ADDITIVE PROPERTIES OF SEQUENCES OF PSEUDO s-TH POWERS

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Abstract. In this paper, we study (random) sequences of pseudo s-th powers, as introduced by Erdős and Rényi in 1960. In 1975, Goguel proved that such a sequence is almost surely not an asymptotic basis of order s. Our first result asserts that it is however almost surely a basis of order $s+\epsilon$ for any $\epsilon > 0$. We then study the s-fold sumset $sA = A + \cdots + A$ (s times) and in particular the minimal size of an additive complement, that is a set B such that $sA+B$ contains all large enough integers. With respect to this problem, we prove quite precise theorems which are tantamount to asserting that a threshold phenomenon occurs.

1. Introduction

In their seminal paper of 1960, Erdős and Rényi [7] proposed a probabilistic model for sequences A growing like the s-th powers. Explicitly, they built a probability space $(\mathcal{U}, \mathcal{T}, \mathbb{P})$ and a sequence of independent Bernoulli random variables $(\xi_n)_{n\in\mathbb{N}}$ with values in $\{0,1\}$ such that

$$\mathbb{P}(\xi_n = 1) = \frac{1}{s} n^{-1+1/s} \quad \text{and} \quad \mathbb{P}(\xi_n = 0) = 1 - \frac{1}{s} n^{-1+1/s}.$$

To any $u \in \mathcal{U}$, they associate the sequence of integers $A = A_u$ such that $n \in A_u$ if and only if $\xi_n(u) = 1$. In other words, the events $\{n \in A\}$ are independent and the probability that n is in A is equal to $\mathbb{P}(n \in A) = n^{-1+1/s}/s$. The counting function of these random sequences A satisfies almost surely the asymptotic relation $|A \cap [1,x]| \sim x^{1/s}$ as x tends to infinity [7] (see also [10]), whence the terminology *pseudo s-th powers*.

In 1975, Goguel [8] proved that, almost surely, the s-fold sumset

$$sA = \{a_1 + \cdots + a_s \text{ with } a_i \in A\}$$

has density $1 - e^{-\lambda_s}$ where

$$\lambda_s = \frac{\Gamma^s(1/s)}{s^s \; s!}$$

(a quantity appearing everywhere in the present study) and thus, almost surely, that A is not an asymptotic basis of order s (from now on, the word 'asymptotic' will be omitted since there is no ambiguity). Indeed it has been proved recently

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(see [4]) that the sequence $(b_n)_{n\in\mathbb{N}}$ of ordered elements in sA has, almost surely, infinitely many gaps of logarithm size that is,

$$\limsup_{n\to+\infty} \frac{b_{n+1}-b_n}{\log b_n} = \frac{1}{\lambda_s}. \tag{1}$$

In contrast to the result of Goguel quoted above, Deshouillers and Iosifescu, as a by-product of their study on the probability that an integer is not a sum of $s+1$ s-th powers, proved in [5], however, that almost surely a sequence of pseudo s-th powers is a basis of order $s+1$. Here we will make more precise this threshold-type phenomenon by using the concept of a basis of order $s+\epsilon$ introduced in [3]: We say that A is a *basis of order* $s+\epsilon$ if any large enough positive integer n can be written in the form

$$n = a_1 + \cdots + a_{s+1}, \qquad \text{with } a_i \in A, \qquad a_{s+1} \le n^{\epsilon}.$$

Our first result is the fact that almost surely a sequence of pseudo s-th powers is a basis of order $s+\epsilon$ for any $\epsilon > 0$. Indeed we prove this result in the following stronger form.

Theorem 1. *Let $s \ge 2$ be an integer and $c > (\lambda_s(1-2\lambda_s))^{-1}$. Almost surely, a sequence of pseudo s-powers A has the following property: any large enough integer n can be written in the form*

$$n = a_1 + \cdots + a_{s+1}, \qquad \textit{with } a_i \in A, \qquad a_{s+1} < (c\log n)^s.$$

We have some reason to believe that the above statement is no longer valid if $c < \lambda_s^{-1}$; this point will be discussed at the end of Section 3. Simply notice now that $\lambda_s < 1/2$ for $s \ge 2$.

A second aim of the paper is the study of how fast an additive complement sequence of sA must grow. We first prove the following theorem.

Theorem 2. *Let s be an integer $s \ge 2$. Let B be a fixed sequence satisfying*

$$\liminf_{n\to\infty} \frac{B(n)}{\log n} > \lambda_s^{-1}.$$

Then a sequence of pseudo s-powers A has, almost surely, the following property: any large enough integer n can be written in the form

$$n = a_1 + \cdots + a_s + b, \qquad \textit{with distinct } a_i \in A \textit{ and with } b \in B.$$

We then prove that Theorem 2 is sharp in the sense that the constant λ_s^{-1} intervening in this result cannot be substituted by a smaller constant.

Theorem 3. *Let s be an integer $s \ge 2$. Let B be a fixed sequence satisfying*

$$\liminf_{n\to\infty} \frac{B(n)}{\log n} < \lambda_s^{-1}.$$

Then a sequence of pseudo s-powers A has, almost surely, the following property: there are infinitely many integers n that cannot be written in the form

$$n = a_1 + \cdots + a_s + b, \qquad \textit{with distinct } a_i \in A \textit{ and with } b \in B.$$

In view of Theorems 2 and 3 it is a natural question to ask for the behaviour of those sequences B with

$$\liminf_{n\to\infty} \frac{B(n)}{\log n} = \lambda_s^{-1}. \tag{2}$$

In our final section, we will show that there are sequences satisfying (2) and the conclusion of Theorem 2 while there are other sequences that satisfy (2) and the conclusion of Theorem 3.

The paper is organized as follows. Section 2 is composed of several lemmas which will be useful in the proofs of the three theorems. Section 3 contains the proof of Theorem 1 which will be presented with precise estimates. Section 4 contains the proof of Theorem 2 and Section 5 that of Theorem 3. Finally, Section 6 contains the discussion about sequences at the threshold (that is, satisfying (2)). In order to avoid overcomplicated and lengthy formulations, these proofs, which rely on analogous but simpler principles and computation, will be written down with slightly less details.

2. Preparatory lemmas and prerequisite

For our purpose, we shall need a few elementary or more or less classical results. The first one is technical and we shall use the standard Vinogradov $\ll$ notation for "less than a constant time"; in the present paper the constants will always depend on the parameter $s \geq 2$, but only on it. We will not recall this dependency in the $\ll$ notation.

Lemma 1. *Let s and t be two integers such that $s \geq 2$ and $1 \leq t \leq s-1$. We have*

(i) *for $z \geq 1$,*
$$\sum_{\substack{1 \leq x_1, \dots, x_t \\ x_1 + \cdots + x_t = z}} (x_1 \cdots x_t)^{-1+1/s} \ll z^{-1+t/s},$$

(ii) *for $z \geq 2$,*
$$\sum_{\substack{1 \leq x_1, \dots, x_t \\ x_1 + \cdots + x_t < z}} (x_1 \cdots x_t)^{-1+1/s} \big(z - (x_1 + \cdots + x_t) \big)^{-2t/s} \ll z^{-1/s} \log z,$$

(iii) *if g is a positive function satisfying $g(z) = o(z)$ as z tends to infinity, then*
$$\lim_{\substack{z \to +\infty \\ z \in \mathbb{N}}} \sum_{\substack{g(z) \leq x_s < \cdots < x_1 \\ x_1 + \cdots + x_s = z}} (x_1 \cdots x_s)^{-1+1/s} = s^s \lambda_s.$$

Proof. Points (i) and (ii) in this lemma appear as Lemma 1 of [4], taking $a_1 = \cdots = a_s = 1$. The special case $g(z) = 1$ of (iii) appears there also. To extend it to our setting, it will be enough to prove that
$$\sum_{\substack{1 \leq x_s \leq g(z) \\ 1 \leq x_{s-1} < \cdots < x_1 \\ x_1 + \cdots + x_s = z}} (x_1 \cdots x_s)^{-1+1/s} = o(1).$$
To see this we use (i) with $t = s-1$ and bound this sum as
$$\begin{aligned}
\sum_{\substack{1 \leq x_s \leq g(z) \\ 1 \leq x_{s-1} < \cdots < x_1 \\ x_1 + \cdots + x_s = z}} (x_1 \cdots x_s)^{-1+1/s} &\leq \sum_{1 \leq x_s < g(z)} x_s^{-1+1/s} \sum_{\substack{1 \leq x_{s-1} < \cdots < x_1 \\ x_1 + \cdots + x_{s-1} = z - x_s}} (x_1 \cdots x_{s-1})^{-1+1/s} \\
&\ll \sum_{1 \leq x_s < g(z)} x_s^{-1+1/s} (z - x_s)^{-1/s} \\
&\ll (z - g(z))^{-1/s} \sum_{1 \leq x_s < g(z)} x_s^{-1+1/s} \\
&\ll \left(\frac{g(z)}{z} \right)^{1/s} \\
&= o(1),
\end{aligned}$$
as needed. $\square$

Here are now a few more or less classical tools from probability theory. The first basic tool is Chebychev's inequality in the following form, suitable for our purpose:

$$\mathbb{P}\left(X < \frac{\mathbb{E}[X]}{2}\right) \leq \frac{4\mathbb{V}(X)}{\mathbb{E}[X]^2}. \tag{3}$$

Here and everywhere in this paper, the symbols $\mathbb{P}, \mathbb{E}$ and $\mathbb{V}$ denote respectively the probability, the mathematical expectation and the variance.

The Borel-Cantelli Lemma is another basic and well known tool in probability (see for instance Lemma 8.6.1 in [1]). We recall it here for the sake of completeness.

Theorem 4 (Borel-Cantelli Lemma). *Let $(F_i)_{i\in\mathbb{N}}$ be a sequence of events. If $\sum_{i=1}^{+\infty} \mathbb{P}(F_i) < +\infty$ then,*

with probability 1, only finitely many of the events F_i occur.

Next, we will need two correlation inequalities due to Janson [9] (see also [2]) which are known as "Janson's correlation inequalities". Up to the ordering of the elements, this is Theorem 8.1.1 in [1].

We shall use the following notation : if Ω is a set, then for any two subsets ω, ω' of Ω, the notation $\omega \sim \omega'$ means that $\omega \neq \omega'$ and $\omega \cap \omega' \neq \emptyset$. Moreover, we use the standard notation E^c for the complementary event of an event E.

Theorem 5 (Janson's inequalities). *Let $(E_\omega)_{\omega\in\Omega}$ be a finite collection of events indexed by subsets of a set Ω and assume that $P(E_\omega) \leq 1/2$ for any $\omega \in \Omega$. Then the quantity $\mathbb{P}\big(\bigcap_{\omega\in\Omega} E_\omega^c\big)$ satisfies*

(i) *the lower bound*

$$\mathbb{P}\Big(\bigcap_{\omega\in\Omega} E_\omega^c\Big) \geq \prod_{\omega\in\Omega} \mathbb{P}(E_\omega^c)$$

and

(ii) *the upper bound*

$$\mathbb{P}\Big(\bigcap_{\omega\in\Omega} E_\omega^c\Big) \leq \Big(\prod_{\omega\in\Omega} \mathbb{P}(E_\omega^c)\Big) \exp\Big(2 \sum_{\substack{\omega,\omega'\in\Omega \\ \omega\sim\omega'}} \mathbb{P}(E_\omega \cap E_{\omega'})\Big).$$

3. Proof of Theorem 1

Let $c > (\lambda_s(1-2\lambda_s))^{-1}$, as in the statement of Theorem 1; we recall that $\lambda_s < 1/2$ when $s \geq 2$.

We represent the sets of $s+1$ distinct elements in the form $\omega = \{x_1, \ldots, x_{s+1}\}$ with

$$x_{s+1} < \cdots < x_1.$$

We also denote $\sigma(\omega) = x_1 + \cdots + x_{s+1}$ and, for each n, we let

$$\Omega_n = \{\omega \text{ such that } \sigma(\omega) = n,\ x_{s+1} < (c\log n)^s \text{ and } (c\log n)^s < x_s\}.$$

If we denote by E_ω the event $\omega \subset A$ and denote $\mathbb{I}$ the indicator function of an event, the function

$$r(n, A) = \sum_{\omega\in\Omega_n} \mathbb{I}(E_\omega)$$

counts the number of representations of n of the form $n = x_1 + \cdots + x_{s+1}$, where

$$x_i \in A, \quad (c\log n)^s < x_s < \cdots < x_1, \quad \text{and} \quad x_{s+1} < (c\log n)^s.$$

By the Borel-Cantelli Lemma, Theorem 1 will be proved as soon as we prove that the series $\mathbb{P}(r(n,A)=0)$ converges. We follow the strategy introduced in [5]. Using our definition and Janson's second correlation inequality, we have

$$\mathbb{P}(r(n,A)=0)=\mathbb{P}\big(\bigcap_{\omega\in\Omega_n}E_\omega^c\big)\le\prod_{\omega\in\Omega_n}\mathbb{P}\big(E_\omega^c\big)\times\exp\left(2\Delta_n\right),$$

with

$$\Delta_n=\sum_{\substack{\omega,\omega'\in\Omega_n\\ \omega\sim\omega'}}\mathbb{P}(E_\omega\cap E_{\omega'}). \tag{4}$$

We first study the product.

Lemma 2. *When n tends to infinity, we have*

$$\prod_{\omega\in\Omega_n}\mathbb{P}\big(E_\omega^c\big)=\exp\big(-(1+o(1))c\lambda_s\log n\big).$$

Proof. We compute

$$\sum_{\omega\in\Omega_n}\mathbb{P}(E_\omega)=\frac{1}{s^{s+1}}\sum_{1\le x_{s+1}<(c\log n)^s}x_{s+1}^{1/s-1}\sum_{\substack{(c\log n)^s<x_s<\cdots<x_1\\ x_1+\cdots+x_s=n-x_{s+1}}}(x_1\ldots x_s)^{1/s-1}.$$

For each $x_{s+1}<(c\log n)^s$, we may apply Lemma 1 (iii) with $z=n-x_{s+1}\sim n$ which gives

$$\sum_{\omega\in\Omega_n}\mathbb{P}(E_\omega)=(1+o(1))\frac{\lambda_s}{s}\sum_{1\le x_{s+1}<(c\log n)^s}x^{1/s-1}=(1+o(1))c\lambda_s\log n,$$

and the result follows from this and the simple relation

$$\prod_{\omega\in\Omega_n}\mathbb{P}\big(E_\omega^c\big)=\exp\left(\sum_{\omega\in\Omega_n}\log(1-(\mathbb{P}(E_\omega))\right)=\exp\left(-(1+o(1))\sum_{\omega\in\Omega_n}\mathbb{P}(E_\omega)\right)$$

□

We now come to the correlation term Δ_n defined in (4).

Lemma 3. *When n tends to infinity, one has*

$$\Delta_n\le(1+o(1))c\lambda_s^2\log n.$$

Proof. In order to decompose the sum defining Δ_n, we introduce

$$\Delta_n(k)=\sum_{\substack{\omega,\omega'\in\Omega_n\\ \omega\sim\omega'\in\Omega_n\\ x_{s+1}=y_{s+1}\\ |\omega\cap\omega'|=k}}\mathbb{P}(E_\omega\cap E_{\omega'})\quad\text{and}\quad\Delta_n'(k)=\sum_{\substack{\omega,\omega'\in\Omega_n\\ \omega\sim\omega'\in\Omega_n\\ x_{s+1}\ne y_{s+1}\\ |\omega\cap\omega'|=k}}\mathbb{P}(E_\omega\cap E_{\omega'})$$

so that

$$\Delta_n=\sum_{k=1}^{s-1}\Delta_n(k)+\sum_{k=1}^{s-1}\Delta_n'(k).$$

We study each term of this formula separately and shall observe that the main contribution comes from $\Delta_n(1)$.

(i) We compute that

$$\begin{aligned}\Delta_n(1) &= \frac{1}{s^{2s+1}} \sum_{\substack{(c\log n)^s<x_s<\cdots<x_1\\ (c\log n)^s<y_s<\cdots<y_1\\ x_{s+1}<(c\log n)^s\\ x_1+\cdots+x_s=y_1+\cdots+y_s=n-x_{s+1}\\ x_i\neq y_j \text{ for any indices } i \text{ and } j}} (x_1\cdots x_{s+1}y_1\cdots y_s)^{-1+1/s}\\ &\leq \frac{1}{s^{2s+1}} \sum_{1\leq x_{s+1}<(c\log n)^s} x_{s+1}^{-1+1/s}\left(\sum_{\substack{1\leq x_s<\cdots<x_1\\ x_1+\cdots+x_s=n-x_{s+1}}} (x_1\cdots x_s)^{-1+1/s}\right)^2.\end{aligned}$$

For each $x_{s+1} < (c\log n)^s$, we may apply Lemma 1 (iii) with $z = n - x_{s+1} \sim n$ which yields

$$\begin{aligned}\Delta_n(1) &\leq (1+o(1))\frac{1}{s^{2s+1}}(s^s\lambda_s)^2 \sum_{1\leq x_{s+1}<(c\log n)^s} x_{s+1}^{-1+1/s}\\ &\leq (1+o(1))c\lambda_s^2\log n\end{aligned}$$

as n tends to infinity.

(ii) For $2 \leq k \leq s-1$, we have

$$\begin{aligned}\Delta_n(k) &= \frac{1}{s^{2s+2-k}} \sum_{\substack{K,K'\subset\{1,\dots,s\}\\ |K|=|K'|=k-1}} \sum_{\substack{(c\log n)^s<x_s<\cdots<x_1\\ (c\log n)^s<y_s<\cdots<y_1\\ 1\leq x_{s+1}<(c\log n)^s\\ \sum_{i\notin K} x_i=\sum_{i\notin K'} y_i=n-\left(\sum_{i\in K} x_i\right)-x_{s+1}\\ x_i\neq y_j \text{ for any indices } i\notin K \text{ and } j\notin K'\\ \{x_i \text{ for } i\in K\}=\{y_i \text{ for } i\in K'\}}} \left(\left(\prod_{i=1}^{s+1} x_i\right)\left(\prod_{i\notin K'} y_i\right)\right)^{-1+1/s}\\ &\ll \sum_{\substack{(c\log n)^s<x_s<\cdots<x_1\\ (c\log n)^s<y_s<\cdots<y_k\\ 1\leq x_{s+1}<(c\log n)^s\\ x_k+\cdots+x_s=y_k+\cdots+y_s=n-(x_1+\cdots+x_{k-1}+x_{s+1})}} (x_1\cdots x_{s+1}y_k\cdots y_s)^{-1+1/s},\end{aligned}$$

after regrouping together similar terms. Thus,

$$\begin{aligned}\Delta_n(k) \ll &\sum_{1\leq x_{s+1}<(c\log n)^s} x_{s+1}^{-1+1/s} \sum_{\substack{(c\log n)^s<x_1,\dots,x_{k-1}\\ x_1+\cdots+x_{k-1}<n-x_{s+1}}} (x_1\cdots x_{k-1})^{-1+1/s}\\ &\times\left(\sum_{\substack{(c\log n)^s<x_k,\dots,x_s\\ x_k+\cdots+x_s=n-x_1-\cdots-x_{k-1}-x_{s+1}}} (x_k\cdots x_s)^{-1+1/s}\right)^2.\end{aligned}$$

We first use Lemma 1 (i) with $z = n - x_1 - \cdots - x_{k-1} - x_{s+1} \geq 1$ to bound the last term. We obtain

$$\begin{aligned}\Delta_n(k) \ll &\sum_{1\leq x_{s+1}<(c\log n)^s} x_{s+1}^{-1+1/s} \sum_{\substack{1\leq x_1,\dots,x_{k-1}\\ x_1+\cdots+x_{k-1}<n-x_{s+1}}} (x_1\cdots x_{k-1})^{-1+1/s}\\ &\times\bigl(n - x_{s+1} - (x_1+\cdots+x_{k-1})\bigr)^{-2(k-1)/s}\end{aligned}$$

and we apply now Lemma 1 (ii) with $z = n - x_{s+1} \geq 2$ which gives

$$\begin{aligned}
\Delta_n(k) &\ll \sum_{1 \leq x_{s+1} < (c \log n)^s} x_{s+1}^{-1+1/s} (n - x_{s+1})^{-1/s} \log(n - x_{s+1}) \\
&\ll n^{-1/s} \log n \sum_{1 \leq x_{s+1} < (c \log n)^s} x_{s+1}^{-1+1/s} \\
&\ll n^{-1/s} \log^2 n.
\end{aligned}$$

(iii) Finally, for $1 \leq k \leq s-1$, using a similar decomposition, we obtain

$$\Delta'_n(k) \ll \sum_{\substack{1 \leq x_s < \cdots < x_1 \\ 1 \leq y_s < \cdots < y_{k+1} \\ 1 \leq x_{s+1}, y_{k+1} < (c \log n)^s \\ x_{k+1} + \cdots + x_{s+1} = y_{k+1} + \cdots + y_{s+1} = n - (x_1 + \cdots + x_k)}} (x_1 \cdots x_{s+1} y_{k+1} \cdots y_{s+1})^{-1+1/s}.$$

Thus,

$$\Delta'_n(k) \ll \sum_{\substack{1 \leq x_1, \ldots, x_k \\ x_1 + \cdots + x_k < n}} (x_1 \cdots x_k)^{-1+1/s} S(n; x_1, \ldots, x_k)^2$$

where

$$S(n; x_1, \ldots, x_k) = \sum_{\substack{1 \leq x_{k+1}, \ldots, x_s \\ 1 \leq x_{s+1} < (c \log n)^s \\ x_{k+1} + \cdots + x_{s+1} = n - (x_1 + \cdots + x_k)}} (x_{k+1} \cdots x_{s+1})^{-1+1/s}.$$

We now study this sum and distinguish two cases.

(a) First, if $x_1 + \cdots + x_k < n - 2(c \log n)^s$ then

$$S(n; x_1, \ldots, x_k) = \sum_{1 \leq x_{s+1} < (c \log n)^s} x_{s+1}^{-1+1/s} \sum_{\substack{1 \leq x_{k+1}, \ldots, x_s \\ x_{k+1} + \cdots + x_s = n - x_{s+1} - (x_1 + \cdots + x_k)}} (x_{k+1} \cdots x_s)^{-1+1/s}$$

which can be bounded above, using Lemma 1 (i) for each internal sum with $z = n - x_{s+1} - (x_1 + \cdots + x_k) \geq 1$ by

$$\begin{aligned}
&\ll \sum_{1 \leq x_{s+1} < (c \log n)^s} x_{s+1}^{-1+1/s} (n - (x_1 + \cdots + x_k) - x_{s+1})^{-k/s} \\
&\ll (n - (x_1 + \cdots + x_k))^{-k/s} \log n.
\end{aligned}$$

(b) Second, in the case $n - 2(c \log n)^s \leq x_1 + \cdots + x_k < n$, we have using Lemma 1 (i) with $z = n - (x_1 + \cdots + x_k) \geq 1$,

$$\begin{aligned}
S(n; x_1, \ldots, x_k) &\leq \sum_{\substack{1 \leq x_{k+1}, \ldots, x_{s+1} \\ x_{k+1} + \cdots + x_{s+1} = n - (x_1 + \cdots + x_k)}} (x_{k+1} \cdots x_{s+1})^{-1+1/s} \\
&\ll (n - (x_1 + \cdots + x_k))^{(1-k)/s} \\
&\ll 1.
\end{aligned}$$

From these bounds (a) and (b) on the sums $S(n; x_1, \dots, x_k)$ we derive

$$\begin{aligned}
\Delta'_n(k) &\ll \sum_{\substack{1\le x_1,\dots,x_k \\ x_1+\cdots+x_k<n-2(c\log n)^s}} (x_1\cdots x_k)^{-1+1/s} S(n; x_1,\dots,x_k)^2 \\
&\qquad + \sum_{\substack{1\le x_1,\dots,x_k \\ n-2(c\log n)^s\le x_1+\cdots+x_k<n}} (x_1\cdots x_k)^{-1+1/s} S(n; x_1,\dots,x_k)^2 \\
&\ll \log^2 n \sum_{\substack{1\le x_1,\dots,x_k \\ x_1+\cdots+x_k<n-2(c\log n)^s}} (x_1\cdots x_k)^{-1+1/s} (n-(x_1+\cdots+x_k))^{-2k/s} \\
&\qquad + \sum_{n-2(c\log n)^s\le r<n} \sum_{\substack{1\le x_1,\dots,x_k \\ x_1+\cdots+x_k=r}} (x_1\cdots x_k)^{-1+1/s} \\
&\ll \log^2 n \sum_{\substack{1\le x_1,\dots,x_k \\ x_1+\cdots+x_k<n}} (x_1\cdots x_k)^{-1+1/s} (n-(x_1+\cdots+x_k))^{-2k/s} \\
&\qquad + \sum_{n-2(c\log n)^s\le r<n} r^{-1+k/s} \\
&\ll n^{-1/s}\log^3 n + n^{-1+k/s}(\log n)^s \\
&\ll n^{-1/s}\log^{s+1} n
\end{aligned}$$

where we use Lemma 1 (ii) applied with $t = k$ and $z = n$ in the first term and Lemma 1 (i) with $t = k$ and $z = r$ for each internal term of the second sum.

The conclusion of the lemma follows from collecting the estimates of (i), (ii) and (iii) just obtained. □

Gathering the results of Lemma 2 and Lemma 3, we obtain

$$\mathbb{P}(r(n, A) = 0) \le \exp\left(-(1 + o(1))c\lambda_s(1 - 2\lambda_s)\log n\right),$$

which is the general term of a convergent series as soon as $c\lambda_s(1-2\lambda_s) > 1$; this ends the proof of Theorem 1. As was noticed in [5], the factor 2 occurring in Janson's inequality may be reduced to any constant larger than 1; however, the correlation term is still of the same order of magnitude as the main term.

What about a reverse result? Janson's first correlation inequality leads to

$$\mathbb{P}(r(n, A) = 0) \ge \exp\left(-(1 + o(1))c\lambda_s \log n\right), \tag{5}$$

which is the general term of divergent series as soon as $c\lambda_s < 1$. A first minor point is that $r(n, A)$ only counts special representations (pairwise distinct summands and only one which is less than $(c \log n)^s$) but it is not difficult to obtain a bound like (5) taking into account all the representations. More seriously, to apply the "reverse" Borel-Cantelli Lemma, some independence between the events $\{r(n, A) = 0\}$ is required; unfortunately, we just miss the condition given in [6].

4. Proof of Theorem 2

By assumption, there is some

$$c > \lambda_s^{-1}$$

such that the fixed sequence B has a counting function satisfying

$$B(n) \ge c(1 + o(1)) \log n. \tag{6}$$

For each integer n, we define $m = m(n)$ to be the smallest positive integer such that

$$B(m) = \left\lfloor \frac{c+\lambda_s^{-1}}{2} \log n \right\rfloor.$$

We observe, by (6) and the definition of m, that

$$m = n^{(1+o(1))\frac{1+\lambda_s^{-1}/c}{2}} = o(n),$$

which will be used through the proof.

We represent the sets of s distinct elements in the form $\omega = \{x_1, \dots, x_s\}$ with $x_1 > \cdots > x_s$. We also denote $\sigma(\omega) = x_1 + \cdots + x_s$ and for each n let

$$\Omega_n = \{\omega \text{ such that } \sigma(\omega) = n - b \text{ for some } b \in B,\ b < m\},$$

where

If we denote by E_ω the event $\omega \subset A$, then the event "n cannot be written in the form $n = a_1 + \cdots + a_s + b$ with $a_1 > \cdots > a_s,\ a_i \in A,\ b \in B,\ b < m$", which we denote by F_n, can be expressed in the form

$$F_n = \bigcap_{\omega\in\Omega_n} E_\omega^c.$$

We start with two lemmas.

Lemma 4. *One has*

$$\sum_{\omega\in\Omega_n} \mathbb{P}(E_\omega) = (1+o(1))\frac{c\lambda_s+1}{2}\log n.$$

Proof. Indeed, using Lemma 1 (iii), we compute

$$\begin{aligned}
\sum_{\omega\in\Omega_n} \mathbb{P}(E_\omega) &= \sum_{b<m} \frac{1}{s^s} \sum_{\substack{1\le x_1<\cdots<x_s\\ x_1+\cdots+x_s=n-b}} (x_1\cdots x_s)^{-1+1/s} \\
&= (1+o(1))B\,(m)\,\lambda_s \\
&= (1+o(1))\frac{c\lambda_s+1}{2}\log n.
\end{aligned}$$

□

Lemma 5. *One has*

$$\sum_{\substack{\omega\sim\omega'\\ \omega,\omega'\in\Omega_n}} \mathbb{P}(E_\omega\cap E_{\omega'}) \ll n^{-1/s}(\log n)^3.$$

Proof. We can write

$$\sum_{\substack{\omega\sim\omega'\\ \omega,\omega'\in\Omega_n}} \mathbb{P}(E_\omega\cap E_{\omega'}) = \sum_{\substack{b\le b'<m\\ b,b'\in B}} \sum_{k=1}^{s-1} \Delta_n(k;b,b')$$

where

$$\Delta_n(k;b,b') = \sum_{\substack{\omega,\omega'\in\Omega_n\\ \sigma(\omega)=n-b,\ \sigma(\omega')=n-b'\\ |\omega\cap\omega'|=k}} \mathbb{P}(E_\omega\cap E_{\omega'}).$$

Thus,

$$\begin{aligned}\Delta_n(k;b,b') \quad &\ll \sum_{\substack{1\le x_1<\cdots<x_k\\ x_1+\cdots+x_k<n-b'}} (x_1\cdots x_k)^{-1+1/s}\left(\sum_{\substack{x_{k+1},\dots,x_s\\ x_{k+1}+\cdots+x_s=n-b-(x_1+\cdots+x_k)}} (x_{k+1}\cdots x_s)^{-1+1/s}\right)\\ &\qquad\times\left(\sum_{\substack{y_{k+1},\dots,y_s\\ y_{k+1}+\cdots+y_s=n-b'-(x_1+\cdots+x_k)}} (y_{k+1}\cdots y_s)^{-1+1/s}\right).\end{aligned}$$

But Lemma 1 (i) gives, for $\zeta = b$ or b',

$$\begin{aligned}\sum_{\substack{1\le x_{k+1},\dots,x_s\\ x_{k+1}+\cdots+x_s=n-\zeta-(x_1+\cdots+x_k)}} (x_{k+1}\cdots x_s)^{-1+1/s} \quad &\ll \quad (n-\zeta-(x_1+\cdots+x_k))^{-k/s}\\ &\ll \quad (n-b'-(x_1+\cdots+x_k))^{-k/s}\end{aligned}$$

and applying this bound and later Lemma 1 (ii) we obtain

$$\begin{aligned}\Delta_n(k;b,b') \quad &\ll \sum_{\substack{1\le x_1<\cdots<x_k\\ x_1+\cdots+x_k<n-b'}} (x_1\cdots x_k)^{-1+1/s}(n-b'-(x_1+\cdots+x_k))^{-2k/s}\\ &\ll \quad (n-b')^{-1/s}\log(n-b').\end{aligned}$$

Adding all the contributions, it follows

$$\sum_{\substack{\omega\sim\omega'\\ \omega,\omega'\in\Omega_n}} \mathbb{P}(E_\omega\cap E_{\omega'}) \ll \sum_{\substack{b\le b'<m\\ b,b'\in B}} (n-b')^{-1/s}\log(n-b') \ll B(m)^2 n^{-1/s}\log n.$$

and using $B(m) \ll \log n$ concludes the proof of the lemma. □

We now come to the very proof of the Theorem. By Janson's second inequality (Theorem 5 (ii)) we obtain the following upper bound for $\mathbb{P}(F_n)$, namely

$$\mathbb{P}(F_n) \le \prod_{\omega\in\Omega_n}(1-\mathbb{P}(E_\omega))\exp\left(2\sum_{\substack{\omega\sim\omega'\\ \omega,\omega'\in\Omega_n}} \mathbb{P}(E_\omega\cap E_{\omega'})\right)$$

which, using the inequality $\log(1-x) < -x$ (valid for $x > 0$) yields

$$\log\mathbb{P}(F_n) \le -\sum_{\omega\in\Omega_n}\mathbb{P}(E_\omega) + 2\sum_{\substack{\omega\sim\omega'\\ \omega,\omega'\in\Omega_n}} \mathbb{P}(E_\omega\cap E_{\omega'}). \tag{7}$$

Plugging in (13) the estimates obtained in Lemmas 4 and 5 we get

$$\log\mathbb{P}(F_n) \le -(1+o(1))\frac{c\lambda_s+1}{2}\log n,$$

so that

$$\mathbb{P}(F_n) \le n^{-(1+o(1))\frac{c\lambda_s+1}{2}}.$$

If $c > \lambda_s^{-1}$ then $(c\lambda_s+1)/2 > 1$ and the sum $\sum_n \mathbb{P}(F_n)$ is finite. The Borel-Cantelli Lemma implies that, almost surely, only a finite number of events F_n can occur and we are done.

5. Proof of Theorem 3

We use the same kind of notation as in the proof of Theorem 2 but now

$$\Omega_n = \{\omega \text{ such that } \sigma(\omega) = n - b \text{ for some } b \in B\}.$$

We define the event F_n: "n cannot be written in the form $n = x_1 + \cdots + x_s + b$ with $x_1, \ldots, x_s \in A$, $x_s < \cdots < x_1$ and $b \in B$." In other words,

$$F_n = \bigcap_{\omega \in \Omega_n} E_\omega^c.$$

The hypothesis of Theorem 3 is tantamount to writing

$$\liminf_{n \to +\infty} \frac{B(n)}{\log n} = c$$

for some $c < \lambda_s^{-1}$. Then there exists a sequence $(N_i)_{i \in \mathbb{N}}$ of integers such that

$$B(N_i) = c(1 + o(1)) \log N_i. \tag{8}$$

In all this proof, if N is some integer, we shall say that a positive integer n is *good (for N)* if $N/2 \leq n \leq N$ and

$$|n - b| > (\log N)^{4s}$$

for all $b \in B$. In the opposite case, n will be said to be *bad (for N)*.

We consider the random variable (recall $\mathbb{I}$ is the indicator function of an event)

$$X_N = \sum_{\substack{N/2 \leq n \leq N \\ n \text{ is good}}} \mathbb{I}(F_n).$$

We use the notation

$$\mu_N = \mathbb{E}(X_N) \quad \text{ and } \quad \sigma_N^2 = \mathbb{V}(X_N).$$

Our strategy is to prove that

$$\lim_{i \to +\infty} \mu_{N_i} = +\infty \tag{9}$$

and that

$$\sigma_{N_i}^2 \ll \frac{\mu_{N_i}^2}{\log N_i}. \tag{10}$$

Then, using Chebychev's inequality in the form (3), we get

$$\mathbb{P}\left(X_{N_i} < \frac{\mu_{N_i}}{2}\right) < \frac{4\sigma_{N_i}^2}{\mu_{N_i}^2} \ll \frac{1}{\log N_i}. \tag{11}$$

Now, Theorem 3 follows immediately from (11) and (9).

From now on, we let N be a term of the sequence $(N_i)_{i \in \mathbb{N}}$.

5.1. Estimate of μ_N.

Proposition 1. *We have*

$$\mu_N \geq N^{(1 - c\lambda_s)(1 + o(1))}.$$

Proof. We have

$$\mu_N = \sum_{\substack{N/2 \leq n \leq N \\ n \text{ good}}} \mathbb{P}(F_n). \tag{12}$$

Let n be a good integer for N. Using Janson's first inequality (Theorem 5 (i)) we observe that

$$\mathbb{P}(F_n) \geq \prod_{\omega\in\Omega_n} (1-\mathbb{P}(E_\omega)).$$

Using that $\log(1-x) = -x + O(x^2)$ we have

$$\log(\mathbb{P}(F_n)) \geq -\sum_{\omega\in\Omega_n} \mathbb{P}(E_\omega) + O\left(\sum_{\omega\in\Omega_n} \mathbb{P}(E_\omega)^2\right). \tag{13}$$

On the one hand, since n is good, we compute

$$\begin{aligned} \sum_{\omega\in\Omega_n} \mathbb{P}(E_\omega) &= \frac{1}{s^s} \sum_{\substack{b<n-(\log N)^{4s} \\ b\in B}} \sum_{\substack{1\leq x_s<\cdots<x_1 \\ x_1+\cdots+x_s=n-b}} (x_1\cdots x_s)^{-1+1/s} \\ &= \frac{1}{s^s} \sum_{\substack{b<n-(\log N)^{4s} \\ b\in B}} s^s\lambda_s(1+o(1)) \\ &\leq \lambda_s(1+o(1))B(N) \\ &\leq c\lambda_s(1+o(1))\log N. \end{aligned} \tag{14}$$

On the other hand,

$$\begin{aligned} \sum_{\omega\in\Omega_n} \mathbb{P}(E_\omega)^2 &= \sum_{\substack{b<n-(\log N)^{4s} \\ b\in B}} \left(\frac{1}{s^s}\right)^2 \sum_{\substack{1\leq x_s<\cdots<x_1 \\ x_1+\cdots+x_s=n-b}} (x_1\cdots x_s)^{-2+2/s} \\ &\ll \sum_{\substack{b<n-(\log N)^{4s} \\ b\in B}} (n-b)^{-2+2/s} \sum_{\substack{1\leq x_s<\cdots<x_2 \\ x_2+\cdots+x_s\leq n-b}} (x_2\cdots x_s)^{-2+2/s} \end{aligned}$$

by noticing that $x_1 \geq (n-b)/s$ in each term of the internal sum. We further compute, n being good,

$$\begin{aligned} \sum_{\omega\in\Omega_n} \mathbb{P}(E_\omega)^2 &\ll \sum_{\substack{b<n-(\log N)^{4s} \\ b\in B}} (n-b)^{-2+2/s} \left(\sum_{x=1}^{n-b} x^{-2+2/s}\right)^{s-1} \\ &\ll \sum_{\substack{b<n \\ b\in B}} (\log^{4s} N)^{-2+2/s} \left(\sum_{x=1}^{n-b} x^{-1}\right)^{s-1} \\ &\ll \sum_{\substack{b<n \\ b\in B}} (\log N)^{-8s+8} (\log N)^{s-1} \\ &\ll (\log N)^{-7s+7} B(N) \\ &\ll (\log N)^{-6}. \end{aligned}$$

Thus, (13) and (14) imply that

$$\mathbb{P}(F_n) \geq N^{-c\lambda_s(1+o(1))} \tag{15}$$

when n is good.

One computes that

$$\begin{aligned} |\{N/2 \leq n \leq N : \ n \text{ bad}\}| &= |\{N/2 \leq n \leq N : \ |n-b| < (\log N)^{4s} \text{ for some } b \in B\}| \\ &\leq \sum_{b<N} 2(\log N)^{4s} \\ &\ll (\log N)^{4s+1}. \end{aligned}$$

Thus, using equations (12), (15) and this, we obtain

$$\begin{aligned}
\mu_N &= \sum_{\substack{N/2\le n\le N\\ n \text{ good}}} N^{-c\lambda_s(1+o(1))}\\
&\ge \sum_{N/2\le n\le N} N^{-c\lambda_s(1+o(1))} - \sum_{\substack{N/2\le n\le N\\ n \text{ bad}}} N^{-c\lambda_s(1+o(1))}\\
&\ge N^{(1+o(1))(1-c\lambda_s)} - O\left((\log N)^{4s+1}\right)\\
&\ge N^{(1+o(1))(1-c\lambda_s)}
\end{aligned}$$

since $1-c\lambda_s>0$. □

5.2. **Estimate of σ_N^2.** Let us recall now that, given a set B, its *difference set* $B-B$ is defined by

$$B-B=\{b-b' \text{ with } b,b'\in B\}.$$

Lemma 6. *Let $B_N=\{b\le N$ with $b\in B\}$. Let $n<m\le N$ be two positive integers such that $m-n\notin B_N-B_N$ then*

$$\mathbb{P}(F_n\cap F_m)\le \mathbb{P}(F_n)\mathbb{P}(F_m)\exp\left(2\sum_{\substack{\omega,\omega'\in\Omega_n\cup\Omega_m\\ \omega\sim\omega'}}\mathbb{P}(E_\omega\cap E_{\omega'})\right).$$

Proof. We observe that

$$F_n\cap F_m=\bigcap_{\omega\in\Omega_n\cup\Omega_m}E_\omega^c$$

and that the condition $m-n\notin B_N-B_N$ implies that $\Omega_n\cap\Omega_m=\emptyset$. Janson's second inequality (Theorem 5 (ii)) applied to $\Omega=\Omega_n\cup\Omega_m$) implies that

$$\begin{aligned}
\mathbb{P}(F_n\cap F_m) &\le \prod_{\omega\in\Omega_n\cup\Omega_m}\mathbb{P}(E_\omega^c)\exp\left(2\sum_{\substack{\omega,\omega'\in\Omega_n\cup\Omega_m\\ \omega\sim\omega'}}\mathbb{P}(E_\omega\cap E_{\omega'})\right)\\
&= \prod_{\omega\in\Omega_n}\mathbb{P}(E_\omega^c)\prod_{\omega\in\Omega_m}\mathbb{P}(E_\omega^c)\exp\left(2\sum_{\substack{\omega,\omega'\in\Omega_n\cup\Omega_m\\ \omega\sim\omega'}}\mathbb{P}(E_\omega\cap E_{\omega'})\right)\\
&\le \mathbb{P}(F_n)\mathbb{P}(F_m)\exp\left(2\sum_{\substack{\omega,\omega'\in\Omega_n\cup\Omega_m\\ \omega\sim\omega'}}\mathbb{P}(E_\omega\cap E_{\omega'})\right)
\end{aligned}$$

using Janson's first inequality (Theorem 5 (i)) applied to Ω_n and to Ω_m. The lemma is proved. □

Lemma 7. *Let N,n,m be integers. If n and m are good for N, then*

$$\sum_{\substack{\omega\in\Omega_n,\omega'\in\Omega_m\\ \omega\sim\omega'}}\mathbb{P}(E_\omega\cap E_{\omega'})\ll\frac{1}{\log N}.$$

Proof. We can write

$$\sum_{\substack{\omega\in\Omega_n,\omega'\in\Omega_m\\ \omega\sim\omega'}} \mathbb{P}(E_\omega\cap E_{\omega'}) = \sum_{\substack{1\le b<n\\ 1\le b'<m\\ b,b'\in B}} \sum_{k=1}^{s-1} \Delta_{n,m}(k;b,b')$$

where, for $k\ge 1$,

$$\Delta_{n,m}(k;b,b') = \sum_{\substack{\omega\in\Omega_n,\omega'\in\Omega_m\\ \sigma(\omega)=n-b,\ \sigma(\omega')=m-b'\\ |\omega\cap\omega'|=k}} P(E_\omega\cap E_{\omega'}).$$

Assume that $n-b\le m-b'$. Thus,

$$\begin{aligned}\Delta_{n,m}(k;b,b') \ll \sum_{\substack{1\le x_1,\dots,x_k\\ x_1+\cdots+x_k<n-b}} (x_1\cdots x_k)^{-1+1/s} &\left(\sum_{\substack{1\le x_{k+1},\dots,x_s\\ x_{k+1}+\cdots+x_s=n-b-(x_1+\cdots+x_k)}} (x_{k+1}\cdots x_s)^{-1+1/s}\right)\\ &\times\left(\sum_{\substack{1\le y_{k+1},\dots,y_s\\ y_{k+1}+\cdots+y_s=m-b'-(x_1+\cdots+x_k)}} (y_{k+1}\cdots y_s)^{-1+1/s}\right).\end{aligned}$$

Lemma 1 (i) applied twice shows that

$$\begin{aligned}\Delta_{n,m}(k;b,b') &\ll \sum_{\substack{1\le x_1,\dots,x_k\\ x_1+\cdots+x_k<n-b}} (x_1\cdots x_k)^{-1+\frac{1}{s}}(n-b-(x_1+\cdots+x_k))^{-\frac{k}{s}}(m-b'-(x_1+\cdots+x_k))^{-\frac{k}{s}}\\ &\ll \sum_{\substack{1\le x_1,\dots,x_k\\ x_1+\cdots+x_k<n-b}} (x_1\cdots x_k)^{-1+1/s}(n-b-(x_1+\cdots+x_k))^{-2k/s}\\ &\ll (n-b)^{-1/s}\log(n-b)\\ &\ll \frac{1}{\log^3 N}.\end{aligned}$$

since $(\log N)^{4s}\le n-b\le N$.

If $m-b'<n-b$ we proceed in the same way. Thus,

$$\begin{aligned}\sum_{\substack{\omega\in\Omega_n,\omega'\in\Omega_m\\ \omega\sim\omega'}} \mathbb{P}(E_\omega\cap E_{\omega'}) &\ll \sum_{\substack{1\le b<n\\ b\in B}}\sum_{\substack{1\le b'<m\\ b'\in B}} \frac{1}{\log^3 N}\\ &\ll \frac{(B(N))^2}{\log^3 N}\\ &\ll \frac{1}{\log N},\end{aligned}$$

hence the result. □

Corollary 1. *Let N,n,m be integers. If n and m are good for N and $m-n\notin B_N-B_N$ then*

$$\mathbb{P}(F_n\cap F_m)-\mathbb{P}(F_n)\mathbb{P}(F_m) \ll \frac{1}{\log N}\mathbb{P}(F_n)\mathbb{P}(F_m).$$

Proof. Lemma 6 implies that

$$\mathbb{P}(F_n \cap F_m) - \mathbb{P}(F_n)\mathbb{P}(F_m) \leq \mathbb{P}(F_n)\mathbb{P}(F_m)\left(\exp\left(2\sum_{\substack{\omega,\omega' \in \Omega_n \cup \Omega_m \\ \omega \sim \omega'}} \mathbb{P}(E_\omega \cap E_{\omega'})\right) - 1\right).$$

We observe that

$$\begin{aligned}\sum_{\substack{\omega,\omega' \in \Omega_n \cup \Omega_m \\ \omega \sim \omega'}} \mathbb{P}(E_\omega \cap E_{\omega'}) &= \sum_{\substack{\omega,\omega' \in \Omega_n \\ \omega \sim \omega'}} \mathbb{P}(E_\omega \cap E_{\omega'}) + \sum_{\substack{\omega,\omega' \in \Omega_m \\ \omega \sim \omega'}} \mathbb{P}(E_\omega \cap E_{\omega'}) \\ &\quad + \sum_{\substack{\omega \in \Omega_n, \omega' \in \Omega_m \\ \omega \sim \omega'}} \mathbb{P}(E_\omega \cap E_{\omega'}).\end{aligned}$$

We finish the proof applying Lemma 7 to the three sums (with $n = m$ or not) and using the estimate $e^x - 1 \sim x$ when x approaches 0. □

Proposition 2. *The following estimate holds*

$$\sigma_N^2 \ll \frac{\mu_N^2}{\log N}.$$

Proof. A standard calculation shows that

$$\sigma_N^2 = 2\sum_{\substack{N/2 \leq n < m \leq N \\ n,m \text{ good}}} \Big(\mathbb{P}(F_n \cap F_m) - \mathbb{P}(F_n)\mathbb{P}(F_m)\Big) + \sum_{\substack{N/2 \leq n \leq N \\ n \text{ good}}} \Big(\mathbb{P}(F_n) - \mathbb{P}^2(F_n)\Big).$$

We decompose

$$\sigma_N^2 = 2\Sigma_1 + 2\Sigma_2 + \Sigma_3,$$

where

$$\begin{aligned}\Sigma_1 &= \sum_{\substack{N/2 < n < m \leq N \\ n-m \notin B_N - B_N \\ n,m \text{ good}}} \Big(\mathbb{P}(F_n \cap F_m) - \mathbb{P}(F_n)\mathbb{P}(F_m)\Big), \\ \Sigma_2 &= \sum_{\substack{N/2 \leq n < m \leq N \\ n-m \in B_N - B_N \\ n,m \text{ good}}} \Big(\mathbb{P}(F_n \cap F_m) - \mathbb{P}(F_n)\mathbb{P}(F_m)\Big), \\ \Sigma_3 &= \sum_{\substack{N/2 \leq n \leq N \\ n \text{ good}}} \Big(\mathbb{P}(F_n) - \mathbb{P}^2(F_n)\Big).\end{aligned}$$

It is clear that

$$\Sigma_3 \leq \mu_N.$$

To bound Σ_2 from above, we use the trivial upper bound

$$\mathbb{P}(F_n \cap F_m) - \mathbb{P}(F_n)\mathbb{P}(F_m) \leq \mathbb{P}(F_m)$$

and get, for Σ_2,

$$\begin{aligned}
\Sigma_2 &\leq \sum_{\substack{N/2\leq m<N\\ m \text{ good}}} \mathbb{P}(F_m)|\{N/2\leq n\leq N \text{ such that } n\in B_N-B_N+m\}|\\
&\leq |B_N-B_N| \sum_{\substack{N/2\leq m<N\\ m \text{ good}}} \mathbb{P}(F_m)\\
&\ll |B_N|^2 \sum_{\substack{N/2\leq m<N\\ m \text{ good}}} \mathbb{P}(F_m)\\
&\ll \log^2 N\ \mu_N.
\end{aligned}$$

Finally, by Corollary 1, we have

$$\Sigma_1 \ll \frac{1}{\log N} \sum_{\substack{N/2<n<m\leq N\\ n-m\notin B_N-B_N\\ n,m \text{ good}}} \mathbb{P}(F_n)\mathbb{P}(F_m) \leq \frac{1}{\log N}\Big(\sum_{\substack{N/2\leq n\leq N\\ n \text{ good}}} \mathbb{P}(F_n)\Big)^2 = \frac{\mu_N^2}{\log N}.$$

Adding the three contributions Σ_1, Σ_2 and Σ_3 we have

$$\sigma_N^2 \ll \frac{\mu_N^2}{\log N} + \log^2 N\ \mu_N + \mu_N \ll \mu_N^2\left(\frac{1}{\log N}+\frac{\log^2 N}{\mu_N}\right). \tag{16}$$

We let

$$\varepsilon = \frac{1-c\lambda_s}{2} > 0$$

and notice that Proposition 1 implies

$$\mu_N \geq N^{2\varepsilon+o(1)} \gg \log^3 N. \tag{17}$$

We obtain the Proposition after plugging (17) in the last term of (16). □

6. The limit case of Theorems 2 and 3: Sequences at the threshold

Theorems 2 and 3 being proved, it is natural to wonder what happens for sequences B at the threshold, namely satisfying

$$\liminf_{n\to\infty} \frac{B(n)}{\log n} = \lambda_s^{-1}.$$

In this paragraph, we show how to build sequences at the threshold satisfying either the conclusion of Theorem 2 or of Theorem 3.

Indeed, consider for example the sequence B defined by the counting function

$$B(n) = \left\lfloor \lambda_s^{-1}\log n + 2\lambda_s^{-1}\log\log n\right\rfloor.$$

We can mimic the proof of Theorem 2 (although we have to change $m = n/2$ now).

We'll use the following refinement of Lemma 1, (iii)

$$\sum_{\substack{1\leq x_s<\cdots<x_1\\ x_1+\cdots+x_s=n}} (x_1\cdots x_s)^{-1+1/s} = s^s\lambda_s + O(n^{-1/(s+1)}). \tag{18}$$

Hint: we let $g(n) = n^{1/(s+1)}$, break the sum over x_s at $g(n)$. In the sum with $x_s \geq g(n)$ we recognize (up to the right gamma factor) a Riemann sum for the integral $\int\cdots\int(t_1\dots t_s)^{-1+1/s}dt_1\dots dt_s$ over the part of the hyperplane $t_1+\cdots+t_s = 1$ limited by $g(n)/n < t_s < \cdots < t_1 \leq 1$; the error in the approximation of the integral by the Riemann sum is $O(1/g(n))$; the error in the truncation of the sum (cf. the proof of part (iii) of Lemma 1) is $O((g(n)/n)^{1/s})$ and so is the error in the truncation of the integral. The resulting global error is $O(n^{1/(s+1)})$, which is enough for our purpose. By looking carefully at what occurs around 0 and integrating the error in the approximation, one can reduce the error term to $O(n^{-1/s})$.

Equation (18) leads to

$$\begin{aligned}\sum_{\omega\in\Omega_n}\mathbb{P}(E_\omega) &= \frac{1}{s^s}\sum_{b<n/2}\sum_{\substack{1\le x_s<\cdots<x_1\\ x_1+\cdots+x_s=n}}(x_1\cdots x_s)^{-1+1/s}\\ &= B(n/2)\left(\lambda_s+O(n^{-1/(s+1)})\right)\\ &= \log n+2\log\log n+O(1).\end{aligned}$$

Following the same reasoning as in Theorem 2 we get

$$\mathbb{P}(F_n)\le e^{-(\log n+2\log\log n+O(1))}\ll\frac{1}{n\log^2 n}.$$

Thus, $\sum_n\mathbb{P}(F_n)<\infty$ and we can apply the Borel Cantelli Lemma to conclude that the sequence B is almost surely complementary sequence of a pseudo s-th power.

Conversely, consider for example a sequence B defined by the counting function

$$B(n)=\left\lfloor\lambda_s^{-1}\log n-t(n)\right\rfloor,$$

where $t(n)$ is an increasing function with $t(n)=o(\log n)$. We can mimic the proof of Theorem 3 with the only difference that now the exponent $2\epsilon+o(1)$ in (17) is $2\epsilon_N\sim\lambda_s t(N)/\log N$. So, we can take for $t(n)$ any function such that $\mu_N\gg N^{\epsilon_N}\gg\log^3 N$. For example, the choice

$$t(N)=4\lambda_s^{-1}\frac{\log\log N}{\log N}$$

is satisfactory.

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