ON THE SUM OF DIGITS OF SOME SEQUENCES OF INTEGERS

JAVIER CILLERUELO, FLORIAN LUCA, JUANJO RUÉ, AND ANA ZUMALACÁRREGUI

ABSTRACT. Let $b \geq 2$ be a fixed positive integer. We show for a wide variety of sequences $\{a_n\}_{n=1}^{\infty}$ that for most n the sum of the digits of a_n in base b is at least $c_b \log n$, where c_b is a constant depending on b and on the sequence. Our approach covers several integer sequences arising from number theory and combinatorics.

1. Introduction

For a positive integer $b \geq 2$ let us denote by $s_b(m)$ the sum of the digits of the positive integer m when written in base b. Lower bounds for $s_b(m)$ when m runs through the members of a sequence with some interesting combinatorial meaning have been investigated before. For example, it follows from a result of Stewart ([14]; see also [9] for a slightly more general result), that in the case of Fibonacci numbers (namely, the sequence defined by $F_0 := 0$, $F_1 := 1$ and $F_{n+2} := F_{n+1} + F_n$ for all $n \geq 0$) the inequality

$$s_b(F_n) > c_1 \frac{\log n}{\log \log n}$$

holds for all $n \geq 3$ for some positive constant $c_1 := c_1(b)$ depending on b. In [10], it is shown that the inequality

$$s_b(n!) > c_2 \log n$$

holds for all $n \geq 1$, where $c_2 := c_2(b)$ is some positive constant depending on b. In [12], it was shown that if we put $C_n := \frac{1}{n+1}\binom{2n}{n}$ and $D_n := \binom{2n}{n}$ for the Catalan number and the middle binomial coefficient, respectively, then both inequalities

$$s_b(C_n) > \varepsilon(n)\sqrt{\log n} \qquad \text{and} \qquad s_b(D_n) \geq \varepsilon(n)\sqrt{\log n} \tag{1}$$

hold on a set of n of asymptotic density equal to 1, where $\varepsilon(n)$ is any function tending to zero when n tends to infinity. In [13], it was shown that there is some positive constant $c_3 := c_3(b)$ depending on b such that if we put

$$A_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

for the nth Apéry number, then the inequality

$$s_b(A_n) > c_3 \left(\frac{\log n}{\log \log n}\right)^{1/4} \tag{2}$$

holds on a set of n of asymptotic density 1. Some of the above results were superseded by the results from the recent paper [8], where it is shown that if

$\mathbf{r} := (r_0, r_1, \ldots, r_m)$ is a fixed vector of nonnegative integers integers with $r_0 > 0$ and if we put

$$S_n(\mathbf{r}) := \sum_{k=0}^{n} \binom{n}{k}^{r_0} \binom{n+k}{k}^{r_1} \cdots \binom{n+km}{k}^{r_m} \quad \text{for} \quad n = 0, 1, \ldots,$$

then for $\mathbf{r} \neq (1)$ there exists a positive constant $c_4 := c_4(b, \mathbf{r})$ depending on both b and $\mathbf{r}$ such that the inequality

$$s_b\left(S_n(\mathbf{r})\right) > c_4 \frac{\log n}{\log\log n} \tag{3}$$

holds for almost all n. Note that inequality (3) improves (1) for the case of the middle binomial coefficients B_n because $C_n = S_n(\mathbf{r})$ for $\mathbf{r} = (2)$, as well as inequality (2) for the case of the Apéry numbers A_n because $A_n = S_n(\mathbf{r})$ for $\mathbf{r} = (2, 2)$.

In [11], it is shown that if P_n is the partition function of n, then the inequality

$$s_b(P_n) > \frac{\log n}{7 \log\log n}$$

holds for almost all positive integers n.

The proofs of such results use a variety of methods from number theory, such as elementary methods, sieve methods, linear forms in logarithms and the subspace theorem of Evertse–Schlickewei–Schmidt [3].

In this work we focus on sequences $\{a_n\}_{n=1}^{\infty}$ of positive integers with a certain growth, and show, independently of the combinatorial properties of the sequence, that $s_b(a_n) > c_b \log n$ for almost every element in the sequence, where c_b is a positive number depending both on b as well as on the sequence $\{a_n\}_{n=1}^{\infty}$. In particular, we concentrate on sequences satisfying the asymptotic behavior

$$a_n = e^{f(n)} \left(1 + O\left(n^{-\alpha}\right)\right), \ \alpha > 0,$$

where $f(x)$ is a two times differentiable function satisfying $f''(x) \asymp \frac{1}{x}$ for large x. Many sequences arising in number theory and combinatorics fit into this scheme. The most basic one, the number of permutations of a set of n elements is clearly a sequence of this kind, since from Stirling's approximation formula we have

$$n! = e^{n\log n - n + \log n + \frac{1}{2}\log 2\pi} \left(1 + O\left(n^{-1}\right)\right). \tag{4}$$

The sequence $a_n = \prod_{k=1}^{n}(k^2+1)$ also has similar behavior: $a_n = c_6 n!^2(1+O(n^{-1}))$. It was proved in [2] that a_n is an square only when $n = 3$.

Other interesting sequences arising from combinatorics have more involved expressions, but they also fit into these hypothesis (see [4] for further details). Examples of them are the Bell numbers (that count the number of partitions of sets), involutions (that count the number of permutations of n elements with either fixed points or cycles of length 2) and fragmented permutations (namely, unordered collections of permutations; in other words, *fragments* are obtained by breaking a permutation into pieces).

In graph enumeration, many important families also follow these asymptotic expressions: the number of labelled trees (Cayley trees) with n vertices is equal to n^{n-1}. More generally, it is shown in [4] that families of labelled trees with degree constraints satisfy asymptotic formulas of the form

$$c_{\mathcal{T}} n^{-3/2} \gamma_{\mathcal{T}}^n \cdot n! \left(1 + O(n^{-1})\right) = e^{f_{\mathcal{T}}(n)} \left(1 + O(n^{-1})\right),$$

where the subindex $\mathcal{T}$ indicates the considered constraint and the function $f_{\mathcal{T}}$ is given by

$$f_{\mathcal{T}}(n) = n\log n - n - \log n + n\log\gamma_{\mathcal{T}} + \log c_{\mathcal{T}} + \frac{1}{2}\log 2\pi.$$

Very recently, many authors have shown that several families of labelled graphs satisfies similar formulas: Giménez and Noy [6] (see also [7]) proved that the number of labelled planar graphs with n vertices follows an asymptotic formula of the form

$$c_0 n^{-7/2}\gamma^n \cdot n!\left(1+O\left(n^{-1}\right)\right),$$

where $\gamma \simeq 27.22687$. More generally, as it is shown in [5] (see also [1]), the number of labelled graphs which can be embedded in a surface of genus g satisfies a very similar formula (with the same growth factor). See Table 1 for the asymptotics of these sequences.

Sequence	Asymptotic
Permutations	$n!$
$\prod_{k=1}^{n}(k^2+1)$	$cn!^2\left(1+O(n^{-1})\right)$
Involutions	$\frac{1}{2\sqrt{\pi}}n^{-1/2}e^{n/2-1/4}n^{-n/2}\cdot n!\left(1+O\left(n^{-1/5}\right)\right)$
Bell numbers	$\frac{e^{e^r-1}}{r^n\sqrt{2\pi r(r+1)e^r}}\cdot n!\left(1+O\left(e^{-r/5}\right)\right),\ re^r=n+1$
Fragmented permutations	$\frac{1}{2\sqrt{\pi}}n^{-3/4}e^{-1/2+2\sqrt{n}}\cdot n!\left(1+O\left(n^{-3/4}\right)\right)$
Cayley trees	$\frac{1}{\sqrt{2\pi}}n^{-3/2}e^{n}\cdot n!\left(1+O\left(n^{-1}\right)\right)$
Labelled trees	$c_{\mathcal{T}}n^{-3/2}\gamma_{\mathcal{T}}^{n}\cdot n!\left(1+O\left(n^{-1}\right)\right)$
Graphs on surfaces	$c_g n^{5(g-1)/2-1}\gamma^n\cdot n!\left(1+O\left(n^{-1}\right)\right)$

TABLE 1. Combinatorial families and their enumerative asymptotic behavior.

Our main result gives a lower bound for $s_b(a_n)$ for sequences of controlled growth described before.

Theorem 1. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers with asymptotic behavior*

$$a_n = e^{f(n)}\left(1+O(n^{-\alpha})\right),\ \text{with } f''(x)\asymp\frac{1}{x}, \tag{5}$$

for some $\alpha>0$ and a two times differentiable function f. For any base $b\geq 2$, the inequality

$$s_b(a_n) > \frac{\beta\log n}{10\log b},\ \beta=\min\left\{\alpha,\frac{2}{3}\right\}$$

holds on a set of positive integers n of asymptotic density 1.

It is a straightforward calculation to check that condition (5) holds for all the sequences in Table 1, except for the Bell numbers which should be studied carefully. We denote by B_n the nth Bell number. In this case, the asymptotic estimate for B_n is given in terms of an implicit function $r=r(n)$ so the analysis of this concrete case should be made in detail. More concretely, we obtain the following corollary, which will be proved in detail in Section 3:

Corollary 2. *Let B_n denote the nth Bell number. For any base $b\geq 2$, the inequality*

$$s_b(B_n) > \frac{\log n}{60\log b}$$

holds on a set of positive integers n of asymptotic density 1.

1.1. **Notation.** We use Landau's symbol O and o as well as the Vinogradov's symbols $\ll$, $\gg$ and $\asymp$ with their usual meanings. Recall that $A = O(B)$, $A \ll B$ and $B \gg A$ are all equivalent to the fact that the inequality $|A| \le cB$ holds with some constant c. The constants implied by these symbols in our arguments might depend in the number b. Furthermore, $A \asymp B$ means that both $A \ll B$ and $B \ll A$ hold. We use $c_1, c_2, \ldots$ for positive constants depending on the number b and the sequence $\{a_n\}_{n=1}^{\infty}$.

2. Proof of Theorem 1

Consider the following set of positive integers:

$$\mathcal{N}_b(x) := \left\{ n \in [x/2, x) : s_b(a_n) < \frac{\beta \log n}{10 \log b} \right\},$$

where $\beta \le \alpha$ will be chosen later. We need to show that $\#\mathcal{N}_b(x) = o(x)$ as $x \to \infty$, since afterwards the conclusion of Theorem 1 will follow by replacing x by $x/2$, then by $x/4$, and so on, and summing up the resulting estimates.

For $n \in \mathcal{N}_b(x)$, we write

$$a_n = d_{k_1} b^{k_1} + d_{k_2} b^{k_2} + \cdots + d_{k_s} b^{k_s}, \tag{6}$$

where $d_{k_1}, \ldots, d_{k_s} \in \{1, \ldots, b-1\}$ and $k_1 > k_2 > \cdots > k_s$. Observe that for $i = 1, \ldots, s$ we have

$$a_n = d_{k_1} b^{k_1} + \cdots + d_{k_i} b^{k_i} \left(1 + E_i(n)\right),$$

where $E_i(n) = 0$, if $i = s$, and

$$E_i(n) = \frac{d_{k_{i+1}} b^{k_{i+1}} + \cdots + d_{k_s} b^{k_s}}{d_{k_1} b^{k_1} + \cdots + d_{k_i} b^{k_i}} = O\left(b^{k_{i+1} - k_1}\right),$$

if $i < s$. We choose $k(n)$ to be the smallest k_i such that $b^{k_i - k_1} > n^{-\beta}$.

From the definition of $k(n)$, we immediately see that

$$a_n = \left(d_{k_1} b^{k_1} + \cdots + d_{k(n)} b^{k(n)}\right)\left(1 + O\left(n^{-\beta}\right)\right) = b^{k(n)} D(n) \left(1 + O\left(n^{-\beta}\right)\right), \tag{7}$$

where $D(n) = d_{k_1} b^{k_1 - k(n)} + d_{k_2} b^{k_2 - k(n)} + \cdots + d_{k(n)}$.

Let $\mathcal{D}_b(x)$ be the subset of all possible values for $D(n)$, $n \in \mathcal{N}_b(x)$. Let us find an upper bound for the cardinality of this set. First observe that

$$D(n) < b^{k_1 - k(n) + 1} \le b^{(\beta \log n / \log b) + 1}.$$

The positive integers $D := D(n)$ bounded by the right hand side of the above inequality have at most $K := \lfloor (\beta \log x / \log b) + 2 \rfloor$ digits in base b. As $n \in \mathcal{N}_b(x)$, the number of nonzero digits of $D(n)$ is bounded by $S := \lfloor (\beta \log x / 10 \log b) \rfloor$, and

$$\begin{aligned} \#\mathcal{D}_b(x) &\le \sum_{i=0}^{S} \binom{K}{i} (b-1)^i \le (S+1) \binom{K}{S} (b-1)^S \le (S+1) \left(\frac{(b-1)eK}{S} \right)^S \\ &\le \left(\frac{\beta \log x}{10 \log b} + 1 \right) \left(10e(b-1) + o(1)\right)^{\frac{\beta \log x}{10 \log b}} = x^{\delta + o(1)} \end{aligned}$$

as $x \to \infty$, where

$$\delta := \frac{\beta \log(10e(b-1))}{10 \log b}.$$

It can be checked that $\delta < \beta/2$ for all integers $b \geq 2$. Thus, we get that

$$\#\mathcal{D}_b(x) \leq x^{\delta+o(1)} \quad \text{as} \quad x \to \infty. \tag{8}$$

Combining the fact that $a_n = e^{f(n)}\,(1+O(n^{-\alpha}))$ with relations (6) and (7) we have

$$e^{f(n)} = b^{k(n)} D(n) \left(1 + O\left(x^{-\beta}\right)\right),$$

since $n \in [x/2, x)$ and $\beta \leq \alpha$ by hypothesis. Taking logarithms, we get that

$$f(n) = k(n)\log b + \log D(n) + O\left(x^{-\beta}\right). \tag{9}$$

We now write

$$\mathcal{N}_b(x) = \bigcup_{D \in \mathcal{D}_b(x)} \mathcal{N}_{b,D}(x),$$

where

$$\mathcal{N}_{b,D}(x) := \{n \in \mathcal{N}_b(x) : D(n) = D\}.$$

Observe that, with this notation, we have

$$\#\mathcal{N}_b(x) \leq \#\mathcal{D}_b(x) \max_{D \in \mathcal{D}_b} \#\mathcal{N}_{b,D}(x),$$

and we must now bound the number of elements lying in each $\mathcal{N}_{b,D}(x)$.

For a fixed $D \in \mathcal{D}_b(x)$ and y depending on x, to be chosen later, we take a look at the elements $n \in \mathcal{N}_{b,D}(x)$. We say that n is *separated* if $[n, n+y] \cap \mathcal{N}_{b,D}(x) = \{n\}$. It is clear that there are at most $x/2y+1$ elements on $\mathcal{N}_{b,D}(x)$ which are separated.

Let us now count the non-separated elements $n \in \mathcal{N}_{b,D}(x)$. For such an n, there exists $1 \leq m \leq y$ with $n+m \in \mathcal{N}_{b,D}(x)$. Taking the difference of the relations (9) in $n,\ n+m \in \mathcal{N}_{b,D}(x)$ we get

$$\begin{aligned}(k(n+m) - k(n))\log b &= (f(n+m) - f(n)) + O(x^{-\beta})\\ &= mf'(\zeta) + O(x^{-\beta}),\end{aligned}$$

where $\zeta \in [n, n+m]$ is some point whose existence is guaranteed by the Intermediate Value Theorem. It follows from condition (5), which in particular implies $f'(x) \asymp \log x$, that $k(n+m) \neq k(n)$ for large x (as $x/2 < n < x$) in the above estimate. Thus, non-separated elements n in $\mathcal{N}_{b,D}(x)$ are characterized by their values $k(n)$. Denoting by $[x]$ the closest integer to x, for a fixed $m \leq y$, the differences

$$k(m+n) - k(n) = \left[\frac{mf'(\zeta)}{\log b}\right] \tag{10}$$

take $O(m)$ integer values, since for two elements $n, n+\ell \in \mathcal{N}_{b,D}(x)$ we have by condition (5)

$$\frac{m}{\log b}\left(f'(\zeta_{n+\ell}) - f'(\zeta_n)\right) \asymp \frac{m\ell}{x\log b} = O(m).$$

For a fixed difference in (10), say M, we must be able to count the number elements $n \in \mathcal{N}_{b,D}(x)$ such that

$$k(n+m) - k(n) = M + O(n^{-\beta}),$$

but it follows from the previous argument that

$$\frac{m}{\log b}\left(f'(\zeta_{n+\ell}) - f'(\zeta_n)\right) = O(x^{-\beta})$$

for at most $O(1+x^{1-\beta}/m)$ values of n. Thus, there are $O(y^2+yx^{1-\beta})$ non-separated elements in $\mathcal{N}_{b,D}(x)$, for an arbitrary $D \in \mathcal{D}_b(x)$. Setting $y := x^{\beta/2}$, we observe that

$$\#\mathcal{N}_{b,D}(x) \ll yx^{1-\beta} + y^2 + \frac{x}{y} + 1 \ll x^{1-\beta/2} + x^{\beta} \ll x^{1-\beta/2},$$

whenever $\beta \le 2/3$. Thus, if we choose $\beta := \min\{\alpha, 2/3\}$ it follows from estimate (8) that

$$\#\mathcal{N}_b(x) = \sum_{D \in \mathcal{D}_b(x)} \#\mathcal{N}_{b,D}(x) \le x^{1-\beta/2} \#\mathcal{D}_b(x) < x^{1-\beta/2+\delta+o(1)} = o(x)$$

as $x \to \infty$, which is what we wanted to prove.

3. Proof of Corollary 2

The study of Bell numbers needs of a more detailed analysis. We start with the following estimate for B_n (see formula (41) on page 562 in [4]).

Lemma 3. *Let $r := r(n)$, defined implicitly by*

$$re^r = n + 1. \tag{11}$$

Then

$$B_n = \frac{n! e^{e^r - 1}}{r^n \sqrt{2\pi r(r+1)e^r}} \left(1 + O\left(e^{-r/5}\right)\right). \tag{12}$$

The number $r := r(n)$ given in (11) satisfies $r = \log n - \log\log n + o(1)$ as $n \to \infty$, therefore

$$e^{-r/5} = \left(\frac{\log n}{n}\right)^{1/5} (1 + o(1)) = O\left(n^{-1/6}\right) \quad \text{as} \quad n \to \infty. \tag{13}$$

Combining Stirling's formula (4) with formula (13) we can rewrite (12) as

$$B_n = e^{f(n)} \left(1 + O\left(n^{-1/6}\right)\right),$$

where

$$f(x) = x\log x - x - \left(\frac{2x+1}{2}\right)\log r + \frac{1}{2}\log x + e^r - \frac{r}{2} - \frac{1}{2}\log(r+1) - 1,$$

and $r := r(x)$ is defined for all real numbers $x \ge 1$ by equation (11) (where n is replaced by x). In particular, $r(x)$ has a derivative for real $x > 1$. In fact, differentiating relation (11) (with x instead of n) with respect to the variable x, we have

$$r'e^r + rr'e^r = 1,$$

or equivalently

$$r'e^r = \frac{1}{r+1}, \tag{14}$$

and, since $e^r = (x+1)/r$,

$$r' = \frac{r}{(x+1)(r+1)}. \tag{15}$$

We get the asymptotic behavior of the second derivative of $f(x)$: observe that differentiating we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x\log x - x - \frac{2x+1}{2}\log r + \frac{1}{2}\log x + e^r - \frac{r}{2} - \frac{1}{2}\log(r+1) - 1\right) \\ &= \log x - \log r - \frac{(2x+1)r'}{2r} + \frac{1}{2x} + r'e^r - \frac{r'}{2} - \frac{r'}{2(r+1)} \\ &= \log x - \log r + \frac{1}{2x} - e^{-r}\left(\frac{1}{2(r+1)^2} + \frac{1}{r+1} - \frac{1}{2r}\right), \end{aligned}$$

since, using equations (14) and (15), we note that

$$r'e^r - r'\left(\frac{(2x+1)(r+1)+r(r+1)+r}{2r(r+1)}\right) = \frac{1}{r+1} - \frac{(2x+1)(r+1)+r(r+2)}{2(r+1)^2(x+1)}$$
$$= -\frac{r^2+r-1}{2(x+1)(r+1)^2}$$
$$= -e^{-r}\left(\frac{1}{2(r+1)^2} + \frac{1}{r+1} - \frac{1}{2r}\right). \tag{16}$$

Differentiating expression (16) we obtain

$$\frac{d}{dx}\left[-e^{-r}\left(\frac{1}{2(r+1)^2} + \frac{1}{r+1} - \frac{1}{2r}\right)\right] =$$
$$= r'e^{-r}\left(\frac{1}{(r+1)^3} + \frac{3}{2(r+1)^2} + \frac{1}{r+1} - \frac{1}{2r^2} - \frac{1}{2r}\right)$$
$$= \frac{r^2}{(x+1)^3}\left(\frac{1}{2(r+1)^3} + \frac{3}{2(r+1)^2} + \frac{1}{r+1} - \frac{1}{2r^2} - \frac{1}{2r}\right) = O(x^{-2}),$$

therefore we can conclude that

$$f''(x) = \frac{1}{x} + \frac{r'}{r} + O(x^{-2}) = \frac{1}{x} + \frac{1}{(x+1)(r+1)} + O(x^{-2}) \asymp \frac{1}{x},$$

and we are under the assumptions of Theorem 1, and Corollary 2 holds.

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J. Cilleruelo: Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM) and Departamento de Matemáticas Universidad, Autónoma de Madrid, 28049 Madrid, Spain

E-mail address: franciscojavier.cilleruelo@uam.es

F. Luca: Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México, 58089 Morelia, Michoacán, Mexico

E-mail address: fluca@matmor.unam.mx

J. Rué: Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), 28049 Madrid, Spain

E-mail address: juanjo.rue@icmat.es

A. Zumalacárregui: Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM) and Departamento de Matemáticas Universidad Autónoma de Madrid, 28049 Madrid, Spain

E-mail address: ana.zumalacarregui@uam.es