DENSE INFINITE $B_h$ SEQUENCES

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Abstract. For $h = 3$ and $h = 4$ we prove the existence of infinite $B_h$ sequences $B$ with counting function

$$B(x) = x\sqrt{(h-1)x + (h-1) + o(1)}.$$ 

This result extends a construction of I. Ruzsa for $B_2$ sequences.

1. Introduction

Let $h \geq 2$ be an integer. We say that a sequence $B$ of positive integers is a $B_h$ sequence if all the sums

$$b_1 + \cdots + b_h, \quad (b_k \in B, \ 1 \leq k \leq h),$$

are distinct subject to $b_1 \leq b_2 \leq \ldots \leq b_h$. The study of the size of finite $B_h$ sets (or the growing of the counting function of infinite $B_h$ sequences) is a classic topic in combinatorial number theory. We define

$$F_h(n) = \max\{|B| : B \text{ is } B_h, B \subset [1,n]\}.$$ 

A trivial counting argument proves that $F_h(n) \leq (C_h + o(1))n^{1/h}$ for a constant $C_h$ (see [3] and [7] for non trivial upper bounds for $C_h$) and consequently that $B(x) \ll x^{1/h}$ when $B$ is an infinite $B_h$ sequence.

There are three algebraic constructions ([2], [12] and [6]) of finite $B_h$ sets showing that $F_h(n) \geq n^{1/h}(1 + o(1))$. It is probably true that $F_h(n) \sim n^{1/h}$ but this is an open problem, except for the case $h = 2$ for which Erdős and Turán [5] did prove that $F_2(n) \sim n^{1/2}$. It is unknown whether $\lim_{n \to \infty} F_h(n)/n^{1/h}$ exists for $h \geq 3$. For further information about $B_h$ sequences see [8, § II.2] or [10].

Erdős conjectured for all $\epsilon > 0$ the existence of an infinite $B_h$ sequence $B$ with counting function $B(x) \gg x^{1/h-\epsilon}$. It is believed that $\epsilon$ cannot be removed from the last exponent, however this has only been proved for $h$ even. On the other hand, the greedy algorithm produces an infinite $B_h$ sequence $B$ with

$$(1.1) \quad B(x) \gg x^{1/(2h-1)} \quad (h \geq 2).$$

Until now the exponent $1/(2h-1)$ has been the largest known for the growth of a $B_h$ sequence when $h \geq 3$. For the case $h = 2$, Atjai, Komlós and Szemerédi [1] proved that there exists a $B_2$ sequence (also called a Sidon sequence) with $B(x) \gg (x \log x)^{1/3}$, improving by a power of logarithm the lower bound [1.1]. So

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far the highest improvement of (1.1) for the case $h = 2$ was achieved by Ruzsa (11). He constructed, in a clever way, an infinite Sidon sequence $B$ satisfying

$$B(x) = x \sqrt{2 - o(1)}.$$ 

Our aim is to adapt Ruzsa’s ideas to build dense infinite $B_3$ and $B_4$ sequences and so improve the lower bound (1.1) for $h = 3$ and $h = 4$.

**Theorem 1.1.** For $h = 2, 3, 4$ there is an infinite $B_h$ sequence $B$ with counting function

$$B(x) = x \sqrt{(h-1)^2 + 1 - (h-1) + o(1)}.$$ 

The starting point in Ruzsa’s construction were the numbers $\log p$, $p$ prime, which form an infinite Sidon set of real numbers. Instead we part from the arguments of the Gaussian primes, which also have the same $B_h$ property with the additional advantage of being a bounded sequence. This idea was suggested in [4] to simplify the original construction of Ruzsa and was written in detail for $B_2$ sequences in [9].

We believe that the theorem can be extended to all $h$, but we have not found yet a proof. Indeed we have written the core of the proof for all $h \geq 2$ except for Lemma 3.3 where we have considered only the cases $h = 2, 3, 4$ since the technical difficulties become significantly more involved as $h$ increases.

2. The Gaussian arguments

For each rational prime $p \equiv 1 \pmod{4}$ we consider the Gaussian prime $p$ of $\mathbb{Z}[i]$ such that

$$p := a + bi, \quad p = a^2 + b^2, \quad a > b > 0,$$

so the argument of $p$ defined by $p = \sqrt{p} e^{2\pi i \theta(p)}$ is a real number in the interval $(0, 1/8)$. We will use several times through the paper the following lemma that can be seen as a measure of the quality of the $B_h$ property of this sequence of real numbers.

**Lemma 2.1.** Let $p_1, \ldots, p_h, p'_1, \ldots, p'_h$ be distinct Gaussian primes satisfying $0 < \theta(p_r), \theta(p'_r) < 1/8$, $r = 1, \ldots, h$. The following inequality holds:

$$\left| \sum_{r=1}^{h} (\theta(p_r) - \theta(p'_r)) \right| > \frac{1}{7|p_1 \cdots p_h p'_1 \cdots p'_h|}.$$ 

**Proof.** It is clear that

$$\left(2.1\right) \quad \sum_{r=1}^{h} (\theta(p_r) - \theta(p'_r)) \equiv \theta(p_1 \cdots p_h p'_1 \cdots p'_h) \pmod{1}.$$  

Since $\mathbb{Z}[i]$ is a unique factorization domain, all the primes are in the first octant and are all distinct, the Gaussian integer $p_1 \cdots p_h p'_1 \cdots p'_h$ cannot be a real integer.
Using this fact and the inequality $\arctan(1/x) > 0.99/x$ for $x \geq \sqrt{5 \cdot 13}$ we have

\begin{equation}
(2.2) \quad |\theta(p_1 \cdots p_h p'_1 \cdots p'_h)| \geq \frac{1}{2\pi} \arctan \left( \frac{1}{|p_1 \cdots p_h p'_1 \cdots p'_h|} \right) > \frac{1}{7|p_1 \cdots p_h p'_1 \cdots p'_h|},
\end{equation}

where $\| \cdot \|$ means the distance to $\mathbb{Z}$. The lemma follows from (2.1) and (2.2). □

We illustrate the $B_h$ property of the arguments of the Gaussian primes with a quick construction, based on them, of a finite $B_h$ set which is only a log $x$ factor below the optimal bound.

**Theorem 2.2.** The set $A = \{ \lfloor x\theta(p) \rfloor, |p| \leq (\frac{1}{\sqrt{h}})^{1/2} \} \subset [1, x]$ is a $B_h$ set with $|A| \gg x^{1/h}/\log x$.

**Proof.** If $|\lfloor x\theta(p_1) \rfloor + \cdots + \lfloor x\theta(p_h) \rfloor| = |\lfloor x\theta(p'_1) \rfloor + \cdots + \lfloor x\theta(p'_h) \rfloor|$ then $x|\theta(p_1) + \cdots + \theta(p_h) - \theta(p'_1) - \cdots - \theta(p'_h)| \leq h$.

If the Gaussian primes are distinct then Lemma 2.1 implies that $|\theta(p_1) + \cdots + \theta(p_h) - \theta(p'_1) - \cdots - \theta(p'_h)| > \frac{1}{7|p_1 \cdots p_h p'_1 \cdots p'_h|} \geq h/x,$ which is a contradiction. □

### 3. Proof of Theorem 1.1

We start following the lines of [11] with several adjustments. In the sequel we will write $p$ for a Gaussian prime in the first octant ($0 < \theta(p) < 1/8$).

We fix a number $c_h > h$ which will determine the growth of the sequence we construct. Indeed we will take $c_h = \sqrt{(h - 1)^2 + 1} + (h - 1)$.

#### 3.1. The construction

We will construct for each $\alpha \in [1, 2]$ a sequence of positive integers indexed with the Gaussian primes

$$B_{\alpha} := \{ b_{\alpha} \},$$

where each $b_{\alpha}$ will be built using the development to base 2 of $\alpha \theta(p)$:

$$\alpha \theta(p) = \sum_{i=1}^{\infty} \delta_{ip}2^{-i} \quad (\delta_{ip} \in \{0, 1\}).$$

The role of the parameter $\alpha$ will be clear at a later stage, for the moment it is enough to note that the set $\{ \alpha \theta(p) \}$ obviously keeps the same $B_h$ property of the set $\{ \theta(p) \}$. 
To organize the construction we describe the sequence $B_\alpha$ as a union of finite sets according with the sizes of the indexes:

$$B_\alpha = \bigcup_K B_{\alpha,K}$$

where

$$B_{\alpha,K} = \{ b_p : p \in P_K \}$$

and

$$P_K := \{ p : 2^{-\frac{(K-3)^2}{8h}} \leq |p|^2 < 2^{-\frac{(K-1)^2}{8h}} \}. $$

Now we build the positive integers $b_p \in B_{\alpha,K}$. For any $p \in P_K$ we define $\alpha\theta(p)$ the truncated series of $\alpha \theta(p)$ at the $K^2$-place:

$$\hat{\alpha\theta}(p) := \sum_{i=1}^{K^2} \delta_i p 2^{-i}$$

and combine the digits at places $(j - 1)^2 + 1, \cdots, j^2$ into a single number

$$\Delta_j p = \sum_{i=(j-1)^2+1}^{j^2} \delta_i p 2^{-i} (j = 1, \cdots, K),$$

so that we can write

$$\hat{\alpha \theta}(p) = \sum_{j=1}^{K} \Delta_j p 2^{-j^2}. $$

We observe that if $p \in P_K$ then

$$|\hat{\alpha \theta}(p) - \alpha \theta(p)| \leq 2^{-K^2}. $$

The definition of $b_p$ is informally outlined as follows. We consider the sequence of blocks $\Delta_1 p, \cdots, \Delta_K p$ and re-arrange them opposite to the original left to right arrangement. Then we insert at the left of each $\Delta_j p$ an additional filling block of $2d + 1$ digits, with $d = \lceil \log_2 h \rceil$. At the filling blocks the digits will be always 0 except for only one exception: in the middle of the first filling block (placed to the left of the $\Delta_K$ block) we put the digit 1. This digit will mark which subset $P_K$ the prime $p$ belongs to.

$$\alpha \theta(p) = 0.1001\cdots011\cdots011\cdots \Delta_1 \cdots \Delta_K$$

$$b_p = 0.1001\cdots110\cdots00\cdots0\cdots0\cdots0\cdots0\cdots00\cdots01\cdots0\cdots0\cdots0\cdots0\cdots0\cdots0\cdots0\cdots0\cdots0\cdots0\cdots0\cdots1,$$

The reason to add the blocks of zeroes and the value of $d$ will be clarified just before Lemma 3.2.

More formally, for $p \in P_K$ we define

$$t_p = 2^{K^2+(2d+1)K+(d+1)},$$
and
\[ b_p = t_p + \sum_{j=1}^{K} \Delta_{jp} 2^{(j-1)^2 + (2j+1)(j-1)}. \]

Furthermore we define \( \Delta_{jp} = 0 \) for \( j > K \).

**Remark 3.1.** The construction at [11] was based on the numbers \( \alpha \log p \), with \( p \) rational prime, hence the digits of their integral parts had to be included also in the corresponding integers \( b_p \). Ruzsa solved that problem reserving fixed places for these digits. Since in our construction the integral part of \( \alpha \theta(p) \) is zero we don’t need to care about this.

We observe that distinct primes \( p, q \) provide distinct \( b_p, b_q \). Indeed if \( b_p = b_q \) then \( \Delta_{ip} = \Delta_{iq} \) for all \( i \leq K \). Also \( t_p = t_q \) which means \( p, q \in P_K \), and so
\[ |\theta(p) - \theta(q)| = \alpha^{-1} \sum_{j> K} (\Delta_{jp} - \Delta_{jq}) < 2^{-K^2}. \]

Now if \( p \neq q \) then Lemma [2.1] implies that \( |\theta(p) - \theta(q)| > \frac{1}{2p^2q^2} > 2^{-(K-1)^2 - 3} \). Combining both inequalities we have a contradiction for \( K \geq h + 1 \geq 3 \). So we assume \( K \geq h + 1 \) through all the paper.

Since all the integers \( b_p \) are distinct, we have that
\[
|B_{\alpha, K}| = |P_K| = \pi \left( 2^{\frac{(K-1)^2}{c_K}}; 1, 4 \right) - \pi \left( 2^{\frac{(K-2)^2}{c_K}}; 1, 4 \right) \geq \frac{2K^2}{K^2}.
\]

We observe also that
\[ b_p < 2^{K^2 + (2d+1)K + (d+1) + 1}. \]

Using these estimates we can easily prove that \( B_{\alpha}(x) = x^{\frac{1}{c_K} + o(1)} \). Indeed, if \( K \) is the integer such that \( 2^{K^2 + (2d+1)K + (d+1) + 1} < x \leq 2^{K^2 + (2d+1)(K+1) + (d+1) + 1} \) then we have
\[
B_{\alpha}(x) \geq |B_{\alpha, K}| \geq 2^{\frac{1}{c_K} K^2 (1 + o(1))} = x^{\frac{1}{c_K} + o(1)}.
\]

For the upper bound we have
\[
B_{\alpha}(x) \leq \# \{ p : |p|^2 \leq 2^{\frac{K^2}{c_K}} \} \leq 2^{\frac{K^2}{c_K}} = x^{\frac{1}{c_K} + o(1)}.
\]

There is a compromise at the choice of a particular value of \( c_K \) for the construction. On one hand larger values of \( c_K \) capture more information from the Gaussian arguments which brings the sequence \( B_{\alpha} = \{ b_p \} \) closer to being a \( B_h \) sequence. On the other hand smaller values of \( c_K \) provide higher growth of the counting function of \( B_{\alpha} \).

Clearly \( B_{\alpha} \) would be a \( B_h \) sequence if for all \( l = 2, \cdots, h \) it does not contain \( b_{p_1}, \cdots, b_{p_l}, b_{p'_1}, \cdots, b_{p'_l} \) satisfying
\[
\begin{align*}
   b_{p_1} + \cdots + b_{p_l} &= b_{p'_1} + \cdots + b_{p'_l}, \\
   \{b_1, \cdots, b_l\} \cap \{b'_1, \cdots, b'_l\} &= \emptyset, \\
   b_{p_1} &\geq \cdots \geq b_{p_l} \quad \text{and} \quad b_{p'_1} \geq \cdots \geq b_{p'_l}.
\end{align*}
\]
We say that \((p_1,\ldots,p_l,p'_1,\ldots,p'_l)\) is a bad 2l-tuple if the equation \((3.7)\) is satisfied by the corresponding \(b_p\).

The sequence \(B_h = \{b_p\}\) we have constructed is not properly a \(B_h\) sequence. Some repeated sums as in \((3.7)\) will eventually appear, but the particular way to construct the elements \(b_p\) will allow us to study these bad 2l-tuples and to prove that there are not too many repeated sums. Then removing the bad elements involved in these bad 2l-tuples we obtain a true \(B_h\) sequence.

Now we will see why blocks of zeroes were added to the binary development of \(b_p\). We can identify each \(b_p\) with a vector as follows:

\[
\begin{align*}
b_{p_1} &= (\cdots, 1, \Delta_{K_1p_1}, 0, \cdots, 0, \Delta_{K_2p_1}, 0, \cdots, 0, \Delta_{K_3p_1}, 0, \cdots, 0, \Delta_{2p_1}, 0, \Delta_{1p_1}) \\
b_{p_2} &= (\cdots, 0, \cdots, \cdots, 1, \Delta_{K_2p_2}, 0, \cdots, 0, \Delta_{K_3p_2}, 0, \cdots, 0, \Delta_{2p_2}, 0, \Delta_{1p_2}) \\
&\vdots \\
b_{p_l} &= (\cdots, 0, \cdots, \cdots, 0, \cdots, 0, \Delta_{K_lp_l}, 0, \cdots, 0, \Delta_{2p_l}, 0, \Delta_{1p_l}),
\end{align*}
\]

where each comma represents one block of \(d\) zeroes. Note that the leftmost part of each vector is null. The value of \(d = \lfloor \log_2 h \rfloor\) has been chosen to prevent the propagation of the carry between any two consecutive coordinates separated by a comma in the above identification. So when we sum no more than \(h\) integers \(b_p\) we can just sum the corresponding vectors coordinate-wise. This argument implies the following lemma.

**Lemma 3.2.** Let \((p_1,\ldots,p_l,p'_1,\ldots,p'_l)\) be a bad 2l-tuple. Then there are integers \(K_1 \geq \cdots \geq K_l\) such that \(p_1,p'_1 \in P_{K_1},\ldots,p_l,p'_l \in P_{K_l}\), and we have

\[(3.9) \quad \alpha\theta(p_1) + \cdots + \alpha\theta(p_l) = \alpha\theta(p'_1) + \cdots + \alpha\theta(p'_l).\]

**Proof.** Note that \((3.7)\) implies \(t_{p_1} + \cdots + t_{p_l} = t_{p'_1} + \cdots + t_{p'_l}\) and \(\Delta_{j,p_1} + \cdots + \Delta_{j,p_l} = \Delta_{j,p'_1} + \cdots + \Delta_{j,p'_l}\) for each \(j\). Using \((3.2)\) we conclude \((3.9)\). As the bad 2l-tuple satisfies condition \((3.8)\) we deduce that \(p_r,p'_r\) belongs to the same \(P_{K_r}\) for all \(r\). \(\square\)

According with the lemma above we will write \(E_{2l}(\alpha;K_1,\ldots,K_l)\) for the set of bad 2l-tuples \((p_1,\ldots,p_l)\) with \(p_r,p'_r \in P_{K_r}, 1 \leq r \leq l\) and

\[
E_{2l}(\alpha; K) = \bigcup_{K_1 \leq \cdots \leq K_l = K} E_{2l}(\alpha; K_1,\ldots,K_l),
\]

where \(K = K_1\). Also we define the set

\[
\text{Bad}_{\alpha,K} = \{b_p \in B_{\alpha,K} : b_p \text{ is the largest element involved in some equation } (3.7)\}.
\]

It is clear that \(\sum_{t \leq h} |E_{2l}(\alpha,K)|\) is an upper bound for \(|\text{Bad}_{\alpha,K}|\), the number of elements that we need to remove from each \(B_{\alpha,K}\) to get a \(B_h\) sequence.

We do know how to obtain a good upper bound for \(|E_{2l}(\alpha,K)|\) for a concrete \(\alpha\), but we can do it for almost \(\alpha\).
Lemma 3.3. For \( l = 2, 3, 4 \) we have
\[
\int_1^2 |E_{2l}(\alpha, K)| \, d\alpha \ll K^{m_1} 2^{\left(\frac{2l-1}{c_h} - 1\right)}(K-1)^2 - 2K
\]
for some \( m_1 \).

The proof of Lemma 3.3 is involved and we postpone it to section §4. We think that Lemma 3.3 holds for any \( l \) but we have not found a proof.

3.2. Last step in the proof of the theorem. For \( h = 2, 3, 4 \) we have that
\[
\int_1^2 \sum_{K} |\text{Bad}_{\alpha,K}| \, d\alpha \ll \sum_{K} \sum_{l \leq h} K^{m_1} 2^{\left(\frac{2l-1}{c_h} - 1\right)}(K-1)^2 - 2K
\]

The last sum is finite for \( c_h = \sqrt{(h-1)^2 + 1} + (h-1) \) which is the largest number for which \( \frac{2(h-1)}{c_h} - 1 - \frac{1}{c_h} \leq 0 \). So for this \( c_h \) the sum \( \sum_K \frac{|\text{Bad}_{\alpha,K}|}{|B_{\alpha,K}|} \) is convergent for almost all \( \alpha \in [1,2] \). We take one of these \( \alpha \), say \( \alpha_0 \), and consider the sequence
\[
\mathcal{B} = \bigcup_K (\mathcal{B}_{\alpha_0,K} \setminus \text{Bad}_{\alpha_0,K}).
\]

We claim that this sequence satisfies the condition of the theorem. It is clear that this sequence is a \( B_h \) sequence because we have destroyed all the repeated sums of \( h \) elements of \( \mathcal{B}_{\alpha_0} \) removing all the bad elements from each \( \mathcal{B}_{\alpha_0,K} \).

On the other hand, the convergence of \( \sum_K \frac{|\text{Bad}_{\alpha_0,K}|}{|B_{\alpha_0,K}|} \) implies that \( |\text{Bad}_{\alpha_0,K}| = o(|B_{\alpha_0,K}|) \). We proceed as in (3.6) to estimate the counting function of \( \mathcal{B} \). For any \( x \) let \( K \) the integer such that
\[
2^{K^2 + (2d+1)K + (d+1) + 1} < x \leq 2^{(K+1)^2 + (2d+1)(K+1)+(d+1)+1}.
\]

We have
\[
\mathcal{B}(x) \geq |\mathcal{B}_{\alpha_0,K}| - |\text{Bad}_{\alpha_0,K}| = |\mathcal{B}_{\alpha_0,K}|(1 + o(1)) \gg K^{-\frac{1}{c_h} + o(1)}.
\]

For the upper bound, we have
\[
\mathcal{B}(x) \leq \mathcal{B}_{\alpha_0}(x) = x^{\frac{1}{c_h} + o(1)}.
\]

Thus
\[
\mathcal{B}(x) = x^{\sqrt{(h-1)^2 + 1} - (h-1) + o(1)}.
\]

4. Proof of Lemma 3.3

The proof of Lemma 3.3 will be a consequence of Propositions 4.5, 4.6 and 4.7. Before proving these propositions we need to study some properties of the bad \( 2l \)-tuples and an auxiliary lemma about visible lattice points.
4.1. Some properties of the 2l-tuples. For any 2l-tuple \((p_1, \ldots, p'_l)\) we define the numbers \(\omega_s = \omega_s(p_1, \ldots, p'_l)\) by

\[
\omega_s = \sum_{r=1}^{s} (\theta(p_r) - \theta(p'_r)) \quad (s \leq l).
\]

The next two lemmas contain several properties of the bad 2l-tuples.

**Lemma 4.1.** Let \((p_1, \ldots, p_l, p'_1, \ldots, p'_l)\) be a bad 2l-tuple with \(K_1 \geq \cdots \geq K_l\) given by Lemma 3.2. We have

\[
\omega_s = \sum_{r=1}^{s} (\theta(p_r) - \theta(p'_r)) \quad (s \leq l).
\]

Proof. i) This is a consequence of (3.9) and (3.3):

\[
|\omega_l| = \frac{1}{\alpha} \left| \sum_{r=1}^{l} (\alpha \theta(p_r) - \alpha \theta(p'_r)) \right| \leq \frac{1}{\alpha} \left( 2^{-K_1^2} + \cdots + 2^{-K_l^2} \right) \leq l2^{-K_i^2},
\]

since \(\alpha \geq 1\).

ii) Lemma 2.1 implies

\[
|\theta(p_l) - \theta(p'_l)| \geq \frac{1}{l |p_l||p'_l|} \geq 2^{-3 - \frac{1}{c_h} (K_1 - 1)^2}
\]

and so,

\[
|\omega_{l-1}| = |\omega_l + \theta(p'_l) - \theta(p_l)| \geq |\theta(p'_l) - \theta(p_l)| - |\omega_l| \geq 2^{-3 - \frac{1}{c_h} (K_1 - 1)^2} - l2^{-K_i^2} \geq 2^{-3 - \frac{1}{c_h} (K_1 - 1)^2 - 3},
\]

since \(K_l \geq h + 1 \geq l + 1\).

iii) Lemma 2.1 implies also that

\[
|\omega| = \frac{1}{l} \left| \sum_{r=1}^{l} (\theta(p_r) - \theta(p'_r)) \right| > \frac{1}{l |p_1 \cdots p'_l|} > 2^{-3 - \frac{1}{c_h} \sum_{r=1}^{l} (K_r - 1)^2}.
\]

Combining this with i) we obtain

\[
(K_l - 1)^2 \leq \frac{1}{c_h - 1} \left( (K_1 - 1)^2 + \cdots + (K_{l-1} - 1)^2 \right) + \frac{\log_2 l - 2K_l + 4}{1 - 1/c_h}.
\]

The last term is negative because \(K_l \geq h + 1 \geq l + 1\) and \(l \geq 2\). □

**Lemma 4.2.** Suppose that \((p_1, \ldots, p_l, p'_1, \ldots, p'_l)\) is a bad 2l-tuple. Then for any \(\omega_s = \sum_{r=1}^{s} (\theta(p_r) - \theta(p'_r))\) with \(1 \leq s \leq l - 1\) we have

\[
\|\alpha 2^{K_s^2} + \omega_s\| \leq s 2^{K_s^2 - K_l^2} \quad (s = 1, \ldots, l - 1),
\]

where \(\| \cdot \|\) means the distance to the nearest integer.
Proof. Since $0 \leq \alpha \theta(p) - \alpha \theta(p') \leq 2^{-K^2}$ when $p \in P_K$, we can write

$$2^{K^2+1} \sum_{r=1}^{s} (\theta(p_r) - \theta(p'_r)) = 2^{K^2+1} \sum_{r=1}^{s} (\alpha \theta(p_r) - \alpha \theta(p'_r)) + \epsilon_s,$$

with $|\epsilon_s| \leq s2^{K^2+1-K^2}$. By Lemma 3.2 we know that $\sum_{r=1}^{s} (\alpha \theta(p_r) - \alpha \theta(p'_r)) = 0$ when $(p_1, \cdots, p_l, p'_1, \cdots, p'_l)$ is a bad 2-tuple. Using this and (3.1) we have that

$$2^{K^2+1} \sum_{r=s+1}^{s+1} (\alpha \theta(p'_r) - \alpha \theta(p_r)) = \sum_{r=s+1}^{l} \sum_{i=1}^{K^2} \sum_{r=1}^{K^2} (\delta(p'_r) - \delta(p_r))$$

is an integer, which proves (4.3).

□

Lemma 4.3.

$$\int_{1}^{2} |E_2(\alpha; K_1, \ldots, K_l)| d\alpha \ll 2^{K^2-K^2} \sum_{(p_1, \ldots, p_l)} \sum_{|\omega| < 2^{-K^2}} \left| \frac{|\omega_j|}{|\omega|} \prod_{j=1}^{l-2} \left( \frac{|\omega_j|}{|\omega_{j+1}|} + 1 \right) \right|$$

Proof. We have seen that if $(p_1, \ldots, p'_l) \in E_2(\alpha; K_1, \ldots, K_l)$, then

$$\left| \alpha 2^{K^2+1}\omega_s \right| \leq s2^{K^2+1-K^2}, s = 1, \ldots, l-1.$$  (4.3)

Then there exists integers $j_s$, $s = 1, \cdots, l-1$ such that

$$\left| \alpha - \frac{j_s}{2^{K^2+1}\omega_s} \right| \leq \frac{s2^{-K^2}}{|\omega_s|}.  \tag{4.4}$$

Writing $I_{j_1}, \cdots, I_{j_s}$ for the intervals defined by the inequalities (4.4) we have

$$\mu\{ \alpha : (p_1, \ldots, p'_l) \in E_2(\alpha; K_1, \ldots, K_l) \} \leq \sum_{j_1, \ldots, j_{l-1}} |I_{j_1} \cap \cdots \cap I_{j_{l-1}}|$$

$$\leq \frac{2^{-K^2+1}}{|\omega|} \# \{(j_1, \ldots, j_{l-1}) : \bigcap_{i=1}^{l-1} I_{j_i} \neq \emptyset \}.$$

To estimate this last cardinality note that for all $s = 1, \ldots, l-2$ we have

$$\left| \frac{j_s}{2^{K^2+1}\omega_s} - \frac{j_{s+1}}{2^{K^2+2}\omega_{s+1}} \right| < \left| \alpha - \frac{j_s}{2^{K^2+1}\omega_s} \right| + \left| \alpha - \frac{j_{s+1}}{2^{K^2+2}\omega_{s+1}} \right| < \frac{s2^{-K^2}}{|\omega_s|} + \frac{(s+1)2^{-K^2+1}}{|\omega_{s+1}|}.$$

Thus,

$$j_s - j_{s+1} \frac{2^{K^2+1}\omega_s}{2^{K^2+2}\omega_{s+1}} < s2^{-K^2+K^2} + \frac{(s+1)|\omega_{s+1}|}{|\omega_s|}.$$  (4.5)

We observe that for each $s = 1, \ldots, l-2$ and for each $j_{s+1}$, the number of $j_s$ satisfying (4.5) is bounded by $2 \left( s2^{-K^2+K^2} + \frac{(s+1)|\omega_{s+1}|}{|\omega_s|} \right) + 1 \ll \frac{|\omega_{s+1}|}{|\omega_s|} + 1$. 

The number of $j_{s+1}$ is bounded by $2^{K^2+1} - 1$ and for each $j_{s+1}$, the number of $j_s$ satisfying (4.5) is bounded by $2 \left( s2^{-K^2+K^2} + \frac{(s+1)|\omega_{s+1}|}{|\omega_s|} \right) + 1 \ll \frac{|\omega_{s+1}|}{|\omega_s|} + 1$. 

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Note also that

\[ |j_{l-1}| \leq 2^{K_l^2} |\omega_{l-1}| \left( \left| \frac{j_{l-1}}{2^{K_l^2} \omega_{l-1}} - \alpha \right| + |\alpha| \right) \]

\[ \leq 2^{K_l^2} |\omega_{l-1}| \left( (l-1)2^{K_{l-1}^2} + 2 \right) \]

\[ \leq l - 1 + 2^{K_l^2+1} |\omega_{l-1}| \]

\[ \ll 2^{K_l^2+1} |\omega_{l-1}| \]

In the last step we have used the condition iii).

Putting all these observations together we complete the proof. 

4.2. Visible points. We will denote by \( V \) the set of lattice points visible from the origin excluding \((1,0)\). In the next subsection we will use several times the following lemma.

**Lemma 4.4.** The number of integral lattice points visible from the origin that are contained in a circular sector centred at the origin of radius \( R \) and angle \( \epsilon \) is at most \( \epsilon R^2 + 1 \). In other words, for any real number \( t \)

\[ \# \{ \nu \in V, |\nu| < R, \|\theta(\nu) + t\| < \epsilon \} \leq \epsilon R^2 + 1. \]

Furthermore,

\[ \# \{ \nu \in V, |\nu| < R, \|\theta(\nu)\| < \epsilon \} \leq \epsilon R^2. \]

**Proof.** We arrange the \( N \) lattice points inside de sector \( \nu_1, \nu_2, \cdots, \nu_N \) that are visible from the origin \( O \) by the value of their argument so that \( \theta(\nu_i) < \theta(\nu_j) \) for \( 1 \leq i < j \leq N \). For each \( i = 1, \ldots, N - 1 \) the three lattice points \( O, \nu_i, \nu_{i+1} \) define a triangle \( T_i \) with \( \text{Area}(T_i) \geq 1/2 \), that does not contain any other lattice point.

Since all \( T_i \) are inside the circular sector their union covers at most the area of the sector. They don’t overlap pairwise, thus

\[ N - 1 \leq \sum_{i=1}^{N} 2 \cdot \text{Area}(T_i) = 2 \cdot \text{Area} \left( \bigcup_{i=1}^{N} T_i \right) \leq R^2 \epsilon. \]

For the last statement we add \( \nu_0 = (1,0) \) to our \( N \) visible points \( \nu_1, \ldots, \nu_N \) and we repeat the argument. 

4.3. Estimates for the number of bad 2l-tuples \( (l = 2, 3, 4) \). We start with the case \( l = 2 \) which was considered by Ruzsa for \( B_2 \) sequences. In the sequel all lattice points \( \nu \) appearing in the proofs belong to \( V \) and Lemma 4.4 applies.

**Proposition 4.5.** For any \( c_h > 2 \) we have

\[ \int_1^2 |E_4(\alpha; K)| \, d\alpha \ll K^{2\left( \frac{2}{8-h-1} - (K-1)^2 \right) - 2K}. \]

**Proof.** Lemma 4.3 implies that

\[ \int_1^2 |E_4(\alpha; K_1, K_2)| \, d\alpha \ll 2^{K_2^2 - K_1^2} \# \{ (p_1, p_2^1, p_2) : |\omega_2| \leq 2 \cdot 2^{-K_2^2} \}. \]
We get an upper bound for the second factor here by using Lemma 4.4 to estimate the number of lattice points of the form \( \nu_2 = p_1 p_1^1 p_2 p_2^1 \) such that \( \| \theta(\nu_2) \| < \epsilon, |\nu_2| < R \), with \( \epsilon = 2 \cdot 2^{-K_2^2} \) and \( R = 2^{2n [(K_1 - 1)^2 + (K_2 - 1)^2]} \). We have

\[
\int_1^2 |E_4(\alpha; K_1, K_2)| \, d\alpha \ll 2^{K_2^2 - K_1^2} \cdot 2^{\frac{n}{2} [(K_1 - 1)^2 + (K_2 - 1)^2] - K_2^2} \\
\ll 2^{\frac{n}{2} [(K_1 - 1)^2 + (K_2 - 1)^2] - K_2^2}.
\]

By Lemma 4.1 iv) we also have \((K_2 - 1)^2 \leq \frac{(K_1 - 1)^2}{c_h - 1}\), thus

\[
\int_1^2 |E_4(\alpha; K_1, K_2)| \, d\alpha \ll 2^{\frac{n}{2} [(K_1 - 1)^2 - 2K_1},
\]

and

\[
\int_1^2 |E_4(\alpha; K)| \, d\alpha = \sum_{K_2 \leq K} \int_1^2 |E_4(\alpha; K, K_2)| \, d\alpha \ll K^2 2^{\frac{n}{2} [(K_1 - 1)^2 - 2K_1}.
\]

Proposition 4.6. For any \( c_h > 3 \) we have

\[
\int_1^2 |E_0(\alpha; K)| \, d\alpha \ll K^2 2^{\frac{n}{2} [(K_1 - 1)^2 - 2K_1}.
\]

Proof. Lemma 4.3 says that

\[
\int_1^2 |E_0(\alpha; K_1, K_2, K_3)| \, d\alpha \ll 2^{K_2^2 - K_1^2} \sum_{(\mathfrak{p}_1, \ldots, \mathfrak{p}_3)} \frac{1}{|\omega_1|}.
\]

Applying Lemma 4.4 by writing \( \nu_1 = p_1 \overline{p}_1 \) and \( \nu_2 = p_2 \overline{p}_2 \overline{p}_2', \) we have that

\[
\sum_{(\mathfrak{p}_1, \ldots, \mathfrak{p}_3)} \frac{1}{|\omega_1|} \ll \sum_m 2^m \# \{(\mathfrak{p}_1, \ldots, \mathfrak{p}_3') : |\omega_1| \leq 2^{-m}, |\omega_3| \leq 3 \cdot 2^{-K_2^2}
\ll \sum_m 2^m \# \{(\nu_1, \nu_2) : \|\theta(\nu_1)\| \leq 2^{-m}, \|\theta(\nu_1) + \theta(\nu_2)\| \leq 3 \cdot 2^{-K_2^2}\}
\ll \sum_m 2^m \sum_{|\theta(\nu_1)| \leq 2^{-m}} \# \{\nu_2 : \|\theta(\nu_1) + \theta(\nu_2)\| \leq 3 \cdot 2^{-K_2^2}\}
\ll \sum_m 2^m \cdot 2^{\frac{n}{2} [(K_2 - 1)^2 + (K_3 - 1)^2] - K_2^2}.
\]
Thus, using the inequalities $K_3 \leq K_2 \leq K_1$ and $(K_3 - 1)^2 \leq \frac{(K_2 - 1)^2 + (K_1 - 1)^2}{c_h - 1}$ we have

$$
\int_1^2 |E_0(\alpha; K_1, K_2, K_3)| \, d\alpha \ll K_1^4 K_2^2 K_3^2 - K_3^7 + \frac{\epsilon}{K_3} (K_1 - 1)^2 \left( \frac{2}{\epsilon} (K_2 - 1)^2 + (K_1 - 1)^2 - K_3^2 + 1 \right)
\ll
K_1^2 K_2^2 - K_3^7 + \frac{\epsilon}{K_3} ((K_1 - 1)^2 + (K_2 - 1)^2 + (K_3 - 1)^2) + K_1^2 (K_2 - 1)^2 - K_3^2 + \frac{\epsilon}{K_3} (K_1 - 1)^2
\ll
K_1^2 K_2^2 (\frac{\epsilon}{K_3} - 1) (K_1 - 1)^2 - 2K_1 + K_1^2 (\frac{\epsilon}{K_3} - 1) (K_1 - 1)^2 - \frac{2}{\epsilon} (K_1 - 1)^2
\ll
K_1^2 (\frac{\epsilon}{K_3} - 1) (K_1 - 1)^2 - 2K_1.
$$

Then we can write

$$
\int_1^2 |E_0(\alpha; K)| \, d\alpha = \sum_{K_3 \leq K_2 \leq K_1} \int_1^2 |E_0(\alpha; K, K_2, K_3)| \, d\alpha \ll K^4 (\frac{\epsilon}{K_3 - 1} (K_1 - 1)^2 - 2K).
$$

as claimed.

\[\square\]

**Proposition 4.7.** For any $c_h > 4$ we have

$$
\int_1^2 |E_0(\alpha; K)| \, d\alpha \ll K^2 2 (\frac{\epsilon}{K_3 - 1} (K_1 - 1)^2 - 2K).
$$

**Proof.** Considering the two possibilities $|\omega_1| < |\omega_2|$ and $|\omega_1| \geq |\omega_2|$ we get the inequality $|\omega|_h (|\omega_1| + 1) (|\omega_2| + 1) \ll \frac{|\omega_1|}{|\omega|_h} (|\omega_1| + 1) \frac{1}{|\omega|_h} \ll \max \left( \frac{1}{|\omega_1|}, \frac{1}{|\omega_2|} \right)$. This combined with Lemma 4.3 implies that

$$
\int_1^2 |E_0(\alpha, K_1, K_2, K_3, K_4)| \, d\alpha \ll 2^{-K_3^2 + K_4^2} \left( \sum_{|\omega_4| \leq 4 \cdot 2^{-K_2^2}} \frac{1}{|\omega_1|} \right) + \sum_{|\omega_4| \leq 4 \cdot 2^{-K_2^2}} \frac{1}{|\omega_2|}.
$$

Applying Lemma 4.4 with the notation $\nu_1 = p_1 \bar{p}_1$ and $\nu_2 = p_2 p_3 p_4 \bar{p}_2 \bar{p}_3 \bar{p}_4$, we have that

$$
\sum_{|\omega_4| \leq 4 \cdot 2^{-K_2^2}} \frac{1}{|\omega_1|} \ll \sum_m 2^m \# \{(p_1, \ldots, p_4) : |\omega_1| < 2^{-m}, |\omega_4| \leq 4 \cdot 2^{-K_2^2} \}
\ll \sum_m 2^m \# \{(\nu_1, \nu_2) : \|\theta(\nu_1)\| \leq 2^{-m}, \|\theta(\nu_1) + \theta(\nu_2)\| \leq 4 \cdot 2^{-K_2^2} \}
\ll \sum_m \# \{(\nu_1, \nu_2) : \|\theta(\nu_1) + \theta(\nu_2)\| \leq 4 \cdot 2^{-K_2^2} \}
\ll \sum_m 2 \frac{1}{\epsilon} (K_1 - 1)^2 \left( \frac{2}{\epsilon} (K_2 - 1)^2 + (K_3 - 1)^2 + (K_4 - 1)^2 - K_3^2 + 1 \right)
\ll \frac{1}{\epsilon} (K_1 - 1)^2 \left( \frac{2}{\epsilon} (K_2 - 1)^2 + (K_3 - 1)^2 + (K_4 - 1)^2 - K_3^2 + 1 \right)
\ll \frac{1}{\epsilon} (K_1 - 1)^2 \left( \frac{2}{\epsilon} (K_2 - 1)^2 + (K_3 - 1)^2 + (K_4 - 1)^2 - K_3^2 + 1 \right).$$
Similarly, but writing now $\nu_1 = p_1p_2p_1p_2$ and $\nu_2 = p_3p_4p_3p_4$, we have

$$
\sum_{\{\nu_1, \nu_2\}} \frac{1}{|\omega_2|} \ll \sum_{m} 2^n \#\{ (\nu_1, \nu_2) : |\omega_2| \leq 2^{-m}, |\omega_4| \leq 4 \cdot 2^{-K^2_2} \}
$$

$$
\ll \sum_{m \leq K^2_2} 2^n \#\{ (\nu_1, \nu_2) : \|\theta(\nu_1)\| \leq 2^{-m}, \|\theta(\nu_1) + \theta(\nu_2)\| \leq 4 \cdot 2^{-K^2_2} \}
$$

$$
+ \sum_{m > K^2_2} 2^n \#\{ (\nu_1, \nu_2) : \|\theta(\nu_1)\| \leq 2^{-m}, \|\theta(\nu_1) + \theta(\nu_2)\| \leq 4 \cdot 2^{-K^2_2} \}
$$

$$
= S_1 + S_2
$$

We observe that if $m \leq K^2_2$ then $\|\theta(\nu_2)\| \leq \|\theta(\nu_1) + \theta(\nu_2)\| + \|\theta(\nu_1)\| \leq 5 \cdot 2^{-m}$.

Thus

$$
S_1 \ll \sum_{m \leq K^2_2} 2^n \#\{ (\nu_1, \nu_2) : \|\theta(\nu_2)\| \leq 5 \cdot 2^{-m}, \|\theta(\nu_1) + \theta(\nu_2)\| \leq 4 \cdot 2^{-K^2_2} \}
$$

$$
\ll \sum_{m \leq K^2_2} 2^{2m} \sum_{\|\theta(\nu_1)\| \leq 5 \cdot 2^{-m}} \#\{ \nu_1 : \|\theta(\nu_1) + \theta(\nu_2)\| \leq 4 \cdot 2^{-K^2_2} \}
$$

$$
\ll \sum_{m \leq K^2_2} 2^{2m} \cdot 2^{\frac{5}{n}} ((K_1-1)^2+(K_1-2)^2)^{-m} \left( 2^{\frac{3}{n}} ((K_1-1)^2+(K_2-1)^2)-K^2_2 + 1 \right)
$$

$$
\ll K^2_1 2^{\frac{5}{n}} ((K_1-1)^2+(K_2-1)^2+(K_3-1)^2+(K_4-1)^2)-K^2_2 + K^2_1 2^{\frac{3}{n}} ((K_1-1)^2+(K_4-1)^2),
$$

To estimate $S_2$, we observe that if $m > K^2_2$ then $\|\theta(\nu_2)\| \leq \|\theta(\nu_1) + \theta(\nu_2)\| + \|\theta(\nu_1)\| \leq 5 \cdot 2^{-K^2_2}$.

Thus

$$
S_2 \ll \sum_{m > K^2_2} 2^n \#\{ (\nu_1, \nu_2) : \|\theta(\nu_1)\| \leq 2^{-m}, \|\theta(\nu_2)\| \leq 5 \cdot 2^{-K^2_2} \}
$$

$$
\ll \sum_{m > K^2_2} 2^{2m} \cdot 2^{\frac{5}{n}} ((K_1-1)^2+(K_2-1)^2)^{-m} \cdot 2^{\frac{3}{n}} ((K_1-1)^2+(K_4-1)^2)-K^2_2
$$

$$
\ll K^2_1 K^2_2 2^{\frac{5}{n}} ((K_1-1)^2+(K_2-1)^2+(K_3-1)^2+(K_4-1)^2)-K^2_2.
$$

Putting together the estimates we have obtained for $\sum \frac{1}{|\omega_1|}$ and $\sum \frac{1}{|\omega_2|}$ we get

$$
\int_{1}^{2} |E_n(\alpha, K_1, K_2, K_3, K_4)| \, d\alpha \ll K^2_1 2^{\frac{5}{n}} ((K_1-1)^2+(K_2-1)^2+(K_3-1)^2+(K_4-1)^2)-K^2_2
$$

$$
+ K^2_1 2^{-K^2_2+K^2_2+\frac{5}{n}} (K_1-1)^2
$$

$$
+ K^2_1 K^2_2 2^{\frac{5}{n}} ((K_1-1)^2+(K_4-1)^2)
$$

$$
= T_1 + T_2 + T_3.
$$
Using the inequalities \((K_4 - 1)^2 \leq \frac{1}{c_h - 1} ((K_1 - 1)^2 + (K_2 - 1)^2 + (K_3 - 1)^2)\)
and \(K_4 \leq K_3 \leq K_2 \leq K_1\) we have

\[
T_1 \ll K_2^2 \biggl(-1 + \frac{6}{K-1}\biggr)(K_1-1)^2 - 2K_1
\]
\[
T_2 \ll K_2^2 (-K_1-1)^2 + (K_4-1)^2 + \frac{2}{h}(K_1-1)^2
\]
\[
\ll K_2^2 \biggl(-1 + \frac{3}{K-1} + \frac{2}{h}\biggr)(K_1-1)^2
\]
\[
\ll K_2^2 \biggl(-1 + \frac{6}{hK-1}\biggr)(K_1-1)^2 - 2K_1,
\]
\[
T_3 \ll K_2^2 (K_4-1)^2 - (K_1-1)^2 + \frac{2}{h}((K_3-1)^2 + (K_4-1)^2)
\]
\[
\ll K_2^2 \biggl(1 + \frac{2}{h}\biggr) \frac{1}{K-1} ((K_1-1)^2 + (K_2-1)^2 + (K_3-1)^2) - (K_1-1)^2 + \frac{2}{h}(K_3-1)^2
\]
\[
\ll K_2^2 \biggl(1 + \frac{2}{h}\biggr) \frac{1}{K-1} - 1 + \frac{2}{h}(K_1-1)^2
\]
\[
\ll K_2^2 \biggl(-1 + \frac{6}{hK-1}\biggr)(K_1-1)^2 - 2K_1
\]

since \(c_h > 4\). Finally,

\[
\int_1^2 |E_0(\alpha, K)| \, d\alpha \ll \sum_{K_4 \leq K_3 \leq K_2 \leq K} K^2 \biggl(-1 + \frac{6}{K-1}\biggr)(K-1)^2 - 2K
\]
\[
\ll K^5 \biggl(-1 + \frac{6}{K-1}\biggr)(K-1)^2 - 2K,
\]
as claimed. \(\square\)

References

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