On multiplicative magic squares

Javier Cilleruelo
Departamento de Matemáticas
Facultad de Ciencias
Universidad Autónoma de Madrid
28049, Madrid, Spain
franciscojavier.cilleruelo@uam.es

Florian Luca
Instituto de Matemáticas
Universidad Nacional Autonoma de México
C.P. 58089, Morelia, Michoacán, Mexico
fluca@matmor.unam.mx

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Abstract

In this note, we give a lower bound for the distance between the maximal and minimal element in a multiplicative magic square of dimension r whose entries are distinct positive integers.

1 Introduction

Let $A = [A(i,j)]_{1\le i,j\le r}$ be a square matrix with positive integer entries. We say that A is an additive magic square of order r if the sums of the entries in each row, column, and the two diagonals are all equal. We write $s(A)$ for this common value.

A multiplicative magic square has the property that the products of the entries in each row, column, and the two diagonals are all equal. We shall deal only with magic squares whose entries are positive integers.

The most popular additive magic squares of order r are those whose entries are the first r^2 positive integers. Clearly, there are no multiplicative magic squares with this property and indeed it is not difficult to guess that the entries in a multiplicative magic square cannot be very close.

In this note, we take a closer look at this problem. Let $X = [x_{ij}]_{1\le i,j,\le r}$ be a multiplicative magic square whose entries are distinct positive integers, and let x_M and x_m be the largest and respectively smallest entry in X. We prove some nontrivial lower bound for $x_M - x_m$. For $r = 3$, we get a very precise result.

Theorem 1. *In a multiplicative magic square X of order 3 we have that*

$$x_M - x_m \ge x_m^{3/4}.$$

Furthermore, there exists an infinite family $(X(n))_{n\ge1}$ of multiplicative magic squares of order 3 such that

$$x_M(n) - x_m(n) \le x_m^{3/4}(n)(1 + o(1)) \qquad \text{as } n \to \infty.$$

For $r = 4$, we obtain the true minimal order of magnitude for the above difference.

Theorem 2. *In an multiplicative magic square X of order 4 we have that*

$$x_M - x_m \ge 5^{5/12} x_m^{1/2}.$$

Furthermore, there exists an infinite family of multiplicative magic squares $(X(n))_{n\ge1}$ of order 4 such that

$$x_M(n) - x_m(n) \le 6x_m^{1/2}(n)(1 + o(1)) \qquad \text{as } n \to \infty.$$

An example of such a family is

$$X(n) = \begin{bmatrix} (n+2)(n+4) & (n+3)(n+7) & (n+1)(n+6) & n(n+5) \\ (n+1)(n+5) & n(n+6) & (n+2)(n+7) & (n+3)(n+4) \\ n(n+7) & (n+1)(n+4) & (n+3)(n+5) & (n+2)(n+6) \\ (n+3)(n+6) & (n+2)(n+5) & n(n+4) & (n+1)(n+7) \end{bmatrix}.$$

It should be noted that these multiplicative magic squares are almost additive magic squares since the sums of the entries in each row, column, and diagonal differ by at most 6. We don't know if 6 is the smallest possible value, but it is not difficult to see that a magic square of order 4 cannot be simultaneously multiplicative and additive. To see this, observe that if X is a additive magic square, then

$$\begin{aligned}
2(x_{11}+x_{44}-x_{32}+x_{23}) &= (x_{11}+x_{12}+x_{13}+x_{14})+(x_{41}+x_{42}+x_{43}+x_{44}) \\
&- (x_{12}+x_{22}+x_{32}+x_{42})-(x_{13}+x_{23}+x_{33}+x_{43}) \\
&+ (x_{11}+x_{22}+x_{33}+x_{44})-(x_{14}+x_{23}+x_{32}+x_{41}) \\
&= 0.
\end{aligned}$$

So, $x_{11}+x_{44}=x_{32}+x_{23}$. If in addition X is a multiplicative magic square we have, for similar reasons, that $x_{11}x_{44}=x_{32}x_{23}$, so $\{x_{11},x_{44}\}=\{x_{23},x_{32}\}$, which is impossible since these four entries must be distinct.

The method we use to obtain the lower bounds in the Theorems 2 and 3 turns out to be too complicated for $r \geq 5$. Thus, when $r \geq 5$, we apply a different method which leads to a weaker result.

Theorem 3. *In a multiplicative magic square X of order $r \geq 5$ we have*

$$x_M - x_m \gg_r x_m^{1/(r-1)}.$$

Of course the exponent $1/(r-1)$ is theorem above is not sharp, at least for $r=3$ and $r=4$. It motives the first question we leave:

Problem 1. *What is the best exponent e_r in Theorem 3?*

Our results Theorem 1 and Theorem 2 show that $e_3 = 3/4$ and $e_4 = 1/2$.

Problem 2. *Are there additive-multiplicative magic squares of order $r=5$?*

We have seen that the answer is negative for $r=4$. On the contrary, Horner [2] found an additive and multiplicative magic square of order $r=8$.

2 Proofs

The multiplicative magic squares can be described in terms of the additive ones in the following way:

We write n^A for the multiplicative magic square given by

$$n^A = \left[n^{A(i,j)}\right]_{1\leq i,j\leq r}.$$

If we write $\times$ for the entrywise multiplication of the magic squares, then we have the following properties:

(i) $n^A \times n^B = n^{A+B}$;

(ii) $n^A \times m^A = (nm)^A$.

Each multiplicative magic square can be factored uniquely as $\prod_{s=1}^{t} {p_s}^{A_{p_s}}$, where $p_1 < \cdots < p_t$ are primes and the A_{p_s}'s are additive magic squares for $s = 1, \ldots, t$.

The additive magic squares of nonnegative integers form the set of *integral points* inside a *pointed polyhedral cone* (see [3]). Thus, the additive magic squares of order r have a minimal base of irreducible magic squares called a *Hilbert base* $H_r = \{B_l : l \in L\}$ in such a way that every additive magic square A with nonnegative integer entries can be written as

$$A = \sum_{l \in L} c_l B_l, \qquad \text{for some nonnegative } c_l \in \mathbb{Z}.$$

The Hilbert bases for the magic squares of orders 3 and 4 have been calculated in [1].

The basis H_3 consists of the following magic squares:

$$B_1 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, B_4 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}, B_5 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The basis H_4 consists of the following magic squares

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} B_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} B_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B_6 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} B_7 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} B_8 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} B_9 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix} B_{10} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$B_{11} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} B_{12} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} B_{13} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix} B_{14} = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} B_{15} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$B_{16} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} B_{17} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} B_{18} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} B_{19} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} B_{20} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

We recall that for us $X = [x_{ij}]_{1 \leq i,j, \leq r}$ is a multiplicative magic square whose entries are distinct positive integers and that x_M and x_m denote the largest and smallest entry in X, respectively. We start with the following preliminary result.

Lemma 1. *Let $R = \{(ij, i'j')\}$ be a collection of pairs of positions in a magic square of order r having the following property:*

$$\sum_{(ij,i'j')\in R} \min\{B_l(i,j), B_l(i',j')\} \geq ks(B_l) \qquad \textit{for all } B_l \in H_r.$$

Let $X = [x_{ij}]_{1\leq i,j\leq r}$ be a multiplicative magic square of order r. Then the inequality

$$x_M - x_m \geq x_m^{kr/|R|}$$

holds.

Proof. Write $X = \prod_{s=1}^t p_s^{A_{p_s}} = \prod_{s=1}^t p_s^{\sum_l c_{l,p_s} B_l}$, where $p_1 < \cdots < p_t$ are distinct primes and A_{p_s}'s are additive magic squares for $s = 1, \ldots, t$. Thus, $x_{ij} = \prod_{s=1}^t p^{\sum_l c_{l,p_s} B_l(i,j)}$. Then

$$\begin{aligned}
|x_{ij} - x_{i'j'}| &= |\prod_{s=1}^t {p_s}^{\sum_l c_{l,p_s} B_l(i,j)} - \prod_{s=1}^t {p_s}^{\sum_l c_{l,p_s} B_l(i',j')}| \\
&\geq \prod_{s=1}^t {p_s}^{\min\{\sum_l c_{l,p_s} B_l(i,j), \sum_l c_{l,p_s} B_l(i',j')\}} \\
&\geq \prod_{s=1}^t {p_s}^{\sum_l c_{l,p_s} \min\{B_l(i,j), B_l(i',j')\}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
(x_M - x_m)^{|R|} &\geq \prod_{(ij,i'j')\in R} |x_{ij} - x_{i'j'}| \\
&\geq \prod_{s=1}^t {p_s}^{\sum_l c_{l,p_s} \sum_{(ij,i'j')\in R} \min\{B_l(i,j), B_l(i',j')\}} \\
&\geq \prod_{s=1}^t {p_s}^{\sum_l c_{l,p_s} ks(B_l)} = \prod_{s=1}^t p_s^{s(A_{p_s})k}.
\end{aligned}$$

We finish the proof by noting that

$$x_m^r \leq \prod_{i=1}^r x_{1i} = \prod_{s=1}^t {p_s}^{\sum_{i=1}^r \alpha_{p_s}(1,i)} = \prod_{s=1}^t {p_s}^{s(A_{p_s})}.$$

□

Proof of Theorem 1. We take $R = \{(11,22),(13,22),(31,22),(33,22)\}$ in Lemma 1 for $r = 3$. Observe that $k = 1$.

The family given by

$$X(n) = n^{B_1} \times (n+1)^{B_2} \times (n+2)^{B_3} \times (n+3)^{B_4}$$

for all $n \geq 1$ satisfies the second part of the theorem. $\square$

Proof of Theorem 2. We now take $R = \cup_{m=1}^{8} R_m$, where for each $m = 1, \ldots, 8$, the set R_m consists of all the 6 subsets of pairs of positions $(ij, i'j')$ such that $B_m(i,j) = B_m(i',j') = 1$. Let us observe that in the notations of Lemma 1, we have $k = 6$. Lemma 1 now gives us the inequality

$$x_M - x_m \geq x_m^{1/2}.$$

To improve a bit on this inequality (on the multiplicative constant, not on the exponent $1/2$), observe that we can write

$$\begin{aligned}\prod_{(ij,i'j')\in R} |x_{ij} - x_{i'j'}| &= \prod_{m=1}^{8} \prod_{(ij,i'j')\in R_m} |x_{ij} - x_{i'j'}| \leq \left(\frac{1}{25\sqrt{5}}(x_M - x_m)^6\right)^8 \\ &= \frac{1}{5^{20}}(x_M - x_m)^{48}.\end{aligned}$$

In the above chain of inequalities, we have used the easy exercise (left to the reader) that if $0 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq 1$, then $\prod_{i<j} |\alpha_i - \alpha_j| \leq 1/(25\sqrt{5})$.

The family

$$X(n) = n^{B_6} \times (n+1)^{B_3} \times (n+2)^{B_2} \times (n+3)^{B_7} \times (n+4)^{B_1} \times (n+5)^{B_4} \times (n+6)^{B_8} \times (n+7)^{B_5}$$

for all positive integers n satisfies the second part of the theorem and corresponds to the family described in the introduction. $\square$

Proof of Theorem 3. We proceed by contradiction. We let s be the smallest element in the magic square and assume that s is on row i and column j. Write $x_{kl} = s + s_{kl}$ for all $k,\ l \in \{1, \ldots, r\}$ and expand the products on row i and column j as follows:

$$\begin{aligned}\prod_{l=1}^{r} x_{il} &= \prod_{l=1}^{r} (s + s_{il}) \qquad (1) \\ &= s\left(s^{r-1} + s^{r-2} \sum_{\substack{1\leq l\leq r \\ l\neq j}} s_{il} + s^{r-3} \sum_{\substack{1\leq l_1<l_2\leq r \\ l_1\neq j\neq l_2}} s_{il_1} s_{il_2} + \cdots + \prod_{\substack{1\leq l\leq r \\ l\neq j}} s_{il}\right)\end{aligned}$$

and similarly for column j. Since the two products obtained in this way are equal and since s is a common factor of both of them, we get that

$$
\begin{aligned}
|\sum_{\substack{1\le l\le r\\ l\neq j}} s_{il} - \sum_{\substack{1\le l\le r\\ l\neq i}} s_{lj}| \quad < \quad & s^{-1}\left|\sum_{\substack{1\le l_1<l_2\le r\\ l_1\neq j\neq l_2}} s_{il_1}s_{il_2} - \sum_{\substack{1\le l_1<l_2\le r\\ l_1\neq j\neq l_2}} s_{l_1 j}s_{l_2 j}\right| \\
& + \quad \cdots + s^{-(r-2)}\left|\prod_{\substack{1\le l\le r\\ l\neq j}} s_{il} - \prod_{\substack{1\le l\le r\\ l\neq i}} s_{lj}\right|.
\end{aligned}
$$

We now assume that $0 < s_{kl} < 2^{-(r-1)/2}s^{1/(r-1)}$ holds for all $k,\ l \in \{1,\ldots,r\}$ except for $(k,l) = (i,j)$ in order to get a contradiction. We then get that the right hand side above is

$$
< 2^{-(r-1)}s^{-1+2/(r-1)}\left(\binom{r-1}{2} + \binom{r-1}{3} + \cdots + \binom{r-1}{r-1}\right) < 1,
$$

therefore

$$
\sum_{l=1}^{r} s_{il} = \sum_{l=1}^{r} s_{lj}. \tag{2}
$$

We now proceed by induction on t to show that the two tth symmetric polynomials

$$
\sum_{1\le l_1<\cdots<l_t\le r} s_{il_1}\cdots s_{il_t} = \sum_{1\le l_1<\cdots<l_t\le r} s_{l_1 j}\cdots s_{l_t j} \tag{3}
$$

in the $(s_{il})_{1\le l\le r}$ and $(s_{lj})_{1\le l\le r}$ are equal. Formula (2) shows that this holds when $t = 1$ and by induction it is enough to show that the two tth symmetric polynomials on the sets of $r-1$ dimensional indeterminates $(s_{il})_{\substack{1\le l\le r\\ l\neq j}}$ and $(s_{lj})_{\substack{1\le l\le r\\ l\neq i}}$ are equal. Assuming that $t \ge 2$ and that the above equality holds for $t-1 < r-1$, then equating again the two products shown at (1) for the

ith row with the analogous one obtained for the jth row, we get

$$\left|\sum_{\substack{1\le l_1<\ldots<l_t\le r\\ l_k\ne j,\ 1\le k\le t}} s_{il_1}\cdots s_{il_t} - \sum_{\substack{1\le l_1<\cdots<l_t\le r\\ l_k\ne i,\ 1\le k\le t}} s_{l_1 j}\cdots s_{l_t j}\right|$$
$$\le s^{-1}\left|\sum_{\substack{1\le l_1<\ldots<l_{t+1}\le r\\ l_k\ne j,\ 1\le k\le t+1}} s_{il_1}\cdots s_{il_{t+1}} - \sum_{\substack{1\le l_1<\cdots<l_{t+1}\le r\\ l_k\ne i,\ 1\le k\le t+1}} s_{l_1 j}\cdots s_{l_{t+1} j}\right|$$
$$+\cdots+s^{-(r-1-t)}\left|\prod_{\substack{1\le l\le r\\ l\ne j}} s_{il} - \prod_{\substack{1\le l\le r\\ l\ne i}} s_{lj}\right|.$$

Using again the fact that $s \ge 1$ and $0 < s_{kl} < 2^{-(r-1)/2}s^{1/(r-1)}$ whenever $(k,l) \ne (i,j)$, we get that

$$\left|\sum_{\substack{1\le l_1<\cdots<l_t\le r\\ l_k\ne j,\ 1\le k\le t}} s_{il_1}\cdots s_{il_t} - \sum_{\substack{1\le l_1<\cdots<l_t\le r\\ l_j\ne i,\ 1\le k\le t}} s_{l_1 j}\cdots s_{l_t j}\right|$$
$$< 2^{-(r-1)}s^{-1+(t+1)/(r-1)}\left(\binom{r-1}{t+1}+\cdots+\binom{r-1}{r-1}\right) < 1,$$

therefore relation (3) holds for t as well. Since this is true for all $t = 1, \ldots, r$, we deduce that the two polynomials

$$\prod_{l=1}^{r}(X - s_{il}) \qquad \text{and} \qquad \prod_{l=1}^{r}(X - s_{lj})$$

are equal. In particular, the entries from row i are a permutation of the entries from column j, but this is not allowed since the union of these entries should be a set of $2r - 1$ distinct integers. This completes the proof of Theorem 3. $\square$

References

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[3] R.P. Stanley, *Enumerative Combinatorics*, Vol. I, Cambridge Univ. Press, Cambridge, 1997.