1 - u^2})) = \sin^{-1} u$, and if $\theta = \tan^{-1}((1 - u)/(\sqrt{1 - u^2}))$ then $\tan^2 \theta = (1 - u)/(1 + u)$ and $\sec^2 \theta = 2/(1 + u)$. It follows that $u = 2 \cos^2 \theta - 1 = \cos 2\theta$ and so $\theta = \frac{1}{2} \cos^{-1} u = \frac{\pi}{4} - \frac{1}{2} \sin^{-1} u$. Hence

$$\begin{aligned}\zeta(2) &= 4 \int_0^{1/2} \frac{\sin^{-1} u}{\sqrt{1 - u^2}} du + 4 \int_{1/2}^1 \frac{1}{\sqrt{1 - u^2}} \left(\frac{\pi}{4} - \frac{\sin^{-1} u}{2} \right) du \\ &= \left[2(\sin^{-1} u)^2 \right]_0^{1/2} + \left[\pi \sin^{-1} u - (\sin^{-1} u)^2 \right]_{1/2}^1 \\ &= \frac{\pi^2}{18} + \frac{\pi^2}{2} - \frac{\pi^2}{4} - \frac{\pi^2}{6} + \frac{\pi^2}{36} \\ &= \frac{\pi^2}{6}\end{aligned}$$

as required.

This is taken from an article in the *Mathematical Intelligencer* by Apostol in 1983.

Proof 2: We start in a similar fashion to Proof 1, but we use (2). We get

$$\sum_{r=0}^{\infty} \frac{1}{(2r + 1)^2} = \int_0^1 \int_0^1 \frac{dx\, dy}{1 - x^2 y^2}.$$

We make the substitution

$$(u, v) = \left(\tan^{-1} x \sqrt{\frac{1 - y^2}{1 - x^2}}, \tan^{-1} y \sqrt{\frac{1 - x^2}{1 - y^2}} \right)$$

so that

$$(x, y) = \left(\frac{\sin u}{\cos v}, \frac{\sin v}{\cos u} \right).$$

The Jacobian matrix is

$$\begin{aligned} \frac{\partial(x, y)}{\partial(u, v)} &= \begin{vmatrix} \cos u/ \cos v & \sin u \sin v/ \cos^2 v \\ \sin u \sin v/ \cos^2 u & \cos v/ \cos u \end{vmatrix} \\ &= 1 - \frac{\sin^2 u \sin^2 v}{\cos^2 u \cos^2 v} \\ &= 1 - x^2 y^2. \end{aligned}$$

Hence

$$\frac{3}{4}\zeta(2) = \iint_A du\, dv$$

where

$$A = \{(u, v) : u > 0, v > 0, u + v < \pi/2\}$$

has area $\pi^2/8$, and again we get $\zeta(2) = \pi^2/6$.

This is due to Calabi, Beukers and Kock.

Proof 3: We use the power series for the inverse sine function:

$$\sin^{-1} x = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots (2n)} \frac{x^{2n+1}}{2n + 1}$$

valid for $|x| \leq 1$. Putting $x = \sin t$ we get

$$t = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n - 1)}{2 \cdot 4 \cdots 2n} \frac{\sin^{2n+1} t}{2n + 1}$$

for $|t| \leq \frac{\pi}{2}$. Integrating from 0 to $\frac{\pi}{2}$ and using the formula

$$\int_0^{\pi/2} \sin^{2n+1} x\, dx = \frac{2 \cdot 4 \cdots (2n)}{3 \cdot 5 \cdots (2n + 1)}$$

gives us

$$\frac{\pi^2}{8} = \int_0^{\pi/2} t\, dt = \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2}$$

which is (2).

This comes from a note by Boo Rim Choe in the *American Mathematical Monthly* in 1987.

Proof 4: We use the L^2-completeness of the trigonometric functions. Let $e_n(x) = \exp(2\pi i n x)$ where $n \in \mathbf{Z}$. The e_n form a complete orthonormal set in $L^2[\,0, 1\,]$. If we denote the inner product in $L^2[\,0, 1\,]$ by $\langle\ ,\ \rangle$, then Parseval's formula states that

$$\langle f, f\rangle = \sum_{n=-\infty}^{\infty} |\langle f, e_n\rangle|^2$$

for all $f \in L^2[\,0, 1\,]$. We apply this to $f(x) = x$. We easily compute $\langle f, f\rangle = \frac{1}{3}$, $\langle f, e_0\rangle = \frac{1}{2}$ and $\langle f, e_n\rangle = \frac{1}{2\pi i n}$ for $n \neq 0$. Hence Parseval gives us

$$\frac{1}{3} = \frac{1}{4} + \sum_{n\in\mathbf{Z}, n\neq 0} \frac{1}{4\pi^2 n^2}$$

and so $\zeta(2) = \pi^2/6$.

Alternatively we can apply Parseval to $g = \chi_{[0,1/2]}$. We get $\langle g, g\rangle = \frac{1}{2}$, $\langle g, e_0\rangle = \frac{1}{2}$ and $\langle g, e_n\rangle = ((-1)^n - 1)/2\pi i n$ for $n \neq 0$. Hence Parseval gives us

$$\frac{1}{2} = \frac{1}{4} + 2\sum_{r=0}^{\infty} \frac{1}{\pi^2(2r+1)^2}$$

and using (2) we again get $\zeta(2) = \pi^2/6$.

This is a textbook proof, found in many books on Fourier analysis.

Proof 5: We use the fact that if f is continuous, of bounded variation on $[\,0, 1\,]$ and $f(0) = f(1)$, then the Fourier series of f converges to f pointwise. Applying this to $f(x) = x(1-x)$ gives

$$x(1-x) = \frac{1}{6} - \sum_{n=1}^{\infty} \frac{\cos 2\pi n x}{\pi^2 n^2},$$

and putting $x = 0$ we get $\zeta(2) = \pi^2/6$. Alternatively putting $x = 1/2$ gives

$$\frac{\pi^2}{12} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

which again is equivalent to $\zeta(2) = \pi^2/6$.

Another textbook proof.

Proof 6: Consider the series

$$f(t) = \sum_{n=1}^{\infty} \frac{\cos nt}{n^2}.$$

This is uniformly convergent on the real line. Now if $\epsilon > 0$, then for $t \in [\,\epsilon, 2\pi - \epsilon\,]$ we have

$$\begin{aligned}
\sum_{n=1}^{N} \sin nt &= \sum_{n=1}^{N} \frac{e^{int} - e^{-int}}{2i} \\
&= \frac{e^{it} - e^{i(N+1)t}}{2i(1 - e^{it})} - \frac{e^{-it} - e^{-i(N+1)t}}{2i(1 - e^{-it})} \\
&= \frac{e^{it} - e^{i(N+1)t}}{2i(1 - e^{it})} + \frac{1 - e^{-iNt}}{2i(1 - e^{it})}
\end{aligned}$$

and so this sum is bounded above in absolute value by

$$\frac{2}{|1 - e^{it}|} = \frac{1}{\sin t/2}.$$

Hence these sums are uniformly bounded on $[\,\epsilon, 2\pi - \epsilon\,]$ and by Dirichlet's test the sum

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n}$$

is uniformly convergent on $[\,\epsilon, 2\pi - \epsilon\,]$. It follows that for $t \in (0, 2\pi)$

$$\begin{aligned}
f'(t) &= -\sum_{n=1}^{\infty} \frac{\sin nt}{n} \\
&= -\mathrm{Im}\left(\sum_{n=1}^{\infty} \frac{e^{int}}{n} \right) \\
&= \mathrm{Im}(\log(1 - e^{it})) \\
&= \arg(1 - e^{it}) \\
&= \frac{t - \pi}{2}.
\end{aligned}$$

By the fundamental theorem of calculus we have

$$f(\pi) - f(0) = \int_0^{\pi} \frac{t - \pi}{2} dt = -\frac{\pi^2}{4}.$$

But $f(0) = \zeta(2)$ and $f(\pi) = \sum_{n=1}^{\infty} (-1)^n/n^2 = -\zeta(2)/2$. Hence $\zeta(2) = \pi^2/6$.

Alternatively we can put

$$D(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2},$$

the *dilogarithm* function. This is uniformly convergent on the closed unit disc, and satisfies $D'(z) = -(\log(1-z))/z$ on the open unit disc. Note that $f(t) = \operatorname{Re} D(e^{2\pi i t})$. We may now use arguments from complex variable theory to justify the above formula for $f'(t)$.

This is just the previous proof with the Fourier theory eliminated.

Proof 7: We use the infinite product

$$\sin \pi x = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$$

for the sine function. Comparing coefficients of x^3 in the MacLaurin series of sides immediately gives $\zeta(2) = \pi^2/6$. An essentially equivalent proof comes from considering the coefficient of x in the formula

$$\pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2}.$$

The original proof of Euler!

Proof 8: We use the calculus of residues. Let $f(z) = \pi z^{-2} \cot \pi z$. Then f has poles at precisely the integers; the pole at zero has residue $-\pi^2/3$, and that at a non-zero integer n has residue $1/n^2$. Let N be a natural number and let C_N be the square contour with vertices $(\pm 1 \pm i)(N + 1/2)$. By the calculus of residues

$$-\frac{\pi^2}{3} + 2\sum_{n=1}^{N} \frac{1}{n^2} = \frac{1}{2\pi i} \int_{C_N} f(z)\, dz = I_N$$

say. Now if $\pi z = x + iy$ a straightforward calculation yields

$$|\cot \pi z|^2 = \frac{\cos^2 x + \sinh^2 y}{\sin^2 x + \sinh^2 y}.$$

It follows that if z lies on the vertical edges of C_n then

$$|\cot \pi z|^2 = \frac{\sinh^2 y}{1 + \sinh^2 y} < 1$$

and if z lies on the horizontal edges of C_n

$$|\cot \pi z|^2 \leq \frac{1+\sinh^2 \pi(N+1/2)}{\sinh^2 \pi(N+1/2)} = \coth^2 \pi(N+1/2) \leq \coth^2 \pi/2.$$

Hence $|\cot \pi z| \leq K = \coth \frac{\pi}{2}$ on C_N, and so $|f(z)| \leq \pi K/(N+1/2)^2$ on C_N. This estimate shows that

$$|I_n| \leq \frac{1}{2\pi} \frac{\pi K}{(N+1/2)^2} 8(N+1/2)$$

and so $I_N \to 0$ as $N \to \infty$. Again we get $\zeta(2) = \pi^2/6$.

Another textbook proof, found in many books on complex analysis.

Proof 9: We first note that if $0 < x < \frac{\pi}{2}$ then $\sin x < x < \tan x$ and so $\cot^2 x < x^{-2} < 1 + \cot^2 x$. If n and N are natural numbers with $1 \leq n \leq N$ this implies that

$$\cot^2 \frac{n\pi}{(2N+1)} < \frac{(2N+1)^2}{n^2\pi^2} < 1 + \cot^2 \frac{n\pi}{(2N+1)}$$

and so

$$\begin{aligned} & \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2 \frac{n\pi}{(2N+1)} \\ < & \sum_{n=1}^{N} \frac{1}{n^2} \\ < & \frac{N\pi^2}{(2N+1)^2} + \frac{\pi^2}{(2N+1)^2} \sum_{n=1}^{N} \cot^2 \frac{n\pi}{(2N+1)}. \end{aligned}$$

If

$$A_N = \sum_{n=1}^{N} \cot^2 \frac{n\pi}{(2N+1)}$$

it suffices to show that $\lim_{N\to\infty} A_N/N^2 = \frac{2}{3}$.

If $1 \leq n \leq N$ and $\theta = n\pi/(2N+1)$, then $\sin(2N+1)\theta = 0$ but $\sin\theta \neq 0$. Now $\sin(2N+1)\theta$ is the imaginary part of $(\cos\theta + i\sin\theta)^{2N+1}$, and so

$$\begin{aligned} \frac{\sin(2N+1)\theta}{\sin^{2N+1}\theta} &= \frac{1}{\sin^{2N+1}\theta} \sum_{k=0}^{N} (-1)^k \binom{2N+1}{2N-2k} \cos^{2(N-k)}\theta \sin^{2k+1}\theta \\ &= \sum_{k=0}^{N} (-1)^k \binom{2N+1}{2N-2k} \cot^{2(N-k)}\theta \\ &= f(\cot^2\theta) \end{aligned}$$

say, where $f(x) = (2N+1)x^N - \binom{2N+1}{3}x^{N-1} + \cdots$. Hence the roots of $f(x) = 0$ are $\cot^2(n\pi/(2N+1))$ where $1 \le n \le N$ and so $A_N = N(2N-1)/3$. Thus $A_N/N^2 \to \frac{2}{3}$, as required.

This is an exercise in Apostol's *Mathematical Analysis* (Addison-Wesley, 1974).

Proof 10: Given an odd integer $n = 2m+1$ it is well known that $\sin nx = F_n(\sin x)$ where F_n is a polynomial of degree n. Since the zeros of $F_n(y)$ are the values $\sin(j\pi/n)$ $(-m \le j \le m)$ and $\lim_{y\to 0}(F_n(y)/y) = n$ then

$$F_n(y) = ny \prod_{j=1}^{m} \left(1 - \frac{y^2}{\sin^2(j\pi/n)}\right)$$

and so

$$\sin nx = n \sin x \prod_{j=1}^{m} \left(1 - \frac{\sin^2 x}{\sin^2(j\pi/n)}\right).$$

Comparing the coefficients of x^3 in the MacLaurin expansion of both sides gives

$$-\frac{n^3}{6} = -\frac{n}{6} - n \sum_{j=1}^{m} \frac{1}{\sin^2(j\pi/n)}$$

and so

$$\frac{1}{6} - \sum_{j=1}^{m} \frac{1}{n^2 \sin^2(j\pi/n)} = \frac{1}{6n^2}.$$

Fix an integer M and let $m > M$. Then

$$\frac{1}{6} - \sum_{j=1}^{M} \frac{1}{n^2 \sin^2(j\pi/n)} = \frac{1}{6n^2} + \sum_{j=M+1}^{m} \frac{1}{n^2 \sin^2(j\pi/n)}$$

and using the inequality $\sin x > \frac{2}{\pi}x$ for $0 < x < \frac{\pi}{2}$, we get

$$0 < \frac{1}{6} - \sum_{j=1}^{M} \frac{1}{n^2 \sin^2(j\pi/n)} < \frac{1}{6n^2} + \sum_{j=M+1}^{m} \frac{1}{4j^2}.$$

Letting m tend to infinity now gives

$$0 \le \frac{1}{6} - \sum_{j=1}^{M} \frac{1}{\pi^2 j^2} \le \sum_{j=M+1}^{\infty} \frac{1}{4j^2}.$$

Hence

$$\sum_{j=1}^{\infty} \frac{1}{\pi^2 j^2} = \frac{1}{6}.$$

This comes from a note by Kortram in *Mathematics Magazine* in 1996.

Proof 11: Consider the integrals

$$I_n = \int_0^{\pi/2} \cos^{2n} x \, dx \qquad \text{and} \qquad J_n = \int_0^{\pi/2} x^2 \cos^{2n} x \, dx.$$

By a well-known reduction formula

$$I_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \frac{\pi}{2} = \frac{(2n)!}{4^n n!^2} \frac{\pi}{2}.$$

If $n > 0$ then integration by parts gives

$$\begin{aligned} I_n &= \left[x \cos^{2n} x\right]_0^{\pi/2} + 2n \int_0^{\pi/2} x \sin x \cos^{2n-1} x \, dx \\ &= n \left[x^2 \sin x \cos^{2n-1} x\right]_0^{\pi/2} \\ &\quad - n \int_0^{\pi/2} x^2 (\cos^{2n} x - (2n-1) \sin^2 x \cos^{2n-2} x) \, dx \\ &= n(2n-1) J_{n-1} - 2n^2 J_n. \end{aligned}$$

Hence

$$\frac{(2n)!}{4^n n!^2} \frac{\pi}{2} = n(2n-1) J_{n-1} - 2n^2 J_n$$

and so

$$\frac{\pi}{4n^2} = \frac{4^{n-1}(n-1)!^2}{(2n-2)!} J_{n-1} - \frac{4^n n!^2}{(2n)!} J_n.$$

Adding this up from $n = 1$ to N gives

$$\frac{\pi}{4} \sum_{n=1}^{N} \frac{1}{n^2} = J_0 - \frac{4^N N!^2}{(2N)!} J_N.$$

Since $J_0 = \pi^3/24$ it suffices to show that $\lim_{N\to\infty} 4^N N!^2 J_N/(2N)! = 0$. But the inequality $x < \frac{\pi}{2} \sin x$ for $0 < x < \frac{\pi}{2}$ gives

$$J_N < \frac{\pi^2}{4} \int_0^{\pi_2} \sin^2 x \cos^{2N} x \, dx = \frac{\pi^2}{4} (I_N - I_{N+1}) = \frac{\pi^2 I_N}{8(N+1)}$$

and so

$$0 < \frac{4^N N!}{(2N)!} J_N < \frac{\pi^3}{16(N+1)}.$$

This completes the proof.

This proof is due to Matsuoka (*American Mathematical Monthly*, 1961).

Proof 12: Consider the well-known identity for the Fejér kernel:

$$\left(\frac{\sin nx/2}{\sin x/2}\right)^2 = \sum_{k=-n}^{n} (n - |k|)e^{ikx} = n + 2\sum_{k=1}^{n}(n-k)\cos kx.$$

Hence

$$\begin{aligned}
\int_0^\pi x \left(\frac{\sin nx/2}{\sin x/2}\right)^2 dx &= \frac{n\pi^2}{2} + 2\sum_{k=1}^{n}(n-k)\int_0^\pi x\cos kx\, dx \\
&= \frac{n\pi^2}{2} - 2\sum_{k=1}^{n}(n-k)\frac{1-(-1)^k}{k^2} \\
&= \frac{n\pi^2}{2} - 4n\sum_{1\le k\le n, 2\nmid k}\frac{1}{k^2} + 4\sum_{1\le k\le n, 2\nmid k}\frac{1}{k}
\end{aligned}$$

If we let $n = 2N$ with N an integer then

$$\int_0^\pi \frac{x}{8N}\left(\frac{\sin Nx}{\sin x/2}\right)^2 dx = \frac{\pi^2}{8} - \sum_{r=0}^{N-1}\frac{1}{(2r+1)^2} + O\left(\frac{\log N}{N}\right).$$

But since $\sin\frac{x}{2} > \frac{x}{\pi}$ for $0 < x < \pi$ then

$$\begin{aligned}
\int_0^\pi \frac{x}{8N}\left(\frac{\sin Nx}{\sin x/2}\right)^2 dx &< \frac{\pi^2}{8N}\int_0^\pi \sin^2 Nx\,\frac{dx}{x} \\
&= \frac{\pi^2}{8N}\int_0^{N\pi}\sin^2 y\,\frac{dy}{y} = O\left(\frac{\log N}{N}\right).
\end{aligned}$$

Taking limits as $N \to \infty$ gives

$$\frac{\pi^2}{8} = \sum_{r=0}^{\infty}\frac{1}{(2r+1)^2}.$$

This proof is due to Stark (*American Mathematical Monthly*, 1969).

Proof 13: We carefully square Gregory's formula

$$\frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}.$$

We can rewrite this as $\lim_{N\to\infty} a_N = \frac{\pi}{2}$ where

$$a_N = \sum_{n=-N}^{N} \frac{(-1)^n}{2n+1}.$$

Let

$$b_N = \sum_{n=-N}^{N} \frac{1}{(2n+1)^2}.$$

By (2) it suffices to show that $\lim_{N\to\infty} b_N = \pi^2/4$, so we shall show that $\lim_{N\to\infty} (a_N^2 - b_N) = 0$.

If $n \neq m$ then

$$\frac{1}{(2n+1)(2m+1)} = \frac{1}{2(m-n)}\left(\frac{1}{2n+1} - \frac{1}{2m+1}\right)$$

and so

$$\begin{aligned} a_N^2 - b_N &= \sum_{n=-N}^{N} {\sum_{m=-N}^{N}}' \frac{(-1)^{m+n}}{2(m-n)}\left(\frac{1}{2n+1} - \frac{1}{2m+1}\right) \\ &= \sum_{n=-N}^{N} {\sum_{m=-N}^{N}}' \frac{(-1)^{m+n}}{(2n+1)(m-n)} \\ &= \sum_{n=-N}^{N} \frac{(-1)^n c_{n,N}}{2n+1} \end{aligned}$$

where the dash on the summations means that terms with zero denominators are omitted, and

$$c_{n,N} = {\sum_{m=-N}^{N}}' \frac{(-1)^m}{(m-n)}.$$

It is easy to see that $c_{-n,N} = -c_{n,N}$ and so $c_{0,N} = 0$. If $n > 0$ then

$$c_{n,N} = (-1)^{n+1} \sum_{j=N-n+1}^{N+n} \frac{(-1)^j}{j}$$

and so $|c_{n,N}| \leq 1/(N-n+1)$ as the magnitude of this alternating sum is not more than that of its first term. Thus

$$|a_N^2 - b_N| \leq \sum_{n=1}^{N} \left(\frac{1}{(2n-1)(N-n+1)} + \frac{1}{(2n+1)(N-n+1)}\right)$$

$$\begin{aligned}
&= \sum_{n=1}^{N} \frac{1}{2N+1}\left(\frac{2}{2n-1}+\frac{1}{N-n+1}\right) \\
&\quad + \sum_{n=1}^{N} \frac{1}{2N+3}\left(\frac{2}{2n+1}+\frac{1}{N-n+1}\right) \\
&\leq \frac{1}{2N+1}(2+4\log(2N+1)+2+2\log(N+1))
\end{aligned}$$

and so $a_N^2 - b_N \to 0$ as $N \to \infty$ as required.

This is an exercise in Borwein & Borwein's *Pi and the AGM* (Wiley, 1987).

Proof 14: This depends on the formula for the number of representations of a positive integer as a sum of four squares. Let $r(n)$ be the number of quadruples (x, y, z, t) of integers such that $n = x^2 + y^2 + z^2 + t^2$. Trivially $r(0) = 1$ and it is well known that

$$r(n) = 8 \sum_{m|n, 4\nmid m} m$$

for $n > 0$. Let $R(N) = \sum_{n=0}^{N} r(n)$. It is easy to see that $R(N)$ is asymptotic to the volume of the 4-dimensional ball of radius $\sqrt{N}$, i.e., $R(N) \sim \frac{\pi^2}{2}N^2$. But

$$R(N) = 1 + 8\sum_{n=1}^{N}\sum_{m|n,4\nmid m} m = 1 + 8 \sum_{m\leq N, 4\nmid m} m \left\lfloor \frac{N}{m} \right\rfloor = 1 + 8(\theta(N) - 4\theta(N/4))$$

where

$$\theta(x) = \sum_{m\leq x} m \left\lfloor \frac{x}{m} \right\rfloor .$$

But

$$\begin{aligned}
\theta(x) &= \sum_{mr\leq x} m \\
&= \sum_{r\leq x} \sum_{m=1}^{\lfloor x/r \rfloor} m \\
&= \frac{1}{2}\sum_{r\leq x}\left(\left\lfloor \frac{x}{r}\right\rfloor^2 + \left\lfloor \frac{x}{r}\right\rfloor\right) \\
&= \frac{1}{2}\sum_{r\leq x}\left(\frac{x^2}{r^2} + O\left(\frac{x}{r}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{x^2}{2}(\zeta(2) + O(1/x)) + O(x \log x) \\
&= \frac{\zeta(2)x^2}{2} + O(x \log x)
\end{aligned}
$$

as $x \to \infty$. Hence

$$R(N) \sim \frac{\pi^2}{2} N^2 \sim 4\zeta(2) \left(N^2 - \frac{N^2}{4} \right)$$

and so $\zeta(2) = \pi^2/6$.

This is an exercise in Hua's textbook on number theory.