DENSE INFINITE B_h SEQUENCES

JAVIER CILLERUELO AND RAFAEL TESORO

Abstract. For $h = 3$ and $h = 4$ we prove the existence of infinite B_h sequences $\mathcal{B}$ with counting function

$$\mathcal{B}(x) = x^{\sqrt{(h-1)^2+1}-(h-1)+o(1)}.$$

This result extends a construction of I. Ruzsa for B_2 sequences.

1. Introduction

Let $h \geq 2$ be an integer. We say that a sequence $\mathcal{B}$ of positive integers is a B_h sequence if all the sums

$$b_1 + \cdots + b_h, \qquad (b_k \in \mathcal{B},\ 1 \leq k \leq h),$$

are distinct subject to $b_1 \leq b_2 \leq \ldots \leq b_h$. The study of the size of finite B_h sets (or the growing of the counting function of infinite B_h sequences) is a classic topic in combinatorial number theory. We define

$$F_h(n) = \max\{|\mathcal{B}| :\ \mathcal{B} \text{ is } B_h,\ \mathcal{B} \subset [1, n]\}.$$

A trivial counting argument proves that $F_h(n) \leq (C_h + o(1))n^{1/h}$ for a constant C_h (see [3] and [7] for non trivial upper bounds for C_h) and consequently that $\mathcal{B}(x) \ll x^{1/h}$ when $\mathcal{B}$ is an infinite B_h sequence.

There are three algebraic constructions ([2], [12] and [6]) of finite B_h sets showing that $F_h(n) \geq n^{1/h}(1 + o(1))$. It is probably true that $F_h(n) \sim n^{1/h}$ but this is an open problem, except for the case $h = 2$ for which Erdős and Turan [5] did prove that $F_2(n) \sim n^{1/2}$. It is unknown whether $\lim_{n\to\infty} F_h(n)/n^{1/h}$ exists for $h \geq 3$. For further information about B_h sequences see [8, § II.2] or [10].

Erdős conjectured for all $\epsilon > 0$ the existence of an infinite B_h sequence $\mathcal{B}$ with counting function $\mathcal{B}(x) \gg x^{1/h-\epsilon}$. It is believed that ϵ cannot be removed from the last exponent, however this has only been proved for h even. On the other hand, the *greedy* algorithm produces an infinite B_h sequence $\mathcal{B}$ with

$$\mathcal{B}(x) \gg x^{\frac{1}{2h-1}} \quad (h \geq 2). \tag{1.1}$$

Until now the exponent $1/(2h - 1)$ has been the largest known for the growth of a B_h sequence when $h \geq 3$. For the case $h = 2$, Atjai, Komlós and Szemerédi [1] proved that there exists a B_2 sequence (also called a Sidon sequence) with $\mathcal{B}(x) \gg (x \log x)^{1/3}$, improving by a power of logarithm the lower bound (1.1). So

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far the highest improvement of (1.1) for the case $h = 2$ was achieved by Ruzsa ([11]). He constructed, in a clever way, an infinite Sidon sequence $\mathcal{B}$ satisfying

$$\mathcal{B}(x) = x^{\sqrt{2}-1+o(1)}.$$

Our aim is to adapt Ruzsa's ideas to build dense infinite B_3 and B_4 sequences and so improve the lower bound (1.1) for $h = 3$ and $h = 4$.

Theorem 1.1. *For $h = 2, 3, 4$ there is an infinite B_h sequence $\mathcal{B}$ with counting function*

$$\mathcal{B}(x) = x^{\sqrt{(h-1)^2+1}-(h-1)+o(1)}.$$

The starting point in Ruzsa's construction were the numbers $\log p$, p prime, which form an infinite Sidon set of *real* numbers. Instead we part from the arguments of the Gaussian primes, which also have the same B_h property with the additional advantage of being a bounded sequence. This idea was suggested in [4] to simplify the original construction of Ruzsa and was written in detail for B_2 sequences in [9].

We believe that the theorem can be extended to all h, but we have not found yet a proof. Indeed we have written the core of the proof for all $h \geq 2$ except for Lemma 3.3 where we have considered only the cases $h = 2, 3, 4$ since the technical difficulties become significantly more involved as h increases.

2. The Gaussian arguments

For each rational prime $p \equiv 1 \pmod 4$ we consider the Gaussian prime $\mathfrak{p}$ of $\mathbb{Z}[i]$ such that

$$\mathfrak{p} := a + bi, \qquad p = a^2 + b^2, \quad a > b > 0,$$

so the argument of $\mathfrak{p}$ defined by $\mathfrak{p} = \sqrt{p}\, e^{2\pi i\,\theta(\mathfrak{p})}$ is a real number in the interval $(0, 1/8)$. We will use several times through the paper the following lemma that can be seen as a measure of the quality of the B_h property of this sequence of real numbers.

Lemma 2.1. *Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_h, \mathfrak{p}'_1, \cdots, \mathfrak{p}'_h$ be distinct Gaussian primes satisfying $0 < \theta(\mathfrak{p}_r), \theta(\mathfrak{p}'_r) < 1/8,\ r = 1, \cdots, h$. The following inequality holds:*

$$\left|\sum_{r=1}^{h} (\theta(\mathfrak{p}_r) - \theta(\mathfrak{p}'_r))\right| > \frac{1}{7\,|\mathfrak{p}_1 \cdots \mathfrak{p}_h\ \mathfrak{p}'_1 \cdots \mathfrak{p}'_h|}.$$

Proof. It is clear that

$$\sum_{r=1}^{h} (\theta(\mathfrak{p}_r) - \theta(\mathfrak{p}'_r)) \equiv \theta(\mathfrak{p}_1 \cdots \mathfrak{p}_h \overline{\mathfrak{p}'_1 \cdots \mathfrak{p}'_h}) \pmod 1. \tag{2.1}$$

Since $\mathbb{Z}[i]$ is a unique factorization domain, all the primes are in the first octant and are all distinct, the Gaussian integer $\mathfrak{p}_1 \cdots \mathfrak{p}_h \overline{\mathfrak{p}'_1 \cdots \mathfrak{p}'_h}$ cannot be a real integer.

Using this fact and the inequality $\arctan(1/x) > 0.99/x$ for $x \ge \sqrt{5\cdot 13}$ we have

$$
\begin{aligned}
(2.2)\qquad |\theta(\mathfrak{p}_1\cdots\mathfrak{p}_h\overline{\mathfrak{p}'_1\cdots\mathfrak{p}'_h})| &\ge \|\theta(\mathfrak{p}_1\cdots\mathfrak{p}_h\overline{\mathfrak{p}'_1\cdots\mathfrak{p}'_h})\| \\
&\ge \frac{1}{2\pi}\arctan\left(\frac{1}{|\mathfrak{p}_1\cdots\mathfrak{p}_h\overline{\mathfrak{p}'_1}\cdots\overline{\mathfrak{p}'_h}|}\right) \\
&> \frac{1}{7|\mathfrak{p}_1\cdots\mathfrak{p}_h\overline{\mathfrak{p}'_1}\cdots\overline{\mathfrak{p}'_h}|},
\end{aligned}
$$

where $\|\cdot\|$ means the distance to $\mathbb{Z}$. The lemma follows from (2.1) and (2.2). □

We illustrate the B_h property of the arguments of the Gaussian primes with a quick construction, based on them, of a finite B_h set which is only a $\log x$ factor below the optimal bound.

Theorem 2.2. *The set* $\mathcal{A} = \{\lfloor x\theta(\mathfrak{p})\rfloor,\ |\mathfrak{p}| \le \left(\frac{x}{7h}\right)^{\frac{1}{2h}}\} \subset [1, x]$ *is a* B_h *set with* $|\mathcal{A}| \gg x^{1/h}/\log x$.

Proof. If

$$\lfloor x\theta(\mathfrak{p}_1)\rfloor + \cdots + \lfloor x\theta(\mathfrak{p}_h)\rfloor = \lfloor x\theta(\mathfrak{p}'_1)\rfloor + \cdots + \lfloor x\theta(\mathfrak{p}'_h)\rfloor$$

then

$$x\,|\theta(\mathfrak{p}_1) + \cdots + \theta(\mathfrak{p}_h) - \theta(\mathfrak{p}'_1) - \cdots - \theta(\mathfrak{p}'_h)| \le h.$$

If the Gaussian primes are distinct then Lemma 2.1 implies that

$$|\theta(\mathfrak{p}_1) + \cdots + \theta(\mathfrak{p}_h) - \theta(\mathfrak{p}'_1) - \cdots - \theta(\mathfrak{p}'_h)| > \frac{1}{7|\mathfrak{p}_1\cdots\mathfrak{p}_h\mathfrak{p}'_1\cdots\mathfrak{p}'_h|} \ge h/x,$$

which is a contradiction. □

3. Proof of Theorem 1.1

We start following the lines of [11] with several adjustments. In the sequel we will write $\mathfrak{p}$ for a Gaussian prime in the first octant $(0 < \theta(\mathfrak{p}) < 1/8)$.

We fix a number $c_h > h$ which will determine the growth of the sequence we construct. Indeed we will take $c_h = \sqrt{(h-1)^2+1} + (h-1)$.

3.1. The construction.

We will construct for each $\alpha \in [1, 2]$ a sequence of positive integers indexed with the Gaussian primes

$$\mathcal{B}_\alpha := \{b_\mathfrak{p}\},$$

where each $b_\mathfrak{p}$ will be built using the development to base 2 of $\alpha\,\theta(\mathfrak{p})$:

$$\alpha\,\theta(\mathfrak{p}) = \sum_{i=1}^{\infty}\delta_{i\mathfrak{p}}2^{-i} \qquad (\delta_{i\mathfrak{p}} \in \{0,1\}).$$

The role of the parameter α will be clear at a later stage, for the moment it is enough to note that the set $\{\alpha\theta(\mathfrak{p})\}$ obviously keeps the same B_h property of the set $\{\theta(\mathfrak{p})\}$.

To organize the construction we describe the sequence $\mathcal{B}_\alpha$ as a union of finite sets according with the sizes of the indexes:

$$\mathcal{B}_\alpha = \bigcup_K \mathcal{B}_{\alpha,K}$$

where

$$\mathcal{B}_{\alpha,K} = \{b_\mathfrak{p} : \ \mathfrak{p} \in P_K\}$$

and

$$P_K := \{\mathfrak{p} : \ 2^{\frac{(K-2)^2}{c_h}} \le |\mathfrak{p}|^2 < 2^{\frac{(K-1)^2}{c_h}}\}.$$

Now we build the positive integers $b_\mathfrak{p} \in \mathcal{B}_{\alpha,K}$. For any $\mathfrak{p} \in P_K$ we define $\widehat{\alpha\theta(\mathfrak{p})}$ the truncated series of $\alpha\,\theta(\mathfrak{p})$ at the K^2-place:

$$\widehat{\alpha\theta(\mathfrak{p})} := \sum_{i=1}^{K^2} \delta_{i\mathfrak{p}} 2^{-i} \tag{3.1}$$

and combine the digits at places $(j-1)^2+1, \cdots, j^2$ into a single number

$$\Delta_{j\mathfrak{p}} = \sum_{i=(j-1)^2+1}^{j^2} \delta_{i\mathfrak{p}} 2^{j^2-i} \quad (j = 1, \cdots, K),$$

so that we can write

$$\widehat{\alpha\,\theta(\mathfrak{p})} = \sum_{j=1}^{K} \Delta_{j\mathfrak{p}} 2^{-j^2}. \tag{3.2}$$

We observe that if $\mathfrak{p} \in P_K$ then

$$|\widehat{\alpha\,\theta(\mathfrak{p})} - \alpha\,\theta(\mathfrak{p})| \le 2^{-K^2}. \tag{3.3}$$

The definition of $b_\mathfrak{p}$ is informally outlined as follows. We consider the sequence of blocks $\Delta_{1\mathfrak{p}}, \cdots, \Delta_{K\mathfrak{p}}$ and re-arrange them opposite to the original left to right arrangement. Then we insert at the left of each $\Delta_{j\mathfrak{p}}$ an additional filling block of $2d+1$ digits, with $d = \lceil \log_2 h \rceil$. At the filling blocks the digits will be always 0 except for only one exception: in the middle of the first filling block (placed to the left of the Δ_K block) we put the digit **1**. This digit will mark which subset P_K the prime $\mathfrak{p}$ belongs to.

$$\alpha\,\theta(\mathfrak{p}) = 0.\overset{\Delta_1}{1}\ \overbrace{001}^{\Delta_2}\ldots\overbrace{1\cdots\cdots 0}^{\Delta_j}\ldots\overbrace{01\cdots\cdots\cdots 11}^{\Delta_K}\underset{\substack{\uparrow\\ K^2}}{.}\ldots\ldots$$

$$b_\mathfrak{p} = \underline{\mathbf{0\cdot 1\cdot 0}}\,\overbrace{01\cdots\cdots\cdots 11}^{\Delta_K}\,\underline{\mathbf{0\cdots 0}}\ \ldots\underline{\mathbf{0\cdots 0}}\,\overbrace{1\cdots\cdots 0}^{\Delta_j}\,\underline{\mathbf{0\cdots 0}}\ldots\underline{\mathbf{0\cdots 0}}\,\overbrace{001}^{\Delta_2}\,\underline{\mathbf{0\cdots 0}}\ \overset{\Delta_1}{1},$$

The reason to add the blocks of zeroes and the value of d will be clarified just before Lemma 3.2.

More formally, for $\mathfrak{p} \in P_K$ we define

$$t_\mathfrak{p} = 2^{K^2+(2d+1)K+(d+1)}, \tag{3.4}$$

and

$$b_{\mathfrak{p}} = t_{\mathfrak{p}} + \sum_{j=1}^{K} \Delta_{j\mathfrak{p}} 2^{(j-1)^2+(2d+1)(j-1)}.$$

Furthermore we define $\Delta_{j\mathfrak{p}} = 0$ for $j > K$.

Remark 3.1. The construction at [11] was based on the numbers $\alpha \log p$, with p rational prime, hence the digits of their integral parts had to be included also in the corresponding integers b_p. Ruzsa solved that problem reserving fixed places for these digits. Since in our construction the integral part of $\alpha\theta(\mathfrak{p})$ is zero we don't need to care about this.

We observe that distinct primes $\mathfrak{p}, \mathfrak{q}$ provide distinct $b_{\mathfrak{p}}, b_{\mathfrak{q}}$. Indeed if $b_{\mathfrak{p}} = b_{\mathfrak{q}}$ then $\Delta_{i\mathfrak{p}} = \Delta_{i\mathfrak{q}}$ for all $i \le K$. Also $t_{\mathfrak{p}} = t_{\mathfrak{q}}$ which means $\mathfrak{p}, \mathfrak{q} \in P_K$, and so

$$|\theta(\mathfrak{p}) - \theta(\mathfrak{q})| = \alpha^{-1} \cdot \sum_{j>K} (\Delta_{j\mathfrak{p}} - \Delta_{j\mathfrak{q}}) < 2^{-K^2}.$$

Now if $\mathfrak{p} \neq \mathfrak{q}$ then Lemma 2.1 implies that $|\theta(\mathfrak{p}) - \theta(\mathfrak{q})| > \frac{1}{7|\mathfrak{p}\mathfrak{q}|} > 2^{-\frac{1}{c}(K-1)^2-3}$. Combining both inequalities we have a contradiction for $K \ge h+1 \ge 3$. So we assume $K \ge h+1$ through all the paper.

Since all the integers $b_{\mathfrak{p}}$ are distinct, we have that

$$|\mathcal{B}_{\alpha,K}| = |P_K| = \pi\left(2^{\frac{(K-1)^2}{c_h}}; 1, 4\right) - \pi\left(2^{\frac{(K-2)^2}{c_h}}; 1, 4\right) \gg \frac{2^{\frac{K^2}{c_h}}}{K^2}. \tag{3.5}$$

We observe also that

$$b_{\mathfrak{p}} < 2^{K^2+(2d+1)K+(d+1)+1}.$$

Using these estimates we can easily prove that $\mathcal{B}_\alpha(x) = x^{\frac{1}{c_h}+o(1)}$. Indeed, if K is the integer such that $2^{K^2+(2d+1)K+(d+1)+1} < x \le 2^{(K+1)^2+(2d+1)(K+1)+(d+1)+1}$ then we have

$$\mathcal{B}_\alpha(x) \ge |\mathcal{B}_{\alpha,K}| = 2^{\frac{1}{c_h}K^2(1+o(1))} = x^{\frac{1}{c_h}+o(1)}. \tag{3.6}$$

For the upper bound we have

$$\mathcal{B}_\alpha(x) \le \#\{\mathfrak{p} : \ |\mathfrak{p}|^2 \le 2^{\frac{K^2}{c_h}}\} \le 2^{\frac{K^2}{c_h}} = x^{\frac{1}{c_h}+o(1)}.$$

There is a compromise at the choice of a particular value of c_h for the construction. On one hand larger values of c_h capture more information from the Gaussian arguments which brings the sequence $\mathcal{B}_\alpha = \{b_{\mathfrak{p}}\}$ closer to being a B_h sequence. On the other hand smaller values of c_h provide higher growth of the counting function of $\mathcal{B}_\alpha$.

Clearly $\mathcal{B}_\alpha$ would be a B_h sequence if for all $l = 2, \cdots, h$ it does not contain $b_{\mathfrak{p}_1}, \cdots, b_{\mathfrak{p}_l}, b_{\mathfrak{p}'_1}, \cdots, b_{\mathfrak{p}'_l}$ satisfying

$$\begin{aligned} b_{\mathfrak{p}_1} + \cdots + b_{\mathfrak{p}_l} &= b_{\mathfrak{p}'_1} + \cdots + b'_l, \qquad (3.7)\\ \{b_1, \cdots, b_l\} &\cap \{b'_1, \cdots, b'_l\} = \emptyset, \end{aligned}$$

$$b_{\mathfrak{p}_1} \ge \cdots \ge b_{\mathfrak{p}_l} \quad \text{and} \quad b_{\mathfrak{p}'_1} \ge \cdots \ge b_{\mathfrak{p}'_l}. \tag{3.8}$$

We say that $(\mathfrak{p}_1,\dots,\mathfrak{p}_l,\mathfrak{p}'_1,\dots,\mathfrak{p}'_l)$ is a bad $2l$-tuple if the equation 3.7 is satisfied by the corresponding $b_{\mathfrak{p}_r}$.

The sequence $\mathcal{B}_\alpha = \{b_\mathfrak{p}\}$ we have constructed is not properly a B_h sequence. Some repeated sums as in (3.7) will eventually appear, but the particular way to construct the elements $b_\mathfrak{p}$ will allow us to study these bad $2l$-tuples and to prove that there are not too many repeated sums. Then removing the bad elements involved in these bad $2l$-tuples we obtain a true B_h sequence.

Now we will see why blocks of zeroes were added to the binary development of $b_\mathfrak{p}$. We can identify each $b_\mathfrak{p}$ with a vector as follows:

$$\begin{aligned}
b_{\mathfrak{p}_1} &= (\cdots, \mathbf{1}, \Delta_{K_1\mathfrak{p}_1}, 0, \cdots, 0, \Delta_{K_2\mathfrak{p}_1}, 0, \cdots, 0, \Delta_{K_l\mathfrak{p}_1}, 0, \cdots, 0, \Delta_{2\mathfrak{p}_1}, 0, \Delta_{1\mathfrak{p}_1})\\
b_{\mathfrak{p}_2} &= (\cdots, 0, \cdots\cdots\cdots\cdots, \mathbf{1}, \Delta_{K_2\mathfrak{p}_2}, 0, \cdots, 0, \Delta_{K_l\mathfrak{p}_2}, 0, \cdots, 0, \Delta_{2\mathfrak{p}_2}, 0, \Delta_{1\mathfrak{p}_2})\\
&\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots\\
b_{\mathfrak{p}_l} &= (\cdots, 0, \cdots\cdots\cdots\cdots, 0, \cdots\cdots\cdots\cdots, \mathbf{1}, \Delta_{K_l\mathfrak{p}_l}, 0, \cdots, 0, \Delta_{2\mathfrak{p}_l}, 0, \Delta_{1\mathfrak{p}_l}),
\end{aligned}$$

where each comma represents one block of d zeroes. Note that the leftmost part of each vector is null. The value of $d = \lceil \log_2 h \rceil$ has been chosen to prevent the propagation of the carry between any two consecutive coordinates separated by a comma in the above identification. So when we sum no more than h integers $b_\mathfrak{p}$ we can just sum the corresponding vectors coordinate-wise. This argument implies the following lemma.

Lemma 3.2. *Let $(\mathfrak{p}_1,\cdots,\mathfrak{p}_l,\mathfrak{p}'_1,\cdots,\mathfrak{p}'_l)$ be a bad $2l$-tuple. Then there are integers $K_1 \ge \cdots \ge K_l$ such that $\mathfrak{p}_1,\mathfrak{p}'_1 \in P_{K_1},\cdots,\mathfrak{p}_l,\mathfrak{p}'_l \in P_{K_l}$, and we have*

$$\widehat{\alpha\theta(\mathfrak{p}_1)} + \cdots + \widehat{\alpha\theta(\mathfrak{p}_l)} = \widehat{\alpha\theta(\mathfrak{p}'_1)} + \cdots + \widehat{\alpha\theta(\mathfrak{p}'_l)}. \tag{3.9}$$

Proof. Note that (3.7) implies $t_{\mathfrak{p}_1}+\cdots+t_{\mathfrak{p}_l} = t_{\mathfrak{p}'_1}+\cdots+t_{\mathfrak{p}'_l}$ and $\Delta_{j\mathfrak{p}_1}+\cdots+\Delta_{j\mathfrak{p}_l} = \Delta_{j\mathfrak{p}'_1}+\cdots+\Delta_{j\mathfrak{p}'_l}$ for each j. Using (3.2) we conclude (3.9). As the bad $2l$-tuple satisfies condition (3.8) we deduce that $\mathfrak{p}_r,\mathfrak{p}'_r$ belongs to the same P_{K_r} for all r. □

According with the lemma above we will write $E_{2l}(\alpha;K_1,\cdots,K_l)$ for the set of bad $2l$-tuples $(\mathfrak{p}_1,\cdots,\mathfrak{p}'_l)$ with $\mathfrak{p}_r,\mathfrak{p}'_r \in P_{K_r},\ 1 \le r \le l$ and

$$E_{2l}(\alpha;K) = \bigcup_{K_l\le\cdots\le K_1=K} E_{2l}(\alpha;K_1,\cdots,K_l),$$

where $K = K_1$. Also we define the set

$$\mathrm{Bad}_{\alpha,K} = \{b_\mathfrak{p} \in \mathcal{B}_{\alpha,K} :\ b_\mathfrak{p} \text{ is the largest element involved in some equation 3.7}\}.$$

It is clear that $\sum_{l\le h} |E_{2l}(\alpha,K)|$ is an upper bound for $|\mathrm{Bad}_{\alpha,K}|$, the number of elements that we need to remove from each $\mathcal{B}_{\alpha,K}$ to get a B_h sequence.

We do know how to obtain a good upper bound for $|E_{2l}(\alpha,K)|$ for a concrete α, but we can do it for almost α.

Lemma 3.3. *For $l=2,3,4$ we have*

$$\int_1^2 |E_{2l}(\alpha,K)|\,\mathrm{d}\alpha \ll K^{m_l}2^{\left(\frac{2(l-1)}{c_h}-1\right)(K-1)^2-2K}$$

for some m_l.

The proof of Lemma 3.3 is involved and we postpone it to section §4. We think that Lemma 3.3 holds for any l but we have not found a proof.

3.2. **Last step in the proof of the theorem.** For $h=2,3,4$ we have that

$$\begin{aligned}\int_1^2 \sum_K \frac{|\mathrm{Bad}_{\alpha,K}|}{|\mathcal{B}_{\alpha,K}|}\,\mathrm{d}\alpha &\ll \sum_K \frac{\sum_{l\le h}\int_1^2 |E_{2l}(\alpha,K)|\,\mathrm{d}\alpha}{K^{-2}2^{\frac{1}{c_h}(K-1)^2}} \\ &\ll \sum_K \frac{\sum_{l\le h} K^{m_l}2^{\left(\frac{2(l-1)}{c_h}-1\right)(K-1)^2-2K}}{K^{-2}2^{\frac{1}{c_h}(K-1)^2}} \\ &\ll \sum_K K^{m_l+2}2^{\left(\frac{2(h-1)}{c_h}-1-\frac{1}{c_h}\right)(K-1)^2-2K}.\end{aligned}$$

The last sum is finite for $c_h=\sqrt{(h-1)^2+1}+(h-1)$ which is the largest number for which $\frac{2(h-1)}{c_h}-1-\frac{1}{c_h}\le 0$. So for this c_h the sum $\sum_K \frac{|\mathrm{Bad}_{\alpha,K}|}{|\mathcal{B}_{\alpha,K}|}$ is convergent for almost all $\alpha\in[1,2]$. We take one of these α, say α_0, and consider the sequence

$$\mathcal{B}=\bigcup_K \left(\mathcal{B}_{\alpha_0,K}\setminus \mathrm{Bad}_{\alpha_0,K}\right).$$

We claim that this sequence satisfies the condition of the theorem. It is clear that this sequence is a B_h sequence because we have destroyed all the repeated sums of h elements of $\mathcal{B}_{\alpha_0}$ removing all the bad elements from each $\mathcal{B}_{\alpha_0,K}$.

On the other hand, the convergence of $\sum_K \frac{|\mathrm{Bad}_{\alpha_0,K}|}{|\mathcal{B}_{\alpha,K}|}$ implies that $|\mathrm{Bad}_{\alpha_0,K}| = o\left(|\mathcal{B}_{\alpha,K}|\right)$. We proceed as in (3.6) to estimate the counting function of $\mathcal{B}$. For any x let K the integer such that $2^{K^2+(2d+1)K+(d+1)+1}<x\le 2^{(K+1)^2+(2d+1)(K+1)+(d+1)+1}$. We have

$$\mathcal{B}(x)\ge |\mathcal{B}_{\alpha_0,K}|-|\mathrm{Bad}_{\alpha_0,K}| = |\mathcal{B}_{\alpha_0,K}|(1+o(1)) \gg K^{-2}2^{\frac{1}{c_h}K^2} = x^{\frac{1}{c_h}+o(1)}.$$

For the upper bound, we have

$$\mathcal{B}(x)\le \mathcal{B}_{\alpha_0}(x)=x^{\frac{1}{c_h}+o(1)}.$$

Thus

$$\mathcal{B}(x)=x^{\sqrt{(h-1)^2+1}-(h-1)+o(1)}.$$

4. Proof of Lemma 3.3

The proof of Lemma 3.3 will be a consequence of Propositions 4.5, 4.6 and 4.7. Before proving these propositions we need to study some properties of the bad $2l$-tuples and an auxiliary lemma about visible lattice points.

4.1. **Some properties of the $2l$-tuples.** For any $2l$-tuple $(\mathfrak{p}_1, \cdots, \mathfrak{p}'_l)$ we define the numbers $\omega_s = \omega_s(\mathfrak{p}_1, \cdots, \mathfrak{p}'_l)$ by

$$\omega_s = \sum_{r=1}^{s} (\theta(\mathfrak{p}_r) - \theta(\mathfrak{p}'_r)) \qquad (s \le l).$$

The next two lemmas contain several properties of the bad $2l$-tuples.

Lemma 4.1. *Let $(\mathfrak{p}_1, \cdots, \mathfrak{p}_l, \mathfrak{p}'_1, \cdots, \mathfrak{p}'_l)$ be a bad $2l$-tuple with $K_1 \ge \cdots \ge K_l$ given by Lemma 3.2. We have*

$$\text{i)} \quad |\omega_l| \le l 2^{-K_l^2},$$

$$\text{ii)} \quad |\omega_{l-1}| \ge 2^{-\frac{1}{c_h}(K_l-1)^2-4},$$

$$\text{iii)} \quad (K_l-1)^2 \le \frac{(K_1-1)^2 + \cdots + (K_{l-1}-1)^2}{c_h - 1}.$$

Proof. i) This is a consequence of (3.9) and (3.3):

$$|\omega_l| = \frac{1}{\alpha}\left|\sum_{r=1}^{l}(\alpha\theta(\mathfrak{p}_r) - \alpha\theta(\mathfrak{p}'_r))\right| \le \frac{1}{\alpha}\left(2^{-K_1^2} + \cdots + 2^{-K_l^2}\right) \le l2^{-K_l^2},$$

since $\alpha \ge 1$.

ii) Lemma 2.1 implies

$$|\theta(\mathfrak{p}_l) - \theta(\mathfrak{p}'_l)| \ge \frac{1}{7\,|\mathfrak{p}_l \mathfrak{p}'_l|} \ge 2^{-3-\frac{1}{c_h}(K_l-1)^2} \tag{4.1}$$

and so,

$$\begin{aligned}|\omega_{l-1}| &= |\omega_l + \theta(\mathfrak{p}'_l) - \theta(\mathfrak{p}_l)| \ge |\theta(\mathfrak{p}'_l) - \theta(\mathfrak{p}_l)| - |\omega_l| \\ &\ge 2^{-\frac{1}{c_h}(K_l-1)^2-3} - l2^{-K_l^2} \ge 2^{-\frac{1}{c_h}(K_l-1)^2-4},\end{aligned}$$

since $K_l \ge h+1 \ge l+1$.

iii) Lema 2.1 implies also that

$$|\omega_l| = \left|\sum_{r=1}^{l}(\theta(\mathfrak{p}_r) - \theta(\mathfrak{p}'_r))\right| > \frac{1}{7\,|\mathfrak{p}_1 \cdots \mathfrak{p}'_l|} > 2^{-3-\frac{1}{c_h}\sum_{r=1}^{l}(K_r-1)^2}.$$

Combining this with i) we obtain

$$(K_l-1)^2 \le \frac{1}{c_h-1}\left((K_1-1)^2 + \cdots + (K_{l-1}-1)^2\right) + \frac{\log_2 l - 2K_l + 4}{1 - 1/c_h}.$$

The last term is negative because $K_l \ge h+1 \ge l+1$ and $l \ge 2$. □

Lemma 4.2. *Suppose that $(\mathfrak{p}_1, \cdots, \mathfrak{p}_l, \mathfrak{p}'_1, \cdots, \mathfrak{p}'_l)$ is a bad $2l$-tuple. Then for any $\omega_s = \sum_{r=1}^{s}(\theta(\mathfrak{p}_r) - \theta(\mathfrak{p}'_r))$ with $1 \le s \le l-1$ we have*

$$\left\|\alpha 2^{K_{s+1}^2}\omega_s\right\| \le s2^{K_{s+1}^2 - K_s^2} \qquad (s = 1, \cdots, l-1), \tag{4.2}$$

where $\|\cdot\|$ means the distance to the nearest integer.

Proof. Since $0 \le \alpha\theta(\mathfrak{p}) - \widehat{\alpha\theta(\mathfrak{p})} \le 2^{-K^2}$ when $\mathfrak{p} \in P_K$, we can write

$$2^{K_{s+1}^2}\alpha \sum_{r=1}^{s} (\theta(\mathfrak{p}_r) - \theta(\mathfrak{p}'_r)) = 2^{K_{s+1}^2} \sum_{r=1}^{s} \left(\widehat{\alpha\theta(\mathfrak{p}_r)} - \widehat{\alpha\theta(\mathfrak{p}'_r)}\right) + \epsilon_s,$$

with $|\epsilon_s| \le s2^{K_{s+1}^2 - K_s^2}$. By Lemma 3.2 we know that $\sum_{r=1}^{l}\left(\widehat{\alpha\theta(\mathfrak{p}_r)} - \widehat{\alpha\theta(\mathfrak{p}'_r)}\right) = 0$ when $(\mathfrak{p}_1, \cdots, \mathfrak{p}_l, \mathfrak{p}'_1, \cdots, \mathfrak{p}'_l)$ is a bad $2l$-tuple. Using this and (3.1) we have that

$$2^{K_{s+1}^2} \sum_{r=s+1}^{l} \left(\widehat{\alpha\theta(\mathfrak{p}'_r)} - \widehat{\alpha\theta(\mathfrak{p}_r)}\right) = \sum_{r=s+1}^{l} \sum_{i=1}^{K_r^2} 2^{K_{s+1}^2 - i}(\delta_{i\mathfrak{p}'_r} - \delta_{i\mathfrak{p}_r})$$

is an integer, which proves (4.3).

□

Lemma 4.3.

$$\int_1^2 |E_{2l}(\alpha; K_1, \dots, K_l)|\, \mathrm{d}\alpha \ll 2^{K_l^2 - K_1^2} \sum_{\substack{(\mathfrak{p}_1, \dots, \mathfrak{p}'_l) \\ |\omega_l| < l \cdot 2^{-K_l^2}}} \frac{|\omega_{l-1}|}{|\omega_1|} \prod_{j=1}^{l-2} \left(\frac{|\omega_j|}{|\omega_{j+1}|} + 1 \right)$$

Proof. We have seen that if $(\mathfrak{p}_1, \dots, \mathfrak{p}'_l) \in E_{2l}(\alpha; K_1, \dots, K_l)$, then

$$\left\| \alpha 2^{K_{s+1}^2} \omega_s \right\| \le s2^{K_{s+1}^2 - K_s^2}, \ \ s = 1, \dots, l-1. \tag{4.3}$$

Then there exists integers $j_s, \ s = 1, \cdots, l-1$ such that

$$\left| \alpha - \frac{j_s}{2^{K_{s+1}^2}\omega_s} \right| \le \frac{s2^{-K_s^2}}{|\omega_s|}. \tag{4.4}$$

Writing $I_{j_1}, \cdots, I_{j_s}$ for the intervals defined by the inequalities 4.4, we have

$$\begin{aligned} \mu\{\alpha : \ (\mathfrak{p}_1, \dots, \mathfrak{p}'_l) \in E_{2l}(\alpha; K_1, \dots, K_l)\} \ &\le \ \sum_{j_1, \dots, j_{l-1}} \left| I_{j_1} \cap \cdots \cap I_{j_{l-1}} \right| \\ &\le \ \frac{2^{-K_1^2+1}}{|\omega_1|} \#\{(j_1, \dots, j_{l-1}) : \ \bigcap_{i=1}^{l-1} I_{j_i} \ne \emptyset\} \end{aligned}$$

To estimate this last cardinality note that for all $s = 1, \dots, l-2$ we have

$$\left| \frac{j_s}{2^{K_{s+1}^2}\omega_s} - \frac{j_{s+1}}{2^{K_{s+2}^2}\omega_{s+1}} \right| < \left| \alpha - \frac{j_s}{2^{K_{s+1}^2}\omega_s} \right| + \left| \alpha - \frac{j_{s+1}}{2^{K_{s+2}^2}\omega_{s+1}} \right| < \frac{s2^{-K_s^2}}{|\omega_s|} + \frac{(s+1)2^{-K_{s+1}^2}}{|\omega_{s+1}|}$$

Thus,

$$\left| j_s - j_{s+1} \frac{2^{K_{s+1}^2}\omega_s}{2^{K_{s+2}^2}\omega_{s+1}} \right| < s2^{-K_s^2 + K_{s+1}^2} + \frac{(s+1)|\omega_{s+1}|}{|\omega_s|}. \tag{4.5}$$

We observe that for each $s = 1, \dots, l-2$ and for each j_{s+1}, the number of j_s satisfying (4.5) is bounded by $2\left(s2^{-K_s^2 + K_{s+1}^2} + \frac{(s+1)|\omega_s|}{|\omega_{s+1}|}\right) + 1 \ll \frac{|\omega_{s+1}|}{|\omega_s|} + 1$.

Note also that

$$\begin{aligned} |j_{l-1}| &\leq 2^{K_l^2}|\omega_{l-1}|\left(\left|\frac{j_{l-1}}{2^{K_l^2}\omega_{l-1}}-\alpha\right|+|\alpha|\right)\\ &\leq 2^{K_l^2}|\omega_{l-1}|\left(\frac{(l-1)2^{K_{l-1}^2}}{|\omega_{l-1}|}+2\right)\\ &\leq l-1+2^{K_l^2+1}|\omega_{l-1}|\\ &\ll 2^{K_l^2+1}|\omega_{l-1}| \end{aligned}$$

In the last step we have used the condition iii).

Putting all these observations together we complete the proof. □

4.2. **Visible points.** We will denote by $\mathcal{V}$ the set of lattice points visible from the origin excluding $(1,0)$. In the next subsection we will use several times the following lemma.

Lemma 4.4. *The number of integral lattice points visible from the origin that are contained in a circular sector centred at the origin of radius R and angle ϵ is at most ϵR^2+1. In other words, for any real number t*

$$\#\{\nu\in\mathcal{V},\ |\nu|<R,\ \|\theta(\nu)+t\|<\epsilon\}\leq \epsilon R^2+1.$$

Furthermore,

$$\#\{\nu\in\mathcal{V},\ |\nu|<R,\ \|\theta(\nu)\|<\epsilon\}\leq \epsilon R^2.$$

Proof. We arrange the N lattice points inside de sector $\nu_1,\nu_2,\cdots,\nu_N$ that are visible from the origin O by the value of their argument so that $\theta(\nu_i)<\theta(\nu_j)$ for $1\leq i<j\leq N$. For each $i=1,\dots,N-1$ the three lattice points O,ν_i,ν_{i+1} define a triangle T_i with $\mathrm{Area}(T_i)\geq 1/2$, that does not contain any other lattice point.

Since all T_i are inside the circular sector their union covers at most the area of the sector. They don't overlap pairwise, thus

$$N-1\leq\sum_{i=1}^{N}2\cdot\mathrm{Area}(T_i)=2\cdot\mathrm{Area}\left(\bigcup_{i=1}^{N}T_i\right)\leq R^2\epsilon.$$

For the last statement we add $\nu_0=(1,0)$ to our N visible points $\nu_1,\dots,\nu_N$ and we repeat the argument. □

4.3. **Estimates for the number of bad $2l$-tuples ($l=2,3,4$).** We start with the case $l=2$ which was considered by Ruzsa for B_2 sequences. In the sequel all lattice points ν appearing in the proofs belong to $\mathcal{V}$ and Lemma 4.4 applies.

Proposition 4.5. *For any $c_h>2$ we have*

$$\int_1^2|E_4(\alpha;K)|\,\mathrm{d}\alpha\ll K2^{(\frac{2}{c_h-1}-1)(K-1)^2-2K}.$$

Proof. Lemma 4.3 implies that

$$\int_1^2|E_4(\alpha;K_1,K_2)|\,\mathrm{d}\alpha\ll 2^{K_2^2-K_1^2}\,\#\{(\mathfrak{p}_1,\mathfrak{p}_1',\mathfrak{p}_2,\mathfrak{p}_2')\colon|\omega_2|\leq 2\cdot 2^{-K_2^2}\}.$$

We get an upper bound for the second factor here by using Lemma 4.4 to estimate the number of lattice points of the form $\nu_2 = \mathfrak{p}_1\mathfrak{p}_1'\overline{\mathfrak{p}_2\mathfrak{p}_2'}$ such that $\|\theta(\nu_2)\| < \epsilon$, $|\nu_2| < R$, with $\epsilon = 2\cdot 2^{-K_2^2}$ and $R = 2^{\frac{1}{c_h}((K_1-1)^2+(K_2-1)^2)}$. We have

$$\begin{aligned}\int_1^2 |E_4(\alpha;K_1,K_2)|\,\mathrm{d}\alpha &\ll 2^{K_2^2-K_1^2}\cdot 2^{\frac{2}{c_h}((K_1-1)^2+(K_2-1)^2)-K_2^2} \\ &\ll 2^{\frac{2}{c_h}((K_1-1)^2+(K_2-1)^2)-K_1^2}.\end{aligned}$$

By Lemma 4.1 iv) we also have $(K_2-1)^2 \le \frac{(K_1-1)^2}{c_h-1}$, thus

$$\int_1^2 |E_4(\alpha;K_1,K_2)|\,\mathrm{d}\alpha \ll 2^{\left(\frac{2}{c_h-1}-1\right)K_1^2-2K_1}$$

and

$$\int_1^2 |E_4(\alpha;K)|\,\mathrm{d}\alpha = \sum_{K_2\le K}\int_1^2 |E_4(\alpha;K,K_2)|\,\mathrm{d}\alpha \ll K2^{\left(\frac{2}{c_h-1}-1\right)(K-1)^2-2K}.$$

□

Proposition 4.6. *For any $c_h > 3$ we have*

$$\int_1^2 |E_6(\alpha;K)|\,\mathrm{d}\alpha \ll K^2\,2^{(\frac{4}{c_h-1}-1)(K-1)^2-2K}.$$

Proof. Lemma 4.3 says that

$$\int_1^2 |E_6(\alpha;K_1,K_2,K_3)|\,\mathrm{d}\alpha \ll 2^{K_3^2-K_1^2}\sum_{\substack{(\mathfrak{p}_1,\dots,\mathfrak{p}_3')\\ |\omega_3|\le 3\cdot 2^{-K_3^2}}}\frac{1}{|\omega_1|}.$$

Applying Lemma 4.4 by writing $\nu_1 = \mathfrak{p}_1\overline{\mathfrak{p}_1'}$ and $\nu_2 = \mathfrak{p}_2\mathfrak{p}_3\overline{\mathfrak{p}_2'\mathfrak{p}_3'}$, we have that

$$\begin{aligned}\sum_{\substack{(\mathfrak{p}_1,\dots,\mathfrak{p}_3')\\ |\omega_3|\le 3\cdot 2^{-K_3^2}}}\frac{1}{|\omega_1|} &\ll \sum_m 2^m\#\{(\mathfrak{p}_1,\cdots,\mathfrak{p}_3'):\ |\omega_1|\le 2^{-m}, |\omega_3|\le 3\cdot 2^{-K_3^2}\} \\ &\ll \sum_m 2^m\#\{(\nu_1,\nu_2):\ \|\theta(\nu_1)\|\le 2^{-m},\ \|\theta(\nu_1)+\theta(\nu_2)\|\le 3\cdot 2^{-K_3^2}\} \\ &\ll \sum_m 2^m \sum_{|\theta(\nu_1)|\le 2^{-m}}\#\{\nu_2:\ \|\theta(\nu_1)+\theta(\nu_2)\|\le 3\cdot 2^{-K_3^2}\} \\ &\ll \sum_m 2^m\cdot 2^{\frac{2}{c_h}(K_1-1)^2-m}\left(2^{\frac{2}{c_h}\left((K_2-1)^2+(K_3-1)^2\right)-K_3^2}+1\right).\end{aligned}$$

Thus, using the inequalities $K_3 \le K_2 \le K_1$ and $(K_3-1)^2 \le \frac{(K_2-1)^2+(K_1-1)^2}{c_h-1}$ we have

$$\begin{aligned}\int_1^2 |E_6(\alpha;K_1,K_2,K_3)|\,\mathrm{d}\alpha &\ll K_1^2 2^{K_3^2-K_1^2+\frac{2}{c_h}(K_1-1)^2}\left(2^{\frac{2}{c_h}\left((K_2-1)^2+(K_3-1)^2\right)-K_3^2}+1\right)\\ &\ll K_1^2 2^{-K_1^2+\frac{2}{c_h}\left((K_1-1)^2+(K_2-1)^2+(K_3-1)^2\right)}+K_1^2 2^{K_3^2-K_1^2+\frac{2}{c_h}(K_1-1)^2}\\ &\ll K_1^2 2^{-(K_1-1)^2+\frac{2}{c_h}\left((K_1-1)^2+(K_2-1)^2+(K_3-1)^2\right)-2K_1}\\ &\quad+\ K_1^2 2^{(K_3-1)^2-(K_1-1)^2+\frac{2}{c_h}(K_1-1)^2}\\ &\ll K_1^2 2^{\left(\frac{4}{c_h-1}-1\right)(K_1-1)^2-2K_1}+K_1^2 2^{\left(\frac{4}{c_h-1}-1\right)(K_1-1)^2-\frac{2}{c_h(c_h-1)}(K_1-1)^2}\\ &\ll K_1^2 2^{\left(\frac{4}{c_h-1}-1\right)(K_1-1)^2-2K_1}.\end{aligned}$$

Then we can write

$$\int_1^2 |E_6(\alpha;K)|\,\mathrm{d}\alpha=\sum_{K_3\le K_2\le K}\int_1^2 |E_6(\alpha;K,K_2,K_3)|\,\mathrm{d}\alpha \ll K^4 2^{\left(\frac{4}{c-1}-1\right)(K-1)^2-2K}.$$

as claimed. □

Proposition 4.7. *For any $c_h>4$ we have*

$$\int_1^2 |E_8(\alpha;K)|\,\mathrm{d}\alpha \ll K^2\,2^{(\frac{6}{c_h-1}-1)(K-1)^2-2K}.$$

Proof. Considering the two possibilities $|\omega_1|<|\omega_2|$ and $|\omega_1|\ge|\omega_2|$ we get the inequality $\frac{|\omega_3|}{|\omega_1|}\left(\frac{|\omega_1|}{|\omega_2|}+1\right)\left(\frac{|\omega_2|}{|\omega_3|}+1\right)\ll\frac{|\omega_3|}{|\omega_1|}\left(\frac{|\omega_1|}{|\omega_2|}+1\right)\frac{1}{|\omega_3|}\ll\max\left(\frac{1}{|\omega_1|},\frac{1}{|\omega_2|}\right)$. This combined with Lemma 4.3 implies that

$$\int_1^2 |E_8(\alpha,K_1,K_2,K_3,K_4)|\,\mathrm{d}\alpha \ \ll\ 2^{-K_1^2+K_4^2}\left(\sum_{\substack{(\mathfrak{p}_1,\dots,\mathfrak{p}_4')\\ |\omega_4|\le 4\cdot 2^{-K_4^2}}}\frac{1}{|\omega_1|}+\sum_{\substack{(\mathfrak{p}_1,\dots,\mathfrak{p}_4')\\ |\omega_4|\le 4\cdot 2^{-K_4^2}}}\frac{1}{|\omega_2|}\right)$$

Applying Lemma 4.4 with the notation $\nu_1=\mathfrak{p}_1\overline{\mathfrak{p}_1'}$ and $\nu_2=\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4\overline{\mathfrak{p}_2'\mathfrak{p}_3'\mathfrak{p}_4'}$, we have that

$$\begin{aligned}\sum_{\substack{(\mathfrak{p}_1,\dots,\mathfrak{p}_4')\\ |\omega_4|\le 4\cdot 2^{-K_4^2}}}\frac{1}{|\omega_1|} &\ll \sum_m 2^m\#\{(\mathfrak{p}_1,\dots,\overline{\mathfrak{p}_4}):\ |\omega_1|<2^{-m},\ |\omega_4|\le 4\cdot 2^{-K_4^2}\}\\ &\ll \sum_m 2^m\#\{(\nu_1,\nu_2):\ \|\theta(\nu_1)\|\le 2^{-m},\ \|\theta(\nu_1)+\theta(\nu_2)\|\le 4\cdot 2^{-K_4^2}\}\\ &\ll \sum_m\sum_{\|\theta(\nu_1)\|<2^{-m}}\#\{\nu_2:\ \|\theta(\nu_1)+\theta(\nu_2)\|\le 4\cdot 2^{-K_4^2}\}\\ &\ll \sum_m 2^{\frac{2}{c_h}(K_1-1)^2}\left(2^{\frac{2}{c_h}((K_2-1)^2+(K_3-1)^2+(K_4-1)^2)-K_4^2}+1\right)\\ &\ll K_1^2 2^{\frac{2}{c_h}((K_1-1)^2+(K_2-1)^2+(K_3-1)^2+(K_4-1)^2)-K_4^2}+K_1^2 2^{\frac{2}{c_h}(K_1-1)^2}.\end{aligned}$$

Similarly, but writing now $\nu_1 = \mathfrak{p}_1\mathfrak{p}_2\overline{\mathfrak{p}_1'\mathfrak{p}_2'}$ and $\nu_2 = \mathfrak{p}_3\mathfrak{p}_4\overline{\mathfrak{p}_3'\mathfrak{p}_4'}$ we have

$$\begin{aligned}
\sum_{\substack{(\mathfrak{p}_1,\dots,\mathfrak{p}_4')\\ |\omega_4|\le 4\cdot 2^{-K_4^2}}} \frac{1}{|\omega_2|} &\ll \sum_m 2^m \#\{(\mathfrak{p}_1,\dots,\overline{\mathfrak{p}_4}) :\ |\omega_2|\le 2^{-m},\ |\omega_4|\le 4\cdot 2^{-K_4^2}\}\\
&\ll \sum_{m\le K_4^2} 2^m \#\{(\nu_1,\nu_2) :\ \|\theta(\nu_1)\|\le 2^{-m},\ \|\theta(\nu_1)+\theta(\nu_2)\|\le 4\cdot 2^{-K_4^2}\}\\
&+ \sum_{m> K_4^2} 2^m \#\{(\nu_1,\nu_2) :\ \|\theta(\nu_1)\|\le 2^{-m},\ \|\theta(\nu_1)+\theta(\nu_2)\|\le 4\cdot 2^{-K_4^2}\}\\
&= S_1 + S_2
\end{aligned}$$

We observe that if $m \le K_4^2$ then $\|\theta(\nu_2)\| \le \|\theta(\nu_1)+\theta(\nu_2)\| + \|\theta(\nu_1)\| \le 5\cdot 2^{-m}$. Thus

$$\begin{aligned}
S_1 &\ll \sum_{m\le K_4^2} 2^m \#\{(\nu_1,\nu_2) :\ \|\theta(\nu_2)\|\le 5\cdot 2^{-m},\ \|\theta(\nu_1)+\theta(\nu_2)\|\le 4\cdot 2^{-K_4^2}\}\\
&\ll \sum_m 2^m \sum_{\|\theta(\nu_2)\|\le 5\cdot 2^{-m}} \#\{\nu_1 :\ \|\theta(\nu_1)+\theta(\nu_2)\|\le 4\cdot 2^{-K_4^2}\}\\
&\ll \sum_m 2^m \cdot 2^{\frac{2}{c_h}\left((K_3-1)^2+(K_4-1)^2\right)-m}\left(2^{\frac{2}{c_h}\left((K_1-1)^2+(K_2-1)^2\right)-K_4^2}+1\right)\\
&\ll K_1^2 2^{\frac{2}{c_h}((K_1-1)^2+(K_2-1)^2+(K_3-1)^2+(K_4-1)^2)-K_4^2} + K_1^2 2^{\frac{2}{c_h}\left((K_3-1)^2+(K_4-1)^2\right)}.
\end{aligned}$$

To estimate S_2, we observe that if $m > K_4^2$ then $\|\theta(\nu_2)\| \le \|\theta(\nu_1)+\theta(\nu_2)\| + \|\theta(\nu_1)\| \le 5\cdot 2^{-K_4^2}$. Thus

$$\begin{aligned}
S_2 &\ll \sum_{m> K_4^2} 2^m \#\{(\nu_1,\nu_2) :\ \|\theta(\nu_1)\|\le 2^{-m},\ \|\theta(\nu_2)\|\le 5\cdot 2^{-K_4^2}\}\\
&\ll \sum_m 2^m \cdot 2^{\frac{2}{c_h}\left((K_1-1)^2+(K_2-1)^2\right)-m}\cdot 2^{\frac{2}{c_h}\left((K_3-1)^2+(K_4-1)^2\right)-K_4^2}\\
&\ll K_1^2 2^{\frac{2}{c_h}((K_1-1)^2+(K_2-1)^2+(K_3-1)^2+(K_4-1)^2)-K_4^2}.
\end{aligned}$$

Putting together the estimates we have obtained for $\sum \frac{1}{|\omega_1|}$ and $\sum \frac{1}{|\omega_2|}$ we get

$$\begin{aligned}
\int_1^2 |E_8(\alpha,K_1,K_2,K_3,K_4)|\,\mathrm{d}\alpha &\ll K_1^2 2^{\frac{2}{c_h}((K_1-1)^2+(K_2-1)^2+(K_3-1)^2+(K_4-1)^2)-K_1^2}\\
&+ K_1^2 2^{-K_1^2+K_4^2+\frac{2}{c_h}(K_1-1)^2}\\
&+ K_1^2 2^{K_4^2-K_1^2+\frac{2}{c_h}\left((K_3-1)^2+(K_4-1)^2\right)}\\
&= T_1+T_2+T_3.
\end{aligned}$$

Using the inequalities $(K_4-1)^2 \le \frac{1}{c_h-1}\left((K_1-1)^2+(K_2-1)^2+(K_3-1)^2\right)$ and $K_4\le K_3\le K_2\le K_1$ we have

$$\begin{aligned}
T_1 &\ll K_1^2 2^{\left(-1+\frac{6}{c_h-1}\right)(K_1-1)^2-2K_1}\\
T_2 &\ll K_1^2 2^{-(K_1-1)^2+(K_4-1)^2+\frac{2}{c_h}(K_1-1)^2}\\
&\ll K_1^2 2^{\left(-1+\frac{3}{c_h-1}+\frac{2}{c_h}\right)(K_1-1)^2}\\
&\ll K_1^2 2^{\left(-1+\frac{6}{c_h-1}\right)(K_1-1)^2-2K_1}.\\
T_3 &\ll K_1^2 2^{(K_4-1)^2-(K_1-1)^2+\frac{2}{c_h}\left((K_3-1)^2+(K_4-1)^2\right)}\\
&\ll K_1^2 2^{\left(1+\frac{2}{c_h}\right)\frac{1}{c_h-1}\left((K_1-1)^2+(K_2-1)^2+(K_3-1)^2\right)-(K_1-1)^2+\frac{2}{c_h}(K_3-1)^2}\\
&\ll K_1^2 2^{\left(\left(1+\frac{2}{c_h}\right)\frac{3}{c_h-1}-1+\frac{2}{c_h}\right)(K_1-1)^2}\\
&\ll K_1^2 2^{\left(-1+\frac{6}{c_h-1}\right)(K_1-1)^2-2K_1}
\end{aligned}$$

since $c_h>4$. Finally,

$$\begin{aligned}
\int_1^2 |E_8(\alpha,K)|\,\mathrm{d}\alpha &\ll \sum_{K_4\le K_3\le K_2\le K} K^2 2^{\left(-1+\frac{6}{c_h-1}\right)(K-1)^2-2K}\\
&\ll K^5 2^{\left(-1+\frac{6}{c_h-1}\right)(K-1)^2-2K},
\end{aligned}$$

as claimed. □

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Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM) and Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049, Madrid, España

E-mail address: franciscojavier.cilleruelo@uam.es

Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049, Madrid, España

E-mail address: rafael.tesoro@estudiante.uam.es