ON THE LARGEST PRIME FACTOR OF THE PARTITION FUNCTION OF n

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For Andrzej Schinzel on his seventy-fifth birthday

Abstract. Let $p(n)$ be the function that counts the number of partitions of n. For a positive integer m, let $P(m)$ be the largest prime factor of m. Here, we show that $P(p(n))$ tends to infinity when n tends to infinity through some set of asymptotic density 1. In fact, we show that the inequality $P(p(n)) > \log\log n$ holds for almost all positive integers n. This improves a result of the second author from [3].

1. Introduction

Let $p(n)$ be the partition function of n, which is the number of ways of writing $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$, where $k \geq 1$ and $1 \leq \lambda_1 \leq \cdots \leq \lambda_k$ are positive integers. There is a huge literature on this function with respect to its size, congruence properties, recurrence relations, and so on. Put $P(m)$ for the largest prime factor of the positive integer m with the convention that $P(1) = 1$ and let $\omega(m)$ be the number of distinct prime factors of m. In response to a question of Erdős and Ivić, Schinzel showed $\omega(\prod_{m=1}^{N} p(m))$ tends to infinity with N (this is Lemma 2 in [2]). His method used lower bounds for nonzero linear forms in logarithms of algebraic numbers. Later, Schinzel and Wirsing [6] proved the effective result

$$\omega\Big(\prod_{m=1}^{N} p(m)\Big) \geq (1-\varepsilon)\frac{\log N}{\log 2} \qquad \text{if} \qquad N > N_0(\varepsilon) \tag{1.1}$$

valid for all $\varepsilon > 0$. The proof of estimate (1.1) does not use linear forms in logarithms.

Here, we visit Schinzel's original argument and prove the following result.

Theorem 1. *The set of n for which the inequality*

$$P(p(n)) > \log\log n$$

holds is of asymptotic density 1.

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This improves a result of the second author from [3], where it is proved by a different method that the inequality $P(p(n)) > \log\log\log\log\log\log n$ holds for almost all positive integers n.

Notation. We use $c_1,\ c_2,\ \ldots$ for computable positive constants that appear increasingly throughout the paper. We use the Landau symbols O and o and the Vinogradov symbols $\ll$, $\gg$ and $\asymp$ with their usual menainings. Recall that $A = O(B)$, $A \ll B$ and $B \gg A$ are all equivalent to the fact that the inequality $|A| \le cB$ holds with some constant c. The constants implied by these symbols in our arguments are absolute. Furthermore, $A \asymp B$ means that both $A \ll B$ and $B \ll A$ hold, and $A = o(B)$ and $A \sim B$ mean that A/B tends to 0 and to 1, respectively.

2. Preliminary results

We start with Rademacher's formula for $p(n)$ (Chapter 5 in [1]).

Lemma 1. *We have*

(2.1)
$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\left[\frac{d}{dx}\left(\frac{\sinh((\pi/k)\sqrt{2/3(x-1/24)})}{\sqrt{x-1/24}}\right)\right]_{x=n},$$

where

$$A_k(n) := \sum_{\substack{1\le h\le k\\ \gcd(h,k)=1}} \omega_{h,k} e^{-2\pi i n h/k}$$

with $\omega_{h,k}$ being the root of unity of order 24 given by

$$\omega_{h,k} := e^{\pi i s(h,k)},$$

and $s(h,k)$ is the Dedekind sum

$$s(h,k) := \sum_{\mu=1}^{k-1}\left(\frac{\mu}{k} - \left[\frac{\mu}{k}\right] - \frac{1}{2}\right)\left(\frac{h\mu}{k} - \left[\frac{h\mu}{k}\right] - \frac{1}{2}\right).$$

In practice, one may truncate the sum appearing in (2.1) at $k := \lfloor\sqrt{n}\rfloor$ and then the nearest integer to this partial sum is exactly the value of $p(n)$ when $n > n_0$ is sufficiently large. Since in the range $k \le \sqrt{n}$ the kth term of the expansion (2.1) is of order of magnitude $O(\exp(c_1\sqrt{n}/k)$, where $c_1 := \pi\sqrt{2/3}$ and $A_1(n) = 1$, we get that

$$p(n) = \frac{1}{\pi\sqrt{2}}\left[\frac{d}{dx}\left(\frac{\sinh(\pi\sqrt{2/3(x-1/24)})}{\sqrt{x-1/24}}\right)\right]_{x=n} + O(\exp(c_1\sqrt{n}/2). \tag{2.2}$$

The first term of the expansion (2.2) is, after some calculation

$$\frac{1}{\pi\sqrt{2}}\left[\frac{d}{dx}\left(\frac{\sinh(\pi\sqrt{2/3(x-1/24)})}{\sqrt{x-1/24}}\right)\right]_{x=n}$$

$$(2.3)\quad = \frac{e^{c_1\sqrt{n-1/24}}}{2\pi\sqrt{2}(n-1/24)}\left(\frac{\pi}{\sqrt{6}}-\frac{1}{2\sqrt{n-1/24}}\right)+O(\exp(-c_1\sqrt{n})).$$

Putting together (2.2) and (2.3), we get our working formula

$$(2.4)\qquad p(n)=e^{c_1\sqrt{n-1/24}}f(n)+O(e^{c_1\sqrt{n}/2}),$$

where

$$(2.5)\qquad f(n):=\frac{1}{4\sqrt{3}(n-1/24)}\left[1-\frac{c_2}{\sqrt{n-1/24}}\right]$$

and $c_2=\dfrac{\sqrt{3/2}}{\pi}$.

We shall also need a result of Matveev [4] from transcendental number theory. But first, some notation. For an algebraic number η having

$$F(X):=a_0\prod_{i=1}^{d}(X-\eta^{(i)})\in\mathbb{Z}[X]$$

as minimal polynomial over the integers, the logarithmic height of η is defined as

$$h(\eta):=\frac{1}{d}\left(\log|a_0|+\sum_{i=1}^{d}\log\max\left\{\left|\eta^{(i)}\right|,1\right\}\right).$$

With this notation, Matveev [4] proved the following deep theorem.

Lemma 2. *Let $\mathbb{K}$ be a field of degree D, $\eta_1,\ldots,\eta_k$ be nonzero elements of $\mathbb{K}$, and $b_1,\ldots,b_k$ be integers. Put $B:=\max\{|b_1|,\ldots,|b_k|\}$, and $\Lambda:=1-\prod_{i=1}^{k}\eta_i^{b_i}$. Let $A_1,\ldots,A_k$ be real numbers such that*

$$A_j\geq\max\{Dh(\eta_j),|\log\eta_j|,0.16\},\qquad j=1,\ldots,k.$$

Then, assuming that $\Lambda\neq 0$, we have

$$\log|\Lambda|>-3\cdot 30^{k+4}(k+1)^{5.5}D^2(1+\log D)(1+\log(kB))\prod_{i=1}^{k}A_i.$$

We shall use the above result only when $\eta_1,\ldots,\eta_k$ are rational. So, $\mathbb{K}:=\mathbb{Q}$, $D=1$, and the logarithmic height of $\eta:=r/s$, with nonzero coprime integers r and s is just $\log(\max\{|r|,|s|\})$.

3. The proof of Theorem 1

We let x be a large positive real number. Let $2 = p_1 < p_2 < \cdots < p_k < \cdots$ be the increasing sequence of prime numbers. We put $r := r(x)$ for a function tending slowly to infinity and let

$$\mathcal{N}_r(x) := \{n \in [x, 2x) : P(p(n)) \le p_r\}. \tag{3.1}$$

Our goal is to show that if $r(x)$ is chosen such that $p_r \le \log\log x$ then $\#\mathcal{N}_r(x) = o(x)$ as $x \to \infty$, since once we have done that then Theorem 1 will follow by replacing x with $x/2$, then with $x/4$, and so on, and then summing up all these estimates.

Well, let us assume that $n \in \mathcal{N}_r(x)$ and write

$$p(n) =: p_1^{a_1} \cdots p_r^{a_r}. \tag{3.2}$$

Comparing relation (3.2) with (2.4), we get

$$e^{c_1\sqrt{n-1/24}} f(n) - p_1^{a_1} \cdots p_r^{a_r} = O(e^{(c_1/2)\sqrt{n}}).$$

Dividing across by $e^{c_1\sqrt{n-1/14}} f(n)$, we get

$$1 - e^{-c_1\sqrt{n-1/24}} f(n)^{-1} p_1^{a_1} \cdots p_r^{a_r} = O\left(ne^{-(c_1/2)\sqrt{n}}\right) = O\left(e^{-c_3\sqrt{n}}\right),$$

where $c_3 := c_1/3$. Taking logarithms, we get

$$\left|c_1\sqrt{n-1/24} + \log f(n) - a_1 \log p_1 - \cdots - a_r \log p_r\right| = O\left(e^{-c_3\sqrt{n}}\right). \tag{3.3}$$

We let $z := \log x$, $K := \lfloor z^{1/2} \rfloor$, and assume that there exists an interval $[n, n+z) \subset [x, 2x)$ containing K numbers $n_1 < n_2 < \cdots < n_K$ such that $P(p(n_i)) \le p_r$ for all $i = 1, \ldots, K$.

Indeed, if this is not the case, then we can split $[x, 2x)$ in $O(x/z)$ intervals of length z, each one containing at most $K-1$ elements of $\mathcal{N}_r(x)$ and then it would follow that

$$\#\mathcal{N}_r(x) \ll \left(\frac{x}{z}\right) \cdot (K-1) = O\left(\frac{x}{(\log x)^{1/2}}\right) = o(x) \quad \text{as} \quad x \to \infty, \tag{3.4}$$

which is what we want to prove.

For $i = 1, \ldots, K$, let us write

$$p(n_i) = \prod_{j=1}^{r} p_j^{\alpha_{i,j}}.$$

Put

$$g(x) := c_1\sqrt{x - 1/24} + \log f(x).$$

We let $y = \lfloor x^{1/4} \rfloor$, $y_i \in \{0, 1, \ldots, \lfloor y \rfloor\}$ and compute

$$\sum_{i=1}^{K} y_i g(n_i). \tag{3.5}$$

The absolute value of a vector shown in (3.5) is $O(Kyx^{1/2})$ and there are $(\lfloor y \rfloor + 1)^K$ such vectors. Thus, by the Pigeon Hole Principle, there is a nonzero vector $\mathbf{y} := (y_1, \ldots, y_K)$ with integer components $|y_i| \le y$ for all $i = 1, \ldots, K$, such that

$$\left| \sum_{i=1}^{K} y_i g(n_i) \right| \ll \frac{Ky\sqrt{x}}{(\lfloor y \rfloor + 1)^K - 1} \ll \frac{x}{y^K} = \frac{1}{x^{K/4-1}} \ll \frac{1}{x^{K/5}}. \tag{3.6}$$

Writing down relations (3.3) for $n := n_i$ for $i = 1, \ldots, K$ we get

$$\left| g(n_i) - \sum_{j=1}^{r} \alpha_{i,j} \log p_j \right| = O(\exp(-c_3\sqrt{x})), \quad \text{for} \quad i = 1, \ldots, K,$$

and taking linear combinations of the above relations with the coefficients $\mathbf{y} = (y_1, \ldots, y_K)$, we get that for large x we have

$$\left| \sum_{i=1}^{K} y_i g(n_i) - \sum_{j=1}^{r} \beta_j \log p_j \right| \ll Ky \exp(-c_3\sqrt{x}) \le \exp(-c_4\sqrt{x}), \tag{3.7}$$

where we can take $c_4 := c_3/2$ and

$$\beta_j := \sum_{i=1}^{K} y_i \alpha_{i,j} \quad \text{for all} \quad j = 1, \ldots, r. \tag{3.8}$$

Comparing the upper bounds from (3.6) and (3.7), we get that

$$\begin{aligned} \left| \sum_{j=1}^{r} \beta_j \log p_j \right| \quad &\le \quad \left| \sum_{i=1}^{K} y_i g(n_i) \right| + O(\exp(-c_4\sqrt{x})) \\ &= \quad O\left(\frac{1}{x^{K/5}} + \frac{1}{\exp(c_4\sqrt{x})} \right) = O\left(\frac{1}{x^{K/5}} \right). \end{aligned} \tag{3.9}$$

We distinguish two cases. In the first case, we assume that

$$\Gamma := \sum_{j=1}^{r} \beta_j \log p_j$$

is nonzero. Hence, we have that the inequality

$$|\Gamma| \le \frac{1}{x^{K/6}} \tag{3.10}$$

holds for all large enough x. Now Γ is nonzero but $\Gamma = o(1)$, so $\Gamma \sim e^{\Gamma} - 1 =: \Lambda \neq 0$ as $x \to \infty$, and we can use Matveev's result Lemma 2 to find a lower bound on this last expression.

We take, in the notations of Lemma 2,

$$k := r, \quad \eta_j := p_j \quad \text{and} \quad b_j := \beta_j \quad \text{for} \quad j = 1, \ldots, r.$$

Clearly, $\mathbb{K} := \mathbb{Q}$, so $D = 1$, and $A_j := \log p_j$ for $j = 1, \ldots, r$.

We also use the fact that the inequality $p_m \leq (m+1)^2$ holds for all positive integers m (see, for example, (3.13) in [5]).

As for B, observe that

$$\alpha_{i,j} \leq \log p(n)/\log p_j \ll x^{1/2} \quad \text{holds for all} \quad j = 1, \ldots, r \quad \text{and} \quad i = 1, \ldots, K,$$

therefore, using (3.8), we deduce that

$$|\beta_j| \ll yKx^{1/2} = o(x) \qquad \text{as} \quad x \to \infty.$$

So, we can take $B := x$ for all sufficiently large x, and then we have that indeed

$$B \geq \max\{|\beta_j| : j = 1, \ldots, r\}$$

holds. Lemma 2 shows that there exists some absolute constant c_5 such that the inequality

$$|\Lambda| > \exp\left(-c_5^r (\log x)(\log(r+1)^2)^r\right) \tag{3.11}$$

holds. Comparing the last estimate (3.11) above with estimate (3.10) and using the fact that $|\Lambda| \sim |\Gamma|$ as $x \to \infty$, we get that the inequality

$$(2c_5 \log(r+1))^r \geq \frac{K}{7}$$

holds for large values of x. In turn, this implies that the inequality

$$r \log\log(r+1) \geq \log K + O(1) \geq c_6 \log\log x$$

holds for large x, where we can take $c_6 := 1/3$. Hence,

$$r \gg \frac{\log\log x}{\log\log\log\log x}.$$

With the Prime Number Theorem (or with the Chebyshev estimates), we get that

$$p_r \gg r \log r \gg \log\log x \left(\frac{\log\log\log x}{\log\log\log\log x}\right).$$

Note that the function appearing in the right–hand side above is of order at least $\log\log x$, which for large x contradicts our assumption that $p_r \leq \log\log x$. Thus, we get a contradiction assuming that $\Gamma \neq 0$.

Now we deal with the harder case when $\Gamma = 0$. Well, in this case, inequality (3.7) becomes

$$\left|\sum_{i=1}^{K} y_i g(n_i)\right| = O(\exp(-c_4\sqrt{x})). \tag{3.12}$$

We write each $n_i := n + \lambda_i$ for $i = 1, \ldots, K$ (note that $\lambda_1 = 0$, although we will not use this information), and write the Taylor series

$$g(n_i) = \sum_{k=0}^{\infty} \frac{g^{(k)}(n)}{k!}\lambda_i^k \quad \text{for all} \quad i = 1, \ldots, K,$$

which, via estimate (3.12), yields

$$\left| \sum_{k=0}^{\infty} \frac{g^{(k)}(n)}{k!} \sum_{i=1}^{K} y_i \lambda_i^k \right| = O(\exp(-c_4\sqrt{x})). \tag{3.13}$$

We need the derivatives of $g(y)$. Observe that

$$g(t) = \sqrt{t - 1/24} - \log\left(t - \frac{1}{24}\right) - \log c_7 + \log\left(1 - \frac{c_2}{\sqrt{t-1/24}}\right),$$

where $c_7 := 4\sqrt{3}$ and $c_2 := \sqrt{3/2}/\pi$. For $k \geq 1$, one checks easily, by induction, that

$$\begin{aligned} \frac{d^k}{dt^k}\sqrt{t-1/24} &= (-1)^{k-1}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\cdots\left(\frac{2k-3}{2}\right)\frac{1}{(t-1/24)^{(2k-1)/2}} \\ &= (-1)^{k-1}\frac{(2k-2)!}{(k-1)!2^{2k-1}(t-1/24)^{(2k-1)/2}}, \end{aligned} \tag{3.14}$$

and that

$$\frac{d^k}{dt^k}\log\left(t - \frac{1}{24}\right) = (-1)^{k-1}\frac{(k-1)!}{(t-1/24)^k}. \tag{3.15}$$

Finally, using the Taylor series expansion for $\log(1-y)$, we get easily that

$$\log\left(1 - \frac{c_2}{\sqrt{t-1/24}}\right) = -\sum_{j\geq 1}\frac{c_2^j}{j(t-1/24)^{j/2}}, \tag{3.16}$$

and taking derivatives, we arrive at

$$\begin{aligned} \frac{d^k}{dt^k} \left(\log\left(1 - \frac{c_2}{\sqrt{t-1/24}}\right)\right) &= -\sum_{j\geq 1}\frac{c_2^j}{j}\frac{d^k}{dy^k}\left(\frac{1}{(t-1/24)^{j/2}}\right) \\ &= (-1)^{k+1}\sum_{j\geq 1}\frac{c_2^j}{j}\left(\frac{j(j+2)\cdots(j+2(k-1))}{2^k(t-1/24)^{j/2+k}}\right) \end{aligned} \tag{3.17}$$

To get a contradiction, we shall show that for large x, inequality (3.13) leads to the conclusion that

$$\sum_{i=1}^{K} y_i\lambda_i^k = 0 \quad \text{for} \quad k = 0, 1, \ldots, K-1. \tag{3.18}$$

Assuming that we proved that, it follows that $\mathbf{y}$ is a zero of the linear map whose matrix A has as ith row the vector $(\lambda_1^{i-1}, \cdots, \lambda_K^{i-1})$ for all

$i=1,\ldots,K$. However, A is nonsingular because its determinant is a Vandermonde whose value is $\prod_{1\le i<j\le K}(\lambda_j-\lambda_i)\neq 0$, so $\mathbf{y}=\mathbf{0}$, which is the contradiction.

Well, let's get to work and prove that relations (3.18) must hold for large x by induction on k.

Put

$$M_k(t) := \left|\frac{g^{(k)}(t)}{k!}\right| \quad \text{for} \quad t\in[n,n+z].$$

Relations (3.14), (3.15) and (3.17) show easily that

$$M_k(t) \asymp \frac{1}{k^{3/2}n^{k-1/2}} \tag{3.19}$$

uniformly in $k\le K$, $t\in[n,n+z]$ and $n\in[x,2x]$. Indeed, for (3.14), by Stirling's formula, we have

$$\begin{aligned}
\frac{1}{k!} \quad & \left|\frac{d^k}{dt^k}\sqrt{t-1/24}\right| = \frac{(2k-2)!}{(k-1)!k!2^{2k-1}(t-1/24)^{k-1/2}} \\
& \asymp \frac{1}{k^{1/2}2^{2k-1}}\frac{((2k-2)/e)^{2k-2}}{(k/e)^k\,((k-1)/e)^{k-1}}\frac{1}{n^{k-1/2}}\left(1+O\left(\frac{z}{n}\right)\right)^{k-1/2} \\
& \asymp \frac{1}{k^{3/2}}\left(1-\frac{1}{k}\right)^{k-1}\frac{1}{n^{k-1/2}}\left(1+O\left(\frac{zK}{x}\right)\right)\asymp\frac{1}{k^{3/2}n^{k-1/2}},
\end{aligned} \tag{3.20}$$

uniformly for $k\le K$ and $n\in[x,2x]$. From (3.15), we have

$$\begin{aligned}
\frac{1}{k!} \quad & \left|\frac{d^k}{dt^k}\left(\log\left(t-\frac{1}{24}\right)\right)\right| = \frac{1}{k(t-1/24)^k} \\
& = \frac{1}{kn^k}\left(1+O\left(\frac{z}{n}\right)\right)^k = \frac{1}{kn^k}\left(1+O\left(\frac{Kz}{x}\right)\right)\asymp\frac{1}{kn^k}.
\end{aligned} \tag{3.21}$$

For (3.17), put

$$a_{j,k} := \frac{c_2^j}{j}\left(\frac{j(j+2)\cdots(j+2(k-1))}{2^k(t-1/24)^{j/2}}\right) \quad \text{for} \quad j\ge 1.$$

Observe that

$$\begin{aligned}
\frac{a_{j+1,k}}{a_{j,k}} & = c_2\left(\frac{j}{j+1}\right)\left(\frac{(j+1)\cdots(j+1+2(k-1))}{j\cdots(j+2(k-1))}\right)\frac{1}{(t-1/24)^{1/2}} \\
& \ll \left(\frac{j+1+2(k-1)}{j}\right)\frac{1}{(t-1/24)^{1/2}}\ll\frac{K}{x^{1/2}}=o(1)
\end{aligned}$$

uniformly in $j \geq 1$, $k \leq K$, and $n \in [x, 2x]$ as $x \to \infty$, which shows that in the series shown in (3.17), the first term dominates. Thus,

$$\begin{aligned} \frac{1}{k!} \quad & \left| \frac{d^k}{dt^k} \log\left(1 - \frac{c_2}{\sqrt{t-1/24}}\right) \right| \asymp \frac{1 \cdot 3 \cdots (2k-1)}{k! 2^k (t-1/24)^{k+1/2}} \\ &= \frac{2k!}{2^{2k} k!^2 n^{k+1/2}} \left(1 + O\left(\frac{z}{n}\right)\right)^{k+1/2} \\ &\asymp \frac{1}{k^{1/2}} \frac{(2k/e)^{2k}}{2^{2k}(k/e)^{2k}} \frac{1}{n^{k+1/2}} \left(1 + O\left(\frac{Kz}{x}\right)\right) \asymp \frac{1}{k^{1/2} n^{k+1/2}}. \end{aligned} \tag{3.22}$$

Since the terms arising from (3.20), (3.21) and (3.22) are of different orders of magnitude with the term coming from (3.20) dominating, we get estimate (3.19).

Now we are ready to prove that relations (3.18) must hold. Let us take $k = 0$ and use the Taylor's formula with remainder at $k = 1$ in (3.12) getting

$$\begin{aligned} \sqrt{n} \left| \sum_{i=1}^K y_i \right| \quad & \ll \quad |g^{(0)}(n) \sum_{i=1}^K y_i \lambda_i^0| \\ & \ll \quad \max_{n \leq t \leq n+z} \left\{ \left| \frac{g^{(1)}(t)}{1!} \right| \right\} \sum_{i=1}^K |y_i \lambda_i| + \exp(-c_4 \sqrt{x}) \\ & \ll \quad \frac{yKz}{n^{1/2}} + \exp(-c_4\sqrt{x}) \ll \frac{yKz}{n^{1/2}}, \end{aligned} \tag{3.23}$$

giving

$$\left| \sum_{i=1}^k y_i \right| \ll \frac{yKz}{n} \ll \frac{yKz}{x} = o(1) \quad \text{as} \quad x \to \infty, \tag{3.24}$$

and since the left–hand side of the inequality (3.24) above is an integer, we get that

$$\sum_{i=1}^K y_i = 0, \tag{3.25}$$

which is the desired relation (3.12) with $k = 0$. Assume now by induction that relation (3.12) holds for all exponents $0, 1, \ldots, k-1$ for some $k \leq K-1$ and let us prove it for k. Applying again the Taylor formula with remainder at k in (3.12) and the induction hypothesis, as well as calculation (3.19), we get that

$$\begin{aligned}
\frac{1}{k^{3/2}n^{k-1/2}}\left|\sum_{i=1}^{K} y_i\lambda_i^k\right| &\ll \left|\frac{g^{(k)}(n)}{k!}\sum_{i=1}^{K} y_i\lambda_i^k\right| = \left|\sum_{\ell=0}^{k}\frac{g^{(\ell)}(n)}{\ell!}\sum_{i=1}^{K} y_i\lambda_i^{\ell}\right| \\
&\ll \max_{n\le t\le n+z}\left\{\left|\frac{g^{(k+1)}(t)}{(k+1)!}\right|\right\}\sum_{i=1}^{K}|y_i\lambda_i|^{k+1} + \exp(-c_4\sqrt{x}) \\
&\ll \frac{yKz^K}{(k+1)^{3/2}n^{k+1-1/2}} + \exp(-c_4\sqrt{x}) \ll \frac{yKz^K}{(k+1)^{3/2}n^{k+1-1/2}},
\end{aligned}$$

where the last inequality follows because the last term $\exp(-c_4\sqrt{x})$ is of a smaller order than

$$\frac{1}{K^{3/2}n^{K+1}} > \exp\left(-(\log(2x))^{3/2} - (\log\log x)\right).$$

Thus, we get from the above calculation that

$$\left|\sum_{i=1}^{K} y_i\lambda_i^k\right| \ll \frac{yKz^K}{n} \ll \frac{yKz^K}{x} = o(1) \quad \text{as} \quad x\to\infty, \tag{3.26}$$

because the numerator of the right–hand side in inequality (3.26) is

$$\begin{aligned}
yKz^K &\le x^{1/4}(\log x)^{1/2}(\log x)^{(\log x)^{1/2}} = x^{1/4}\exp(O((\log x)^{1/2}\log\log x) \\
&= o(x)
\end{aligned}$$

as $x\to\infty$. Since the right–hand side of the inequality (3.26) above is an integer, we get that $\sum_{i=1}^{K} y_i\lambda_i^k = 0$, as desired. Thus, we obtained a contradiction, assuming that $\Gamma = 0$. Hence, both cases $\Gamma = 0$ and $\Gamma \ne 0$ yielded contradictions, so the conclusion is that an interval $[n, n+z] \subset [x, 2x)$ cannot contain K members of $\mathcal{N}_r(x)$. Now the argument used previously to derive estimate (3.4) yields the desired conclusion.

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