PALINDROMES IN LINEAR RECURRENCE SEQUENCES

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Abstract. We prove that for any base $b \geq 2$ and for any linear homogeneous recurrence sequence $\{a_n\}_{n\geq 1}$ satisfying certain conditions, there exits a positive constant $c > 0$ such that $\#\{n \leq x : \ a_n \text{ is palindromic in base } b\} \ll x^{1-c}$.

1. Introduction

Probably $F_6 = 55$ is the largest palindromic Fibonacci number. It seems, however, a hard problem to decide if there are only finitely many of these numbers. Luca proved that for any base $b \geq 2$, the set

$$\{n\colon F_n \text{ is palindromic in base } b > 1\}$$

has zero density [L]. We will use a distinct approach to prove a stronger and more general result for a broader class of linear recurrent sequences.

Theorem 1.1. *Let $b \geq 2$ be an integer and let $\{a_n\}_{n\geq 1}$ be the linear recurrent sequence of integers of minimal recurrence relation*

$$a_{n+k} = c_1 a_{n+k-1} + \cdots + c_k a_n, \qquad (n \geq 1), \tag{1.1}$$

where $c_i \in \mathbb{Z}$ for $1 \leq i \leq k$. If the polynomial $C(X) = X^k - c_1 X^{k-1} - \cdots - c_k$ has a unique dominant root $\alpha_1 > 0$ which is multiplicatively independent with b, then there exists $c = c(b) > 0$ such that

$$\#\{n \leq x : \ a_n \textit{ is palindromic in base } b\} = O(x^{1-c}).$$

An inmediate corollary is that the number of Fibonnaci numbers up to x which are palindromes in any base is $O(x^{1-c})$, for some constant $c > 0$. We prove that in this case we can take $c = 10^{-11}$.

Corollary 1.2. *We have that*

$$\#\{n \leq x : \ F_n \textit{ is palindrome in base } 10\} \ll x^{1-10^{-11}}.$$

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2. Preliminary results

In this section, we recall several well known results that will be used in the paper. The linear recurrence sequence given by (1.1) can be *solved* as follows.

Theorem 2.1. *The general solution of* (1.1) *is given by*

$$a_n = \sum_{i=1}^{R} \alpha_i^n p_i(n), \tag{2.1}$$

where the corresponding characteristic polynomial

$$X^k - c_1 X^{k-1} - \cdots - c_{k-1}X - c_k = \prod_{i=1}^{R} (X - \alpha_i)^{m_i},$$

has R distinct complex roots α_i with multiplicity m_i, and $p_i(X)$ is a polynomial of degree $m_i - 1$ and coefficients determined by the first k terms of the sequence $\{a_n\}_{n\geq 1}$ for $i = 1, \ldots, R$.

For more details refer to [E, §2.3, 2.5].

We say that α_1 is *dominant* if $|\alpha_1| > |\alpha_i|$ for all $1 < i \leq R$ (if the dominant root exists, we can always index it as the first one by rearranging the roots if needed). Clearly, the dominant root is real, has $|\alpha_1| > 1$ and $p_1(X)$ is a polynomial with real coefficients. In particular, the sign of a_n is the same as the sign of the leading term of $p_1(x)$ for all large n when $\alpha_1 > 0$, whereas the sign of a_n is $(-1)^n$ times the sign of the leading term of $p_1(X)$ for all large n when $a_1 < 0$. Thus, by replacing $C(X)$ with $(-1)^k C(-X)$, and simultaneously changing the signs of $p_i(X)$ for all $i = 1, \ldots, R$, if needed (operations which do not change $|a_n|$ for any $n \geq 1$), we may assume that $\alpha_1 > 0$ and that a_n is positive for all large n.

Lemma 2.2. *Let M be an integer greater than 1. Any recurrence sequence satisfying* (1.1) *is periodic modulus M. The period $m = m(M)$ satisfies $m \leq M^k$.*

Proof. Consider the k-tuples $\overline{a}_r = (a_r, a_{r+1}, \ldots, a_{r+k-1})$, $1 \leq r \leq M^k + 1$. By the pigeon-hole principle, two of them are equal modulo M, say $\overline{a}_r \equiv \overline{a}_{r'} \pmod{M}$. Denote $m = r' - r$. Since the value of $a_n \pmod{M}$ is determined by the k previous values $a_{n-i} \pmod{M}$ for $i = 1, \ldots, k$ of the sequence, we have that the two sequences a_n, $n \geq r$ and a_{m+n}, $n \geq r$ are the same sequence $\pmod{M}$. Thus, $a_n \equiv a_{m+n} \pmod{M}$ for all n. □

We say that the sequence $\{s_k\}_{k\geq 1} \subset [0,1]$ is well distributed if for any interval $I \subset [0,1)$ we have that $D_I(x) = o(x)$ as $x \to \infty$, where

$$D_I(x) = \left| \frac{\#\{k \leq x : \ s_k \in I\}}{x} - |I| \right|.$$

Write $D(x) = \sup_{I\subset[0,1)} D_I(x)$. A quantitative version of this definition is the following inequality.

Theorem 2.3 (Erdős-Turán)**.** *For any positive integer M and any sequence $\{s_k\}$*

$$D(y) \leq \frac{y}{M+1} + 3\sum_{m=1}^{M} \frac{1}{m} \left| \sum_{1\leq j\leq y} e(j\, s_k) \right|, \tag{2.2}$$

where $e(x) = e^{2\pi i x}$.

See [K-N, page 112] for more details.

We will write $\|x\|$ for the distance of any real number x to the the nearest integer.

Theorem 2.4 (Baker)**.** *For any algebraic independent numbers y, z there exists $\delta = \delta(y,z) > 0$ such that $\|n \log y/ \log z\| \gg n^{-\delta}$.*

To compute an explicit δ for our example involving the Fibonacci sequence and the base 10, we use the following result due to Matveev [M]. Recall that for an algebraic number η we write $h(\eta)$ for its logarithmic height whose formula is

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right),$$

with d being the degree of η over $\mathbb{Q}$ and

$$f(X) := a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X] \tag{2.3}$$

being the minimal primitive polynomial over the integers having positive leading coefficient a_0 and η as a root.

With this notation, Matveev proved the following deep theorem:

Theorem 2.5 (Matveev)**.** *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\gamma_1, \ldots, \gamma_t$ be positive reals of $\mathbb{K}$, and $b_1, \ldots, b_t$ rational integers. Put*

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1.$$

Let $A_1, \ldots, A_t$ be real numbers such that

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \quad i = 1, \ldots, t.$$

Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > \exp\left(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t\right).$$

Corollary 2.6. *In Theorem 2.4, we can take $\delta((1+\sqrt{5})/2, 10) = 4.92 \times 10^{10}$.*

Proof. In Matveev's theorem, we take $t = 2$, $\gamma_1 = (1+\sqrt{5})/2$, $\gamma_2 = 10$, $\mathbb{K} = \mathbb{Q}[\sqrt{5}]$ for which $D = 2$. We can then take $A_1 = 0.5 > 2h(\gamma_1) = \log((1+\sqrt{5})/2)$, $A_2 = 4.7 > 2h(\gamma_2)$. For an integer $n \geq 2$ consider the expression

$$\|n \log \gamma_1 / \log \gamma_2\| = \frac{1}{\log \gamma_2} |n \log \gamma_1 - m \log \gamma_2|$$

for some integer m. Clearly, $m < n$, for if not the right–hand side above is at least,

$$\frac{n}{\log \gamma_2} |\log \gamma_2 - \log \gamma_1| \geq 2n \geq 6,$$

a contradiction. Thus, $m < n$. Then $B := \max\{m, n\} = n$. Put $z = n \log \gamma_1 - m \log \gamma_2$. Then

$$\frac{|z|}{\log \gamma_2} = \|n \log \gamma_1 / \log \gamma_2\| \leq \frac{1}{2},$$

therefore $|z| \leq (\log \gamma_2)/2 < 1.5$. Thus,

$$\frac{|e^z - 1|}{|z|} \leq \frac{e^{1.5} - 1}{1.5} < 2.5.$$

We thus get that

$$\|n \log \gamma_1 / \log \gamma_2\| = \frac{|z|}{\log \gamma_2} \geq \frac{1}{2.5 \log \gamma_2} |e^z - 1| > \frac{1}{6} |\gamma_1^n \gamma_2^{-m} - 1| := \frac{|\Lambda|}{6}.$$

The right-hand side above is not zero since γ_1 and γ_2 are multiplicatively independent. We apply Theorem 2.5 to get the following inequality:

$$|\Lambda| \geq \exp\left(-1.4 \times 30^5 \times 2^{4.5} \times 2^2 (1 + \log 2)(1 + \log n) A_1 A_2\right).$$

Using the fact that $n \geq 3$, we have $1 + \log n < 2 \log n$, therefore

$$|\Lambda| > n^{-c},$$

where we can take

$$c = 1.4 \times 30^5 \times 2^{4.5} \times 2^2 \times (1 + \log 2) \times 2 \times 0.5 \times 4.7 < 2.46 \times 10^{10}.$$

Hence,

$$\|n \log \gamma_1 / \log \gamma_2\| > \frac{1}{6} n^{-c} > n^{-2c},$$

which completes the proof of this corollary since $2c = 4.92 \times 10^{10}$. □

3. Proof of theorem 1.1

It is enough to prove the estimate for dyadic intervals

$$P(x) = \{n : \ x/2 < n \le x, \ a_n \text{ is palindromic in base } b\}.$$

For any positive integer $t = t(x)$, Lemma 2.2 yields an integer m_t which is the period of the sequence $\{a_n\}$ modulo b^t. The value of $a_n \pmod{b^t}$ is determined by the residue of $n \pmod{b^t}$. We will write ξ_r for the residue of $a_r \pmod{b^t}$ and $\overline{\xi}_r$ for the number obtained from ξ_r by reversing digits. We have that for $n \equiv r \pmod{m_t}$ we have that $a_n \equiv \xi_r \pmod{b^t}$. A typical sufficiently large palindromic number a_n with $n \equiv r \pmod{m_t}$ can be written in base b as

$$a_n = \overline{\xi}_r \cdots \xi_r,$$

where both ξ_r and $\overline{\xi}_r$ are strings of t digits in base b. The value of t will be taken at the end of the proof but certainly it will be $t(x) = O(\log x)$. Thus when $x/2 < n \le x$ we have that $b^t < \sqrt{|a_n|}$ holds for x large enough, so ξ_r and $\overline{\xi}_r$ do not overlap. For short, we call $J = b^t$ throughout this proof. Also we define α by $m^t = J^\alpha$. Note that Lemma 2.2 implies that $\alpha \le k$.

Since the t most significant digits of a_n are coincident with the t digits of the number $\overline{\xi}_r$, we can write

$$a_n = \overline{\xi}_r b^d (1 + \theta b^{-t}) \qquad 0 \le \theta < 1, \tag{3.1}$$

for some positive integer d. By hypothesis, we know that $|\alpha_2/\alpha_1| < 1$. Thus, we have that

$$|\sum_{i=2}^{R} (\alpha_i/\alpha_1)^n p_i(n)/p_1(n)| \ll n^{\max_i(\deg(p_i))} |\alpha_2/\alpha_1|^n \ll x^{O(1)} |\alpha_2/\alpha_1|^{x/2} < b^{-t}$$

for x large enough. By Theorem 2.1, we have

$$a_n = \alpha_1^n p_1(n) + \sum_{i=2}^{R} \alpha_i^n p_i(n) = \alpha_1^n \, p_1(n) \left(1 + O\left(J^{-1}\right)\right).$$

Taking logarithms and inserting (3.1) we have

$$\log \overline{\xi}_r + d \log b + \log\left(1 + \theta J^{-1}\right) = n \log \alpha_1 + \log p_1(n) + O(J^{-1}).$$

We consider first the case when the multiplicity of α_1 is $m_1 = 1$, so the polynomial $p_1(n)$ is a constant, say $p_1(n) = p_1$. Therefore

$$d = n \frac{\log \alpha_1}{\log b} + \frac{\log p_1 - \log \overline{\xi}_r}{\log b} + O(J^{-1}).$$

Thus, when $x/2 < n \le x,\ n \equiv r \pmod{m_t}$ and a_n is palindromic, we have that

$$\left\| n \frac{\log \alpha_1}{\log b} - \gamma_r \right\| \ll J^{-1},$$

where

$$\gamma_r = \frac{\log \overline{\xi}_r - \log p_1}{\log b}.$$

Hence, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |P(x)|^2 &= \left(\sum_{r=0}^{m_t-1} \#\{n \in P(x) :\ n \equiv r \pmod{m_t}\} \right)^2 \\ &\ll m_t \sum_{r=0}^{m_t-1} \#\left\{n, n' \in P(x),\ n, n' \equiv r \pmod{m_t}\right\} \\ &\ll m_t \# \left\{n, n' \in P(x),\ n \equiv n' \pmod{m_t}\right\}. \end{aligned}$$

We observe that if $n, n' \in P(x)$ then

$$\begin{aligned} \left\| |n - n'| \frac{\log \alpha_1}{\log b} \right\| &= \left\| (n - n') \frac{\log \alpha_1}{\log b} \right\| \\ &\le \left\| n \frac{\log \alpha_1}{\log b} - \gamma_r \right\| + \left\| n' \frac{\log \alpha_1}{\log b} - \gamma_r \right\| \\ &\ll J^{-1}. \end{aligned}$$

Furthermore $|n - n'| = l m_t$ for some $0 \le l \le \frac{x}{2m_t}$ and, given l and $n \in P(x)$, there are at most two distinct n' such that $|n - n'| = l m_t$. Thus, for each l, the number of pairs $n, n' \in P(x)$ with $|n - n'| = lm_t$ is bounded by $2|P(x)|$. With these observations we have

(3.2)
$$\begin{aligned}|P(x)| &\ll \frac{m_t}{|P(x)|}\#\left\{n, n' \in P(x),\ n \equiv n' \pmod{m_t},\ \left\| |n-n'| \frac{\log \alpha_1}{\log b} \right\| \ll J^{-1}\right\} \\ &\ll m_t \# \left\{ l:\ 0 \le l \le \frac{x}{2m_t},\ \left\| l m_t \frac{\log \alpha_1}{\log b} \right\| \le \frac{C}{J} \right\},\end{aligned}$$

for some constant C. Now we apply Lemma 2.3 to the sequence $s_l = \| l m_t \frac{\log \alpha_1}{\log b} \|$, the interval $[0, C/J]$ and $y = \frac{x}{2m_t}$. It gives

$$|P(x)| \ll m_j \left(1 + \frac{Cx}{2Jm_t} + D\left(\frac{x}{2m_t}\right)\right).$$

By Theorem 2.2, we have

$$\begin{aligned} D\left(\frac{x}{2m_t}\right) &\ll \frac{x}{2m_tT} + \sum_{i=1}^{T} \frac{1}{i} \left| \sum_{1 \le j \le x/(2m_t)} e\left(ijm_t \frac{\log \alpha_1}{\log b}\right) \right| \\ &\ll \frac{x}{2m_tT} + \sum_{i=1}^{T} \frac{1}{i} \frac{1}{\| i\, m_t \frac{\log \alpha_1}{\log b} \|}. \end{aligned}$$

As α_1 and b are algebraically independent, Theorem 2.4 yields that there exists $\delta = \delta(\alpha_1, b) > 0$ such that

$$\begin{aligned} D\left(\frac{x}{2m_t}\right) &\ll \frac{x}{2m_tT} + \sum_{i=1}^{T} \frac{1}{i}(im_t)^{\delta} \\ &\ll \frac{x}{2m_tT} + T^{\delta} m_t^{\delta} \\ &\ll x^{\frac{\delta}{1+\delta}}, \end{aligned}$$

where we take $T = \lfloor x^{\frac{1}{\delta+1}}/m_t \rfloor$. Indeed as $t = t(x)$ grows with x, inserting this in (3.1) we have

$$\begin{aligned} |P(x)| &\ll \frac{x}{J} + m_j x^{\frac{\delta}{1+\delta}} \\ &\ll \frac{x}{J} + J^{\alpha} x^{\frac{\delta}{1+\delta}} \\ &\ll x^{1 - \frac{1}{(1+\delta)(1+\alpha)}}, \end{aligned} \tag{3.3}$$

where we take $J \sim x^{\frac{1}{(1+\delta)(1+\alpha)}}$ as $x \to \infty$.

Now we consider the case when the multiplicity of α_1 is $m_1 \geq 2$. In this case, we split the interval $[x/2, x]$ in J intervals $I_j = [n_j, n_{j+1}]$ of length $\sim x/(2J)$. We observe that if $n \in I_j$ then $\log p_1(n) = \log p_1(n_j) + O(J^{-1})$. Thus, if $n \in I_j \cap P(x)$, $n \equiv r \pmod{m_t}$, we have that

$$\left\| n\frac{\log \alpha_1}{\log b} - \gamma_{r,j} \right\| \ll J^{-1},$$

where

$$\gamma_{r,j} = \frac{\log \overline{\xi}_r - \log p_1(n_j)}{\log b}.$$

If we denote by $P_j(x) = P(x) \cap I_j$, we proceed as above to get that

$$|P_j(x)| \ll m_t \# \left\{ l : \ 0 \leq l \leq \frac{x}{J2m_t}, \ \left\| l m_t \frac{\log \alpha_1}{\log b} \right\| \leq \frac{C}{J} \right\} \tag{3.4}$$

$$\ll m_j \left(1 + \frac{Cx}{2J^2 m_t} + D\left(\frac{x}{2Jm_t} \right) \right). \tag{3.5}$$

As in the case $m_1 = 1$, we have

$$D\left(\frac{x}{2Jm_t}\right) \ll \frac{x}{2m_t T} + T^\delta m_t^\delta \ll (x/J)^{\frac{\delta}{1+\delta}},$$

therefore

$$\begin{aligned} |P_j(x)| &\ll \frac{x}{J^2} + m_j (x/J)^{\frac{\delta}{1+\delta}} \\ &\ll \frac{x}{J^2} + J^{\alpha - \frac{\delta}{1+\delta}} x^{\frac{\delta}{1+\delta}}. \end{aligned}$$

Thus,

$$|P(x)| = \sum_{j=1}^{J} |P_j(x)| \ll \frac{x}{J} + J^{\alpha+1-\frac{\delta}{1+\delta}} x^{\frac{\delta}{1+\delta}} \ll x^{1 - \frac{1}{(\alpha+1)(1+\delta)+1}}$$

where we take $J \sim x^{\frac{1}{(\alpha+1)(1+\delta)+1}}$, as $x \to \infty$.

3.1. **Proof of Corollary 1.2.** The characteristic polynomial of the Fibonacci recurrence has $\alpha_1 = \frac{1+\sqrt{5}}{2}$ as the unique dominant root. It has multiplicity $m_1 = 1$, so we can apply the estimate (3.3). It is known that for $b = 10$ and $t \geq 2$, the period of the Fibonacci sequence $\pmod{10^t}$ is $m_t = 3 \times 10^t \ll 10^t$, so we can take $\alpha = 1$. Thus, by Corollary 2.6, we have

$$\#\{n \leq x : F_n \text{ is base 10 palindrome}\} \ll x^{1 - \frac{1}{2(1+4.92\times 10^{10})+1}} \ll x^{1-10^{-11}},$$

which is what we wanted to prove.

4. Comments and further problems

Each of the binary recurrent sequences of general term $a_n = 10^n + 1$ or $10^n - 1$ consists of palindromes in base 10. This shows that in the case of the dominant root, the condition that the dominant root and the base be multiplicatively independent cannot be removed without affecting the conclusion of Theorem 1.1. In a related spirit, we mention that in [L-T], it was shown that the largest base 2 palindrome of the form $10^n \pm 1$ is $99 = \overline{110011}_{(2)}$.

We believe the conclusion of the theorem also holds under the somewhat more general hypothesis namely that the sequence is non degenerated (i.e, that α_i/α_j is not a root of 1 for $i \neq j$ in $\{1, \ldots, R\}$, and that the absolute value of the largest root of the characteristic polynomial is multiplicatively independent over b. However, we could not deal with the case when a dominant root is not present and we leave this as an open research problem for the reader.

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