$B_2[g]$ SETS AND A CONJECTURE OF SCHINZEL AND SCHMIDT

JAVIER CILLERUELO AND CARLOS VINUESA

ABSTRACT. A set of integers $\mathcal{A}$ is called a $B_2[g]$ set if every integer m has at most g representations of the form $m = a + a'$, with $a \le a'$ and $a, a' \in \mathcal{A}$. We obtain a new lower bound for $F(g,n)$, the largest cardinality of a $B_2[g]$ set in $\{1,\dots,n\}$. More precisely, we prove that $\liminf_{n\to\infty} \frac{F(g,n)}{\sqrt{gn}} \ge \frac{2}{\sqrt{\pi}} - \varepsilon_g$ where $\varepsilon_g \to 0$ when $g \to \infty$. We show a connection between this problem and another one discussed by Schinzel and Schmidt which can be considered its continuous version.

1. INTRODUCTION

A set of integers $\mathcal{A}$ is called a $B_2[g]$ set if every integer m has at most g representations of the form $m = a + a'$, with $a \le a'$ and $a, a' \in \mathcal{A}$. We write $r_{\mathcal{A}}(m)$ for the number of such representations.

A major problem in additive number theory is the study of the behaviour of the function $F(g,n)$, the largest cardinality of a $B_2[g]$ set in $\{1,\dots,n\}$.

It is a well known result on Sidon sets that $F(1,n) \sim n^{1/2}$, but the asymptotic behavior of $F(g,n)$ is an open problem for $g \ge 2$. The trivial counting argument gives $F(g,n) \le 2\sqrt{gn}$ and it is not too difficult to show (see section 2) that $F(g,n) \gtrsim \sqrt{gn}$.

We define

$$\beta(g) = \liminf_{n\to\infty} \frac{F(g,n)}{\sqrt{gn}} \le \limsup_{n\to\infty} \frac{F(g,n)}{\sqrt{gn}} = \alpha(g).$$

In the last years some progress has been done, improving the easier estimates $1 \le \beta(g) \le \alpha(g) \le 2$. We list below the successive results obtained by several authors including the improvement obtained in this work.

This work was developed during the Doccourse in Additive Combinatorics held in the Centre de Recerca Matemàtica from January to March 2007. Both authors are extremely grateful for their hospitality.

Both authors are supported by Grants CCG07-UAM/ESP-1814 and DGICYT MTM 2005-04730 (Spain) .

$$\begin{aligned}
\alpha(g) &\le 2 \text{ (trivial)}\\
&\le 1.864 \text{ (J. Cilleruelo - I. Ruzsa - C. Trujillo, [1])}\\
&\le 1.844 \text{ (B. Green, [2])}\\
&\le 1.839 \text{ (G. Martin - K. O'Bryant, [5])}\\
&\le 1.789 \text{ (G. Yu, [9])}
\end{aligned}$$

$$\begin{aligned}
\beta(g) &\ge 1 \text{ (M. Kolountzakis, [3])}\\
&\gtrsim 1.060 \text{ (J. Cilleruelo - I. Ruzsa - C. Trujillo, [1])}\\
&\gtrsim 1.122 \text{ (G. Martin - K. O'Bryant, [4])}\\
&\gtrsim 2/\sqrt{\pi} = 1.128... \text{ (Corollary 1.2)}
\end{aligned}$$

The aim of this work is not only to improve the lower bound for $\beta(g)$ but also to show a connection with another problem discussed by Schinzel and Schmidt [7] which can be seen as the continuous version of this problem.

We define the Schinzel-Schmidt's constant S to be the real number

$$S = \sup_{f\in\mathcal{F}} \frac{1}{|f * f|_\infty} \tag{1}$$

where $\mathcal{F} = \{f : \ f \ge 0, \ \text{supp}(f) \subseteq [0,1], \ |f|_1 = 1\}$ and $f * f(x) = \int f(t)f(x-t)\, dt$. We use the notation $|g|_1 = \int_0^1 |g(x)|\, dx$, $|g|_\infty = \sup_x g(x)$ and $\text{supp}(g) = \{x : \ g(x) \ne 0\}$.

Remark 1.1. *The definition in* [7] *is* $S = \sup_{f\in\widetilde{\mathcal{F}}} |f|_1^2/|f * f|_\infty$ *with* $\widetilde{\mathcal{F}} = \{f : \ f \ge 0, \ f \not\equiv 0, \ supp(f) \subseteq [0,1], \ f \in L_1[0,1]\}$*, and we can assume that* $|f|_1 = 1$ *because* $|f|_1^2/|f * f|_\infty$ *is invariant under dilates of* f*.*

It is easy to see that $1 \le S \le 2$ but Schinzel and Schmidt proved in [7] that $4/\pi \le S \le 1.7373$. The witness for the lower found is the function $f(x) = \frac{1}{2\sqrt{x}} \in \mathcal{F}$. They also conjecture that $S = 4/\pi$. Our main theorem relates $\alpha(g)$ and $\beta(g)$ to S.

Theorem 1. $\sqrt{S} \le \liminf_{g\to\infty} \beta(g) \le \limsup_{g\to\infty} \alpha(g) \le \sqrt{2S}$.

Corollary 1.2. $\beta(g) \ge 2/\sqrt{\pi} - \varepsilon_g$*, where* $\varepsilon_g \to 0$ *when* $g \to \infty$*.*

2. Lower bound constructions

At this point, it is convenient to introduce a few definitions.

Definition 1. *We say that* $\mathcal{A}$ *is a* $B_2^*[g]$ *set if any integer* n *has at most* g *representations of the form* $n = a + a'$ *with* $a, a' \in \mathcal{A}$*. We write* $r_{\mathcal{A}}^*(n)$ *for the number of such representations.*

Definition 2. *We say that* $\mathcal{A}$ *is a Sidon set* $\pmod m$ *if* $a_1 + a_2 \equiv a_3 + a_4 \pmod m \implies \{a_1, a_2\} = \{a_3, a_4\}$*, where* $a_i \in \mathcal{A}$*.*

All the known lower bounds for $\beta(g)$ were obtained from the next lemma (see [1]).

Lemma 1. *Let* $\mathcal{A} = \{0 = a_1 < \ldots < a_k\}$ *be a* $B_2^*[g]$ *set and* $\mathcal{C} \subseteq [1, m]$ *a Sidon set* $\pmod m$*. Then* $\mathcal{B} = \cup_{i=1}^k (\mathcal{C} + ma_i)$ *is a* $B_2[g]$ *set in* $[1, m(a_k+1)]$ *with* $k|\mathcal{C}|$ *elements.*

Remark 2.1. *The lemma shows how to obtain a* $B_2[g]$ *set by carefully arranging (with a dilation of a* $B_2^*[g]$ *set) several copies of a Sidon set* $\pmod m$.

Proof. To prove that $\mathcal{B}$ is a $B_2[g]$ set, suppose that we have

$$b_{1,1} + b_{2,1} = \cdots = b_{1,g+1} + b_{2,g+1} \tag{2}$$

for some $b_{1,j}, b_{2,j} \in \mathcal{B}$. We can write each $b_{i,j} = c_{i,j} + ma_{i,j}$ in a unique way with $c_{i,j} \in \mathcal{C}$ and $a_{i,j} \in \mathcal{A}$. Let us order the elements $b_{i,j}$ of each sum in such a way that for any i, j we have $c_{1,j} \le c_{2,j}$, and when $c_{1,j} = c_{2,j}$ we order them so $a_{1,j} \le a_{2,j}$.

To see that $\mathcal{B}$ is a $B_2[g]$ set we need to check that there exist j and j' such that $b_{1,j} = b_{1,j'}$, $b_{2,j} = b_{2,j'}$.

From (2), and since $\mathcal{C}$ is a Sidon set $\pmod m$, we get $\{c_{1,1}, c_{2,1}\} = \{c_{1,j}, c_{2,j}\}$ for every $1 \le j \le g+1$. Moreover, since we ordered the elements of the equalities in that way, we have $c_{1,1} = c_{1,j}$ and $c_{2,1} = c_{2,j}$ for every j.

Then, the equalities (2) imply these other equalities

$$a_{1,1} + a_{2,1} = a_{1,2} + a_{2,2} = \cdots = a_{1,g+1} + a_{2,g+1}. \tag{3}$$

And since $\mathcal{A}$ satisfies the $B_2^*[g]$ condition there exist j and j' such that $a_{1,j} = a_{1,j'}$ and $a_{2,j} = a_{2,j'}$.

Then, for these j and j' we have that $b_{1,j} = b_{1,j'}$ and $b_{2,j} = b_{2,j'}$. This proves that $\mathcal{B} \in B_2[g]$.

Finally, it is clear that $B \subset \{1, \ldots, (a_k+1)m\}$ and $|\mathcal{B}| = k|\mathcal{C}|$. □

In order to apply Lemma 1 in an efficient way, we have to take dense Sidon sets $\pmod m$. For example, for each prime p we consider $\mathcal{C}_p$ the Sidon set $\pmod m$ with $p-1$ elements and $m = p(p-1)$ discovered by Ruzsa (see [6]).

Given a positive integer N, we write

$$(a_k+1)p_n(p_n-1) < N \le (a_k+1)p_{n+1}(p_{n+1}-1)$$

for suitable consecutive primes, p_n and p_{n+1}. Clearly

$$\frac{F(g,N)}{\sqrt{gN}} \ge \frac{|\mathcal{C}_{p_n}|k}{\sqrt{g(a_k+1)p_{n+1}(p_{n+1}-1)}} \ge \frac{k}{\sqrt{g(a_k+1)}} \cdot \frac{p_n - 1}{p_{n+1}}.$$

Thus

$$\beta(g) = \liminf_{N\to\infty} \frac{F(g,N)}{\sqrt{gN}} \geq \frac{k}{\sqrt{g(a_k+1)}} \liminf_{n\to\infty} \frac{p_n - 1}{p_{n+1}}.$$

Since $\liminf_{n\to\infty} \frac{p_n}{p_{n+1}} = 1$, as a consequence of the Prime Number Theorem, we get

$$\beta(g) \geq \frac{k}{\sqrt{g(a_k+1)}}. \tag{4}$$

So, in order to improve the lower bound for $\beta(g)$, we need to find a set $\mathcal{A} = \{0 = a_1 < \ldots < a_k\}$ which satisfies the $B_2^*[g]$ condition and maximizes the quotient $\frac{k}{\sqrt{g(a_k+1)}}$.

The sets

(a) $\mathcal{A} = \{0, 1, \ldots, g-1\}$
(b) $\mathcal{A} = \{0, 1, \ldots, g-1\} \cup \{g+1, g+3, \ldots, g-1+2\lfloor g/2 \rfloor\}$
(c) $\mathcal{A} = [0, \lfloor g/3 \rfloor) \cup (g - \lfloor g/3 \rfloor + 2\cdot[0, \lfloor g/6 \rfloor))$
$\cup\, [g, g+\lfloor g/3 \rfloor) \cup (2g - \lfloor g/3 \rfloor, 3g - \lfloor g/3 \rfloor]$

provide, respectively, the lower bounds

(a) $\beta(g) \geq 1$
(b) $\beta(g) \geq \frac{g+\lfloor g/2 \rfloor}{\sqrt{g^2+2g\lfloor g/2 \rfloor}} \geq \sqrt{\frac{9}{8}} - \varepsilon_g = 1.060\ldots - \varepsilon_g$
(c) $\beta(g) \geq \frac{g+2\lfloor \frac{g}{3} \rfloor + \lfloor \frac{g}{6} \rfloor}{\sqrt{3g^2 - g\lfloor \frac{g}{3} \rfloor + g}} \geq \sqrt{\frac{121}{96}} - \varepsilon_g = 1.122\ldots - \varepsilon_g$,

cited in the introduction. In the next section we will find a denser set $\mathcal{A}$.

3. Schinzel-Schmidt's conjecture

The convolution $f * f$ in Schinzel-Schmidt's problem can be thought as the continuous version of the function $r^*_{\mathcal{A}}(n)$ and $|f * f|_\infty$ as the analog of the maximum of $r^*_{\mathcal{A}}(n)$.

The idea is to start with a function $f \in \mathcal{F}$ such that $1/|f*f|_\infty$ is close to S (see (1)) and use f as a model to construct our set $\mathcal{A}$. We will use the probabilistic method.

An interesting result in [7] relates the constant S with the coefficients of squares of polynomials. We state that result in a more convenient way for our purposes.

Theorem 2. *For any $\varepsilon > 0$, for any $n > n(\varepsilon)$, there exists a sequence of non negative real numbers $c_0, \ldots, c_{n-1}$ such that*

i) $\sum_{j=0}^{n-1} c_j = \sqrt{n}$.
ii) $c_j \leq n^{-1/6}(1+\varepsilon)$ *for all* $j = 0, \ldots, n-1$.

iii) $\sum_{j<m/2} c_j c_{m-j} \le \frac{1}{2S}(1+\varepsilon)$ *for any* $m = 0, \dots, n-1$.

Proof. We follow the ideas of the proof of assertion (iii) of Theorem 1 in [7]. Let $f \in \mathcal{F}$ with $|f * f|_\infty$ close to $1/S$, say $|f * f|_\infty \le 1/S + 1/n$, and define for $j = 0, \dots, n-1$,

$$a_j = \frac{n}{2t} \int_{(j+1/2-t)/n}^{(j+1/2+t)/n} f(x)\, dx$$

where $t = \lceil 2n^{1/3} \rceil$. We have the following estimate

$$\begin{aligned} \left(\int_r^s f(x)\, dx \right)^2 &\le \iint\limits_{2r \le x+y \le 2s} f(x) f(y)\, dx dy \\ &= \int_{2r}^{2s} \left(\int f(x) f(z-x)\, dx \right) dz \\ &= \int_{2r}^{2s} f * f(z)\, dz \le 2(s-r)(1/S + 1/n) \le 4(s-r), \end{aligned}$$

where in the last inequality we have used the fact that $S \ge 1$ and $n \ge 1$.

In particular, we can deduce $a_j \le (2n/t)^{1/2}$. The idea for proving Theorem 1 (iii) in [7] consists of showing that $\sum_{j=0}^{n-1} a_j \ge n + o(n)$ and $\sum_{j=0}^{m} a_j a_{m-j} \le (1/S)(n + o(n))$ for all m. See [7] for the details.

We define $c_j = a_j \rho$, where $\rho = \frac{\sqrt{n}}{\sum_{j=0}^{n-1} a_j}$. Clearly $\rho \le (1/\sqrt{n})(1 + o(1))$, so $c_j \le n^{-1/6}(1 + o(1))$, $\sum_{j=0}^{n-1} c_j = \sqrt{n}$ and $\sum_{j=0}^{m} c_j c_{m-j} \le (1/S)(1 + o(1))$. □

4. The proof

We will use a special case of Chernoff's inequality (see Corollary 1.9 in [8]):

Proposition 4.1. *(Chernoff's inequality) Let* $X = t_1 + \cdots + t_n$ *where the* t_i *are independent Boolean random variables. Then for any* $\delta > 0$

$$\mathbb{P}(|X - \mathbb{E}(X)| \ge \delta \mathbb{E}(X)) \le 2e^{-\min(\delta^2/4, \delta/2)\mathbb{E}(X)}. \tag{5}$$

Given $\varepsilon > 0$ and the constants c_j's defined in Theorem 2, we consider the probability space of all the subsets $\mathcal{A} \subseteq \{0, 1, 2, \dots, n-1\}$ defined by $\mathbb{P}(j \in \mathcal{A}) = \lambda_n c_j$, where $\lambda_n = \lfloor n^{1/6}/(1+\varepsilon) \rfloor$ (observe that $c_j \lambda_n \le 1$ for n large enough).

Lemma 2. *With the conditions above, given $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$*

$$\mathbb{P}\big(|\mathcal{A}| \geq \lambda_n \sqrt{n}(1-\varepsilon)\big) > 0.9.$$

Proof. Since $|\mathcal{A}|$ is a sum of independent boolean variables and $\mathbb{E}(|\mathcal{A}|) = \sum_{j=0}^{n-1} \mathbb{P}(j \in \mathcal{A}) = \lambda_n \sqrt{n}$ we can apply Chernoff's lemma to deduce that

$$\mathbb{P}\Big(|\mathcal{A}| < \lambda_n \sqrt{n}(1-\varepsilon)\Big) \leq 2e^{-\min(\varepsilon^2/4, \varepsilon/2)\lambda_n \sqrt{n}} < 0.1$$

for n large enough. □

Lemma 3. *Again with the same conditions, given $0 < \varepsilon < 1$, there exists n_1 such that for all $n \geq n_1$*

$$r^*_{\mathcal{A}}(m) \leq \frac{\lambda_n^2}{S}(1+\varepsilon)^3 \quad \textit{for all } m$$

with probability > 0.9.

Proof. Since $r^*_{\mathcal{A}}(m) = \sum_{j=0}^{m} \mathbb{I}(j \in \mathcal{A})\mathbb{I}(m-j \in \mathcal{A})$ is a sum of boolean variables which are not independent, it is convenient to define a new variable ${r^*_{\mathcal{A}}}'(m) = \frac{1}{2} r^*_{\mathcal{A}}(m) - \frac{1}{2}\mathbb{I}(m/2 \in \mathcal{A}) = \sum_{j<m/2} \mathbb{I}(j \in \mathcal{A})\mathbb{I}(m-j \in \mathcal{A})$. Now we can apply Chernoff's inequality to this variable.

Let μ_m denote the expected value of ${r^*_{\mathcal{A}}}'(m)$. We observe that, from the independence of the indicator functions, $\mathbb{E}\big(\mathbb{I}(j \in \mathcal{A})\mathbb{I}(m-j \in \mathcal{A})\big) = \mathbb{P}(j \in \mathcal{A})\mathbb{P}(m-j \in \mathcal{A}) = \lambda_n^2 c_j c_{m-j}$ for every $j < m/2$ and so

$$\mu_m = \sum_{j<m/2} \mathbb{E}\big(\mathbb{I}(j \in \mathcal{A})\mathbb{I}(m-j \in \mathcal{A})\big) = \sum_{j<m/2} \lambda_n^2 c_j c_{m-j} \leq \frac{\lambda_n^2}{2S}(1+\varepsilon),$$

by Theorem 2 iii).

- If $\mu_m \geq \dfrac{\lambda_n^2}{6S}(1+\varepsilon)$, we apply Proposition 4.1 (observe that $\varepsilon < 2$ implies that $\varepsilon^2/4 \leq \varepsilon/2$) to obtain

$$\begin{aligned} \mathbb{P}\left({r^*_{\mathcal{A}}}'(m) \geq \frac{\lambda_n^2}{2S}(1+\varepsilon)^2\right) &\leq \mathbb{P}\big({r^*_{\mathcal{A}}}'(m) \geq \mu_m(1+\varepsilon)\big) \\ &\leq 2\exp\left(-\frac{\mu_m \varepsilon^2}{4}\right) \\ &\leq 2\exp\left(-\frac{\lambda_n^2}{24S}(1+\varepsilon)\varepsilon^2\right). \end{aligned}$$

- If $\mu_m = 0$ then ${r^*_{\mathcal{A}}}'(m) = 0$.

- If $0 < \mu_m < \frac{\lambda_n^2}{6S}(1+\varepsilon)$, for $\delta = \frac{\lambda_n^2}{\mu_m 2S}(1+\varepsilon)^2 - 1 \geq 2$ (now $\delta/2 \leq \delta^2/4$) we obtain

$$\begin{aligned}\mathbb{P}\left(r_{\mathcal{A}}^{*\prime}(m) \geq \frac{\lambda_n^2}{2S}(1+\varepsilon)^2\right) &= \mathbb{P}\big(r_{\mathcal{A}}^{*\prime}(m) \geq \mu_m(1+\delta)\big)\\ &\leq 2\exp(-\delta\mu_m/2)\\ &\leq 2\exp\left(-\frac{\lambda_n^2}{4S}(1+\varepsilon)^2 + \frac{\mu_m}{2}\right)\\ &\leq 2\exp\left(-\frac{\lambda_n^2}{4S}(1+\varepsilon)^2 + \frac{\lambda_n^2}{12S}(1+\varepsilon)\right)\\ &\leq 2\exp\left(-\frac{\lambda_n^2}{6S}(1+\varepsilon)^2\right).\end{aligned}$$

Then

$$\begin{aligned}&\mathbb{P}\left(r_{\mathcal{A}}^{*\prime}(m) \geq \frac{\lambda_n^2}{2S}(1+\varepsilon)^2 \text{ for some } m\right)\\ &\leq 2n\left(\exp\left(-\frac{\lambda_n^2}{24S}(1+\varepsilon)\varepsilon^2\right) + \exp\left(-\frac{\lambda_n^2}{6S}(1+\varepsilon)^2\right)\right) < 0.1\end{aligned}$$

for n large enough.

Because of the way we defined $r_{\mathcal{A}}^{*\prime}(m)$, this means

$$\mathbb{P}\left(r_{\mathcal{A}}^{*}(m) \geq \frac{\lambda_n^2}{S}(1+\varepsilon)^2 + \mathbb{I}(m/2 \in \mathcal{A}) \text{ for some } m\right) < 0.1,$$

so

$$\mathbb{P}\left(r_{\mathcal{A}}^{*}(m) \geq \frac{\lambda_n^2}{S}(1+\varepsilon)^3 \text{ for some } m\right) < 0.1$$

for n large enough.

□

Lemmas 1 and 2 imply that for any $0 < \varepsilon < 1$, for $n \geq n(\varepsilon) = \max(n_0, n_1)$ the probability that $|\mathcal{A}| \geq \lambda_n\sqrt{n}(1-\varepsilon)$ and $r_{\mathcal{A}}^*(m) \leq \frac{\lambda_n^2}{S}(1+\varepsilon)^3$ for all m is greater than 0.8. We now choose one of these sets $\mathcal{A} \subset \{0, \dots, n-1\}$ for a suitable n.

Write $g_\varepsilon = \lfloor \frac{\lambda_{n(\varepsilon)}^2}{S}(1+\varepsilon)^3 \rfloor$. For any $g \geq g_\varepsilon$ we take n such that $g = \lfloor \frac{\lambda_n^2}{S}(1+\varepsilon)^3 \rfloor$ (this is possible because $\frac{\lambda_n^2}{S}(1+\varepsilon)^3$ grows slower than n). Thus, for $g \geq g_\varepsilon$,

$$\beta(g) \geq \frac{|\mathcal{A}|}{g^{1/2}n^{1/2}} \geq \frac{\lambda_n\sqrt{n}(1-\varepsilon)}{(\lambda_n/\sqrt{S})(1+\varepsilon)^{3/2}n^{1/2}} = \sqrt{S}\frac{1-\varepsilon}{(1+\varepsilon)^{3/2}},$$

which completes the proof for the lower bound in Theorem 1, since we can take ε arbitrary small.

To obtain the upper bound in Theorem 1, we will use the following result (assertion (ii) of Theorem 1 in [7]):

Theorem 3. *Let S be the Schinzel-Schmidt constant and $\mathcal{Q} = \{Q : Q \in \mathbb{R}_{\geq 0}[x],\ Q \not\equiv 0,\ \deg(Q) < N\}$. Then*

$$\frac{1}{N} \sup_{Q \in \mathcal{Q}} \frac{|Q^2(x)|_1}{|Q^2(x)|_\infty} \leq S,$$

where $|P|_1$ is the sum and $|P|_\infty$ the maximum of the coefficients of a polynomial, P.

Given a $B_2[g]$ set, $\mathcal{A} \subseteq \{0, \dots, N-1\}$, we define the polynomial $Q_{\mathcal{A}}(x) = \sum_{a \in \mathcal{A}} x^a$, so $Q^2_{\mathcal{A}}(x) = \sum_n r^*_{\mathcal{A}}(n) x^n$. Theorem 3 says that, in particular,

$$S \geq \frac{1}{N} \sup_{\mathcal{A} \subseteq \{0,\dots,N-1\}} \frac{|\mathcal{A}|^2}{2g} = \frac{F^2(g,N)}{2gN},$$

and so $\dfrac{F(g,N)}{\sqrt{gN}} \leq \sqrt{2S}$.

References

[1] J. Cilleruelo, I. Ruzsa, C. Trujillo, *Upper and lower bounds for finite $B_2[g]$ sequences*, J. Number Theory **97**, no. 1, 26-34 (2002).

[2] B. Green *The number of squares and $B_h[g]$ sequences*, Acta Arithmetica **100**, no. 4, 365-390 (2001)

[3] M. Kolountzakis, *The density of $B_h[g]$ sequences and the minimun of dense cosine sums*, J. Number Theory **56**, 4-11 (1996).

[4] G. Martin, K. O'Bryant, *Constructions of generalized Sidon sets*, J. Comb. Theory, Ser. A **113**, no 4, 591-607 (2006).

[5] G. Martin, K. O'Bryant, *The Symmetric Subset Problem in Continuous Ramsey Theory*, Experiment. Math. Volume **16**, no 2, 145-166 (2007).

[6] I. Ruzsa, *Solving a linear equation in a set of integers I*, Acta Arithmetica, **65**, 259-282 (1993).

[7] A. Schinzel, W. M. Schmidt, *Comparison of L^1- and $L^\infty-$ norms of squares of polynomials*, Acta Arithmetica, **104**, no 3 (2002).

[8] T. Tao, V. Vu, *Additive Combinatorics.* Cambridge Studies in Advanced Mathematics, **105** (2006).

[9] G. Yu *An upper bound for $B_2[g]$ sets*, Journal of Number Theory, **122**, 211-220 (2007)

Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049-Madrid.

E-mail address: franciscojavier.cilleruelo@uam.es, c.vinuesa@uam.es.