Fibonacci lattice points

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1 Introduction

Let $r(n)$ be the number of integer solutions (x, y) of the Diophantine equation $x^2 + y^2 = n$. It is known that $r(n)$ is an unbounded function. Consider the polygon with vertices in the $r(n)$ lattice points (x, y). Clearly, all these lattice points lie on the circle of radius $\sqrt{n}$ centered in the origin. The distribution of these points on the above circle was studied in [3] and [4]. In order to study the above distribution, let $S(n)$ denotes the area of the polygon whose vertices are the $r(n)$ lattice points. If the above $r(n)$ lattice points are well-distributed, then $S(n)$ should be close to the area of the circle which is πn.

If $r(n) > 0$, then trivially $2/\pi \le S(n)/\pi n < 1$. In [3], it was proved that the inequality $|S(n)/\pi n - 1| \ll (\log\log n/\log n)^2$ holds infinitely often. It [4], it was shown that the inequality $|S(n)/\pi n - 1| \ll (\log\log\log n/\log\log n)^2$ holds for most positive integers n having $r(n) > 0$.

Let $(F_n)_{n\ge 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \ge 0$. Since

$$F_{2n+1} = F_n^2 + F_{n+1}^2, \tag{1}$$

it follows that $r(F_{2n+1}) > 0$ for all $n \geq 0$. In [6] and Lemma 1 in [1], it was shown that the equality $r(F_{2n}) = 0$ holds for most positive integers n.

In this paper, we investigate the size of $S(F_{2n+1})$. We have the following result.

Theorem 1. *There exists a positive constant c_1 such that*

(i) The inequality

$$\left|\frac{S(F_n)}{\pi F_n} - 1\right| \ll \frac{1}{(\log\log n)^{c_1}} \tag{2}$$

holds for most odd integers n.

(ii) The inequality

$$\left|\frac{S(F_n)}{\pi F_n} - 1\right| \ll \left(\frac{\log\log n}{\log n}\right)^{c_1} \tag{3}$$

holds for infinitely many positive integers n.

In [3], it was also proved that the set $\{S(n)/\pi n : r(n) > 0\}$ is dense in $[2/\pi, 1]$. It turns out that this is not the case for the set $\{S(F_{2n+1})/\pi F_{2n+1}\}_{n\geq 0}$. We have the following result.

Theorem 2. *i) For any $\epsilon > 0$, the elements of the set $\{S(F_{2n+1})/\pi F_{2n+1}\}_{n\geq 0}$ lying in $[\frac{2}{\pi}, \frac{6}{\pi\sqrt{5}} - \epsilon]$ form a finite set.*

ii) The number of elements of the set $\{S(F_{2n+1})/\pi F_{2n+1}\}_{n\geq 0}$ lying in $[\frac{2}{\pi}, \frac{6}{\pi\sqrt{5}}]$ is infinite if and only if the sequence F_{4n+3} contains infinitely primes.

We believe that the set $\{S(F_{2n+1})/\pi F_{2n+1}\}_{n\geq 0} \cap [\frac{6}{\pi\sqrt{5}}, 1]$ is a dense set in $[\frac{6}{\pi\sqrt{5}}, 1]$, but this seems to be a very difficult problem to solve.

Throughout this paper, we use the standard notations $\ll$, $\gg$, O and o with their regular meaning.

2 The Proof of Theorem 1

We start with (i). Let x be a large positive real number. Let $n \leq x$ be odd. We show that estimate (2) holds for all such n with $o(x)$ exceptions as $x \to \infty$. We may assume that $n \geq x/\log x$. Let $\omega(n)$ be the number of distinct prime factors of n. By the Turán-Kubilius estimate,

$$\sum_{n\leq x}(\omega(n) - \log\log x)^2 = O(x \log\log x),$$

it follows that the estimate $\omega(n) > 0.5 \log\log x$ holds for all odd $n \le x$ with $o(x)$ exceptions as $x \to \infty$. Since $\pi(10 \log\log x) \ll \log\log x/\log\log\log x = o(\log\log x)$, it follows that for all odd $n \le x$ have at least $K := \lfloor 0.25 \log\log x \rfloor$ distinct prime factors $p > L := \lfloor 10 \log\log x \rfloor$ except for some subset of them of cardinality $o(x)$ as $x \to \infty$. Write

$$n = p_1 \cdots p_K m,$$

where $L < p_1 < \cdots < p_K$ are distinct primes and m is an integer. Since $p_i \mid n$, it follows that $F_{p_i} \mid F_n$ for $i = 1, \ldots, K$. Since F_a and F_b are coprime when a and b are coprime positive integers, it follows that $\prod_{i=1}^K F_{p_i}$ is a divisor of F_n. Write

$$F_n = F_{p_1} \cdots F_{p_K} M,$$

for some positive integer M. Since n is odd, it follows that al divisors of F_n are either 2 or are primes which are congruent to 1 modulo 4. Indeed, this is an easy consequence of representation (1) together with the fact that F_n and F_{n+1} are coprime. Now by (1) we have

$$F_{p_i} = F^2_{(p_i-1)/2} + F^2_{(p_i+1)/2}.$$

Put

$$\Phi_{p_i} := (4/\pi) \tan^{-1}\left(F_{(p_i+1)/2}/F_{(p_i-1)/2}\right) \qquad \text{for} \quad i = 1, \ldots, K.$$

Write also

$$F_m = \frac{\alpha^m - \beta^m}{\alpha - \beta} \quad \text{for} \quad m \ge 0, \quad \text{where} \quad (\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right).$$

Then

$$\frac{F_{m+1}}{F_m} = \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha^m - \beta^m} = \alpha + O\left(\frac{1}{\alpha^{2m}}\right)$$

holds for all $m \ge 1$, therefore

$$\frac{F_{(p_i+1)/2}}{F_{(p_i-1)/2}} = \alpha + O\left(\frac{1}{\alpha^{p_i}}\right) = \alpha + O\left(\frac{1}{(\log n)^2}\right) \quad \text{holds for} \quad i = 1, \ldots, K,$$

where we used the fact that since $p_i > L$, we have that $\alpha^{p_i} > (\log n)^2$. Thus, by Taylor's formula,

$$\Phi_{p_i} = \gamma + O\left(\frac{1}{(\log n)^2}\right) \qquad \text{holds for} \quad i = 1, \ldots, K, \tag{4}$$

where $\gamma := (4/\pi)\tan^{-1}(\alpha)$. It is well-known and easy to prove that γ is irrational. Write also $M = a^2 + b^2$ with some integers $0 \le a \le b$ and put $\Phi = (4/\pi)\tan^{-1}(a/b)$. Proposition 2.4 in [4] implies that the angles

$$(\pi/4)\left(\Phi + \sum_{i=1}^{K} \varepsilon_j \Phi_{p_j}\right), \qquad \varepsilon_j \in \{\pm 1\} \quad \text{for} \quad j = 1, \ldots, K \tag{5}$$

correspond to lattice points on the circle $x^2 + y^2 = F_n^2$. Using estimate (4) in (5), we get that among the angles (5) we also have all the angles

$$(\pi/4)\left(\Phi + (K - 2i)\gamma + O\left(\frac{\log\log n}{(\log n)^2}\right)\right), \qquad i = 0, 1, \ldots, K. \tag{6}$$

Indeed, this can be deduced by taking in (4) $K - i$ of the signs ε_j to equal 1 and the remaining i of them to equal -1. We next show that for some constant $c_2 > 0$ every arc of length $> 1/K^{c_2}$ contains one of the points from list (6). To see why, take $\gamma_1 := \gamma/4$ and put $x_i = \{i\gamma_1\}$ for $i = 1, \ldots, K$, where for a number x we write $\{x\}$ for the fractionary part of x. For each interval $\mathcal{J}$ of $[0, 1)$, let $V(\mathcal{J}, K) = \{1 \le i \le K : x_i \in \mathcal{J}\}$. Let $D(K)$ be the discrepancy of the sequence $\mathbf{x} := (x_i)_{i=1}^K$ defined as

$$D(K) = \sup_{\mathcal{J} \subset [0,1]} \left| \frac{V(\mathcal{J}, K)}{K} - |\mathcal{J}| \right|.$$

Here, $|\mathcal{J}|$ denotes the length of $\mathcal{J}$. By Theorem 3.2 in Chapter 2 in [5], we know that if the type of γ_1 is finite τ, then $D(K) \le K^{-1/\tau + o(1)}$ as $K \to \infty$, where

$$\tau = \sup\left\{\rho \in \mathbb{R} : \liminf_{m \to \infty} m^{\rho} \|\gamma_1 m\| = 0\right\}.$$

Thus, assuming that τ is finite, it follows that if we take $c_2 := 1/(2\tau)$, and keeping in mind that the error in (6) tends to zero much faster than any power of negative exponent of K, we conclude that for large x, any arc of length $> 1/K^{c_2}$ contains one of the points from list (6).

It remains to justify that τ is finite. For this, it suffices to shows that the inequality

$$|2\tan^{-1}(\alpha) - 2p\pi/q| \gg q^{-c_3}$$

holds for all rational numbers p/q with some constant c_3. We may assume that $2p\pi/q$ is very close to $2\tan^{-1}(\alpha)$. Since $\tan(2\tan^{-1}(\alpha)) = 2$, we get that

$$\begin{aligned}
|2\tan^{-1}(\alpha) - 2p\pi/q| &\asymp |e^{i2\tan^{-1}(\alpha)} - e^{2\pi i p/q}| \\
&= |e^{i\tan^{-1}(2)} - e^{2\pi i p/q}| \\
&= \left|\frac{1+2i}{\sqrt{5}} - e^{2p\pi i/q}\right| \\
&\gg \frac{1}{q}\left|\left(\frac{1+2i}{\sqrt{5}}\right)^q - 1\right| \gg \frac{1}{q^{c_3}}
\end{aligned}$$

where all the above inequalities are obvious for $2p\pi/q$ in a small neighborhood $2\tan^{-1}(\alpha)$ except for the last one which follows by a classical application of a lower bound for a linear form in logarithms of algebraic numbers.

Finally, an easy geometrical argument (see the proof of Proposition 3.1 in [4]) now shows that

$$\left|\frac{S(F_n)}{\pi F_n} - 1\right| \ll \frac{1}{K^{2c_2}} \ll \frac{1}{(\log\log n)^{c_1}},$$

with $c_1 := 2c_2$, which is what we wanted to prove. This takes care of (i).

The proof of (ii) is similar, except that for (ii) we start with a large x, put $L := \lfloor 10 \log\log x \rfloor$ and let $p_1 < \cdots < p_K$ be all the primes in $[L, (\log x)/2]$. By the Prime Number Theorem, $K \asymp \log x / \log\log x$. Put

$$n = p_1 \cdots p_K.$$

Again by the Prime Number Theorem, we have

$$n = \prod_{L \le p \le (\log x)/2} p = x^{1/2+o(1)}$$

as $x \to \infty$. We thus have that for large x the number n is odd and smaller than x. Now the previous argument shows that

$$\left|\frac{S(F_n)}{\pi F_n} - 1\right| \ll \frac{1}{K^{2c_2}} \ll \left(\frac{\log\log x}{\log x}\right)^{c_1} \ll \left(\frac{\log\log n}{\log n}\right)^{c_1},$$

which is what we wanted to prove.

3 The proof of Theorem 2

Let us say that m is a 2-prime if $m = 2^k p$, where $k \ge 0$ and $p = 1$, or p is prime. Positive integers m which are 2-primes are characterized by the fact that whenever $m = a^2 + b^2$ with integers a and b, then a and b are uniquely determined up to signs and order. We next record that if F_n is a 2-prime, then $n \in \{1, 2, 3, 4, 6, 8, 9\}$, or $n \geq 5$ is prime and F_n is also prime. Indeed, this follows easily from [2].

The proof of Theorem 2 is a corollary of the following lemma.

Lemma 1. *For any $n \geq 1$, then $\frac{S(F_{2n+1})}{\pi F_{2n+1}} < \frac{6}{\pi\sqrt{5}}$ if and only if n is odd and F_{2n+1} is a 2-prime.*

Proof. Since $F_{2n+1} = F_{n+1}^2 + F_n^2$, the circle $x^2 + y^2 = F_{2n+1}$ contains lattice points at the angles

$$\Psi_n, \pi/2 - \Psi_n, \Psi_n + \pi/2, \pi - \Psi_n, \Psi_n + \pi, 3\pi/2 - \Psi_n, \Psi_n + 3\pi/2, 2\pi - \Psi_n, \quad (7)$$

where $\Psi_n = \tan^{-1}(F_n/F_{n+1})$.

We observe that $\lim_{n\to\infty} \Psi_n = \Psi = \tan^{-1}(\alpha^{-1})$, that $\Psi_n < \Psi$ when n is even, and that $\Psi_n > \Psi$ if n is odd.

If $\phi_1, \ldots, \phi_k$ denote the counter-clockwise ordered angles of the lattice points on the circle of radius $\sqrt{n}$, we then have

$$S(n) = \frac{n}{2} \sum_{i=1}^{k} \sin(\phi_{i+1} - \phi_i), \quad (8)$$

where we make the convention that $\phi_{k+1} = \phi_1$.

In particular, the area of the polygon determined by the angles shown at (7) is

$$2F_{2n+1} \left(\cos(2\Psi_n) + \sin(2\Psi_n)\right).$$

Thus,

$$\frac{S(F_{2n+1})}{\pi F_{2n+1}} \geq \frac{2}{\pi} \left(\cos(2\Psi_n) + \sin(2\Psi_n)\right),$$

and the equality holds if the circle $x^2 + y^2 = F_{2n+1}$ does not contain more angles than the (at most) eight angles described above, and this happens only if F_{2n+1} is a 2-prime.

Since $\tan^{-1}(1/2) \leq \Psi_n \leq \tan^{-1}(1)$ for $n \geq 1$, and the function $f(x) = \cos(2x) + \sin(2x)$ is decreasing in that interval, we deduce that:

- If n is even, then $\frac{S(F_{2n+1})}{\pi F_{2n+1}} > \frac{2}{\pi} f(\Psi) = \frac{6}{\pi\sqrt{5}}$;
- If n is odd and F_{2n+1} is a 2-prime, then $\frac{S(F_{2n+1})}{\pi F_{2n+1}} < \frac{2}{\pi} f(\Psi) = \frac{6}{\pi\sqrt{5}}$.

To conclude the proof, we have to prove that if n is odd and F_{2n+1} is not a 2-prime, then $S(F_{2n+1})/\pi F_{2n+1} > \frac{6}{\pi\sqrt{5}}$. Since $F_3 = 2, F_7 = 13$ and $F_{11} = 89$ are prime numbers, we can assume that $n \geq 7$.

If F_{2n+1} is not prime and n is odd, then besides Ψ_n, which is larger than Ψ when n is odd, there exist other lattice points on the circle $x^2+y^2 = F_{2n+1}$ with an angle $\Phi \in [0, \pi/4]$.

Thus, the circle $x^2+y^2 = F_{2n+1}$ contains lattice points at angles $\Psi_n, \Phi, \frac{\pi}{2} - \Phi, \frac{\pi}{2} - \Psi_n$ and all the translations of these angles by a multiple of $\pi/2$.

Using (8), we can compute that the area of the polygon determined by these angles equals:

$$\begin{cases} \frac{F_{2n+1}}{2}\left(8\sin(\Phi - \Psi_n) + 4\cos(2\Phi) + 4\cos(2\Psi_n)\right), & \text{if } \Phi > \Psi_n; \\ \frac{F_{2n+1}}{2}\left(8\sin(\Psi_n - \Phi) + 4\cos(2\Psi_n) + 4\cos(2\Phi)\right), & \text{if } \Phi < \Psi_n. \end{cases}$$

Using easy trigonometric manipulations, we can resume both formulas above in the common expression

$$2F_{2n+1}\big(2\sin(|\Psi_n - \Phi|)(1 - \sin(\Phi + \Psi_n)) + f(\Psi_n)\big),$$

where $f(x) = \cos(2x) + \sin(2x)$. Thus,

$$\frac{S(F_{2n+1})}{2F_{2n+1}} \geq 2\sin(|\Psi_n - \Phi|)(1 - \sin(\Phi + \Psi_n)) + f(\Psi_n).$$

Since the distance between two lattice points in the circle is $\geq \sqrt{2}$, we have that

$$|\Phi - \Psi_n| > \sqrt{2}/\sqrt{F_{2n+1}}. \tag{9}$$

On the other hand, it is easy to compute that for n odd we have

$$\tan\Psi_n - \tan\Psi = \frac{F_n}{F_{n+1}} - \alpha^{-1} = \frac{\sqrt{5}}{\alpha^{2n+2} - 1}. \tag{10}$$

We have to prove that

$$2\sin(|\Phi - \Psi_n|)(1 - \sin(\Phi + \Psi_n)) + f(\Psi_n) \geq f(\Psi) = \frac{3}{\sqrt{5}},$$

which is equivalent to proving that

$$2\sin(|\Phi - \Psi_n|)(1 - \sin(\Phi + \Psi_n)) > f(\Psi) - f(\Psi_n).$$

To estimate the left hand from below, we will use that $\sin x \geq \frac{2\sqrt{2}}{\pi}x$ for $0 < x \leq \pi/4$, that $\Phi + \Psi_n \leq \pi/4 + \tan^{-1}(13/21)$, when $n \geq 7$ is odd, and that the estimate $|\Phi - \Psi_n| \geq \sqrt{2}/\sqrt{F_{2n+1}}$ holds. Thus,

$$\begin{aligned} 2\sin(\Phi - \Psi_n)\,(1 - \sin(\Phi + \Psi_n)) &\geq \frac{4\sqrt{2}}{\pi}(\Phi - \Psi_n)\left(1 - \sin\left(\frac{\pi}{4} + \tan^{-1}\left(\frac{13}{21}\right)\right)\right) \\ &\geq \frac{8}{\pi}\left(1 - \frac{34}{\sqrt{610}}\right)\frac{1}{\sqrt{F_{2n+1}}}. \end{aligned}$$

To estimate the right hand from above, we use that for $n \geq 7$ odd, we have that $\alpha^{-1} < \tan(\Psi_n) \leq 13/21$, and that $F_{2n+2} < \alpha^{2n+2}/\sqrt{5}$. Hence,

$$\begin{aligned} f(\Psi) - f(\Psi_n) &= \cos(2\Psi) - \cos(2\Psi_n) + \sin(2\Psi) - \sin(2\Psi_n) \\ &= 2\left(\frac{1}{\tan^2(\Psi)+1} - \frac{1}{\tan^2(\Psi_n)+1} + \frac{\tan(\Psi)}{\tan^2(\Psi)+1} - \frac{\tan(\Psi_n)}{\tan^2(\Psi_n)+1}\right) \\ &= (\tan(\Psi_n) - \tan(\Psi))\left(\frac{\tan(\Psi)\tan(\Psi_n) + \tan(\Psi) + \tan(\Psi_n) - 1}{(\tan^2(\Psi)+1)(\tan^2(\Psi_n)+1)}\right) \\ &\leq (\tan(\Psi_n) - \tan(\Psi))\left(\frac{\tan(\Psi)\tan(\Psi_n) + \tan(\Psi) + \tan(\Psi_n) - 1}{(\tan^2(\Psi)+1)^2}\right) \\ &= (\tan(\Psi_n) - \tan(\Psi))\left(\frac{\alpha^{-1}\tan(\Psi_n) + \alpha^{-1} + \tan(\Psi_n) - 1}{(\alpha^{-2}+1)^2}\right) \\ &= (\tan(\Psi_n) - \tan(\Psi))\left(\frac{(2+\sqrt{5})\tan(\Psi_n) - 1}{5}\right) \\ &< \frac{(2+\sqrt{5})\tan(\Psi_n) - 1}{\sqrt{5}(\alpha^{2n+2} - 1)} < \frac{(2+\sqrt{5})(13/21) - 1}{\sqrt{5}(\sqrt{5}F_{2n+2} - 1)} \\ &< \left(\frac{5 + 13\sqrt{5}}{105}\right)\frac{1}{F_{2n+2} - 1}. \end{aligned}$$

Finally, we observe that the inequality

$$\frac{8}{\pi}\left(1 - \frac{34}{\sqrt{610}}\right)\frac{1}{\sqrt{F_{2n+1}}} > \left(\frac{5 + 13\sqrt{5}}{105}\right)\frac{1}{F_{2n+2} - 1}$$

holds for $n \geq 7$. This completes the proof of the lemma. □

To derive Theorem 2 from the lemma above, we observe that only when F_{2n+1} is 2-prime and n is odd (hence, F_{2n+1} is prime), we have that

$$\frac{S(F_{2n+1})}{\pi F_{4n-1}} < \frac{6}{\pi\sqrt{5}},$$

and that

$$\lim_{\substack{n\to\infty \\ F_{2n+1} \text{ 2-prime}}} \frac{S(F_{2n+1})}{\pi F_{2n+1}} = \frac{6}{\pi\sqrt{5}}.$$

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References

[1] C. Ballot and F. Luca, ' On the equation $x^2 + dy^2 = F_n$', *Acta Arith.* **127** (2007), 145–155.

[2] Y. Bugeaud, F. Luca, M. Mignotte and S. Siksek, 'On Fibonacci numbers with few prime divisors', *Proceedings of the Japan Academy* **81** (2005), 17–20.

[3] J. Cilleruelo, 'The distribution of lattice points on circles', *J. Number Theory* **43** (1993), 198–202.

[4] J. Cilleruelo, 'Lattice points on circles', *J. Austral Math. Soc.* **72** (2002), 217–222.

[5] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley-Interscience, New York, 1974.

[6] F. Luca, 'Prime factors of Fibonacci numbers, solution to Advanced Problem $H546$, *Fibonacci Quart.* **42** (2004).