DENSE SETS OF INTEGERS WITH PRESCRIBED REPRESENTATION FUNCTIONS

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ABSTRACT. Let $\mathcal{A}$ be a set of integers and let $h \geq 2$. For every integer n, let $r_{\mathcal{A},h}(n)$ denote the number of representations of n in the form $n = a_1 + \cdots + a_h$, where $a_i \in \mathcal{A}$ for $1 \leq i \leq h$, and $a_1 \leq \cdots \leq a_h$. The function $r_{\mathcal{A},h} : \mathbb{Z} \to \mathbf{N}$, where $\mathbf{N} = \mathbb{N} \cup \{0, \infty\}$, is the *representation function of order h for $\mathcal{A}$*.

We prove that, given a positive integer g, every function $f : \mathbb{Z} \to \mathbf{N}$ satisfying $\liminf_{|n|\to\infty} f(n) \geq g$ is the representation function of order h of a sequence $\mathcal{A}$ of integers "almost" as dense as any given $B_h[g]$ sequence. In particular we prove that, given an integer $h \geq 2$ and $\varepsilon > 0$, there exists $g = g(h, \epsilon)$ such that for any function $f : \mathbb{Z} \to \mathbf{N}$ satisfying $\liminf_{|n|\to\infty} f(n) \geq g$ there exists a sequence $\mathcal{A}$ satisfying $r_{\mathcal{A},h} = f$ and $|\mathcal{A} \cap [1, x]| \gg x^{(1/h)-\varepsilon}$.

Roughly speaking we prove that the problem of finding a dense set of integers with a prescribed representation function f of order h and $\liminf_{|n|\to\infty} f(n) \geq g$ is "equivalent" to the classical problem of finding dense $B_h[g]$ sequences of positive integers.

1. Introduction

Let $\mathcal{A}$ be a set of integers and let $h \geq 2$. For every integer n, let $r_{\mathcal{A},h}(n)$ denote the number of representations of n in the form

$$n = a_1 + \cdots + a_h$$

where $a_1 \leq \cdots \leq a_h$ and $a_i \in \mathcal{A}$ for $1 \leq i \leq h$. The function $r_{\mathcal{A},h} : \mathbb{Z} \to \mathbf{N}$ is the *representation function of order h for $\mathcal{A}$*, where $\mathbf{N} = \mathbb{N} \cup \{0, \infty\}$.

Nathanson proved [8] that any function $f : \mathbb{Z} \to \mathbf{N}$ satisfying $\liminf_{|n|\to\infty} f(n) \geq 1$ is the *representation function of order h* of a set of integers $\mathcal{A}$ such that

$$\mathcal{A}(x) \gg x^{1/(2h-1)}, \tag{1}$$

where $\mathcal{A}(x)$ counts the number of positive elements $a \in \mathcal{A}$ no greater than x and $f(x) \gg g(x)$ means that there exists a constant $C > 0$ such that $f(x) \geq Cg(x)$ for x *large enough*.

It is an open problem to determine how dense these sets $\mathcal{A}$ can be. In this paper we study the connection between this problem and the problem of finding dense $B_h[g]$ sequences. We recall that a set $\mathcal{B}$ of nonnegative integers is called a $B_h[g]$ sequence if

$$r_{\mathcal{B},h}(n) \leq g$$

for every nonnegative integer n. It is usual to write B_h to denote $B_h[1]$ sequences.

Luczak and Schoen [7] proved that any B_h sequence satisfying an additional kind of Sidon property (see [7] for the definition of this property, which they call the S_h property) can be enlarged to obtain a sequence with any prescribed representation function

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f satisfying $\liminf_{|x|\to\infty} f(x) \geq 1$. In particular, since they prove that there exists a B_h sequence $\mathcal{A}$ satisfying the S_h property with $\mathcal{A}(x) \gg x^{1/(2h-1)}$, they recover Nathanson's result.

1.1. **Main result.** In this paper we prove that any $B_h[g]$ sequence, without any additional property, can be modified to have any prescribed representation function f of order h satisfying $\liminf_{|x|\to\infty} f(x) \geq g$. Our main theorem is the following.

Theorem 1.1. *Let $f : \mathbb{Z} \to \mathbf{N}$ be any function such that $\liminf_{|n|\to\infty} f(n) \geq g$ and let $\mathcal{B}$ be any $B_h[g]$ sequence. Then, for any decreasing function $\epsilon(x) \to 0$ as $x \to \infty$, there exists a sequence $\mathcal{A}$ of integers such that*

$$r_{\mathcal{A},h}(n) = f(n) \quad \text{for all} \quad n \in \mathbb{Z} \qquad \text{and} \quad \mathcal{A}(x) \gg \mathcal{B}(x\epsilon(x)).$$

Roughly speaking, theorem above says that the problem of finding dense sets of integers with prescribed representation functions with $\liminf_{|n|\to\infty} f(n) \geq g$ is "equivalent" to the classical problem of finding dense $B_h[g]$ sequences of positive integers.

It is a difficult problem to construct dense $B_h[g]$ sequences. A trivial counting argument gives $\mathcal{B}(x) \ll x^{1/h}$ for these sequences. On the other hand, the greedy algorithm shows that there exists a B_h sequence $\mathcal{B}$ such that

$$\mathcal{B}(x) \gg x^{1/(2h-1)}. \tag{2}$$

For B_2 sequences, also called Sidon sets, Ruzsa proved [10] that there exists a Sidon set $\mathcal{B}$ such that

$$\mathcal{B}(x) \gg x^{\sqrt{2}-1+o(1)}. \tag{3}$$

This result and Theorem 1.1 give the following corollary.

Corollary 1. *Let $f : \mathbb{Z} \to \mathbf{N}$ any function such that $\liminf_{|n|\to\infty} f(n) \geq 1$. Then there exists a sequence of integers $\mathcal{A}$ such that*

$$r_{\mathcal{A},2}(n) = f(n) \quad \text{for all} \quad n \in \mathbb{Z} \qquad \text{and} \qquad \mathcal{A}(x) \gg x^{\sqrt{2}-1+o(1)}.$$

This result gives an affirmative answer to the third open problem in [1], which was also posed previously in [9]. Unfortunately, nothing better than (4) is known for B_h sequences when $h \geq 3$.

Erdős and Renyi [4] proved however that, for any $\epsilon > 0$, there exists a positive integer g and a $B_2[g]$ sequence $\mathcal{B}$ such that $\mathcal{B}(x) \gg x^{1/2-\epsilon}$. They claimed that the probabilistic method they used could be extended to $B_h[g]$ sequences, but a serious problem with non-independent events appears when $h \geq 3$. As an application of a more general theory, Vu [12] overcame this problem. He proved that for any integer $h \geq 2$ and for any $\epsilon > 0$, there exists an integer $g = g(h,\epsilon) \ll_h \epsilon^{1-h}$ and a $B_h[g]$ sequence $\mathcal{B}$ such that

$$\mathcal{B}(x) \gg x^{1/h-\epsilon}.$$

See also [2] for a different proof of this result which improves the upper bound to $g = g(h,\epsilon) \ll_h \epsilon^{-1}$. Vu's result and Theorem 1.1 imply the next corollary.

Corollary 2. *Given $h \geq 2$, for any $\varepsilon > 0$, there exists $g = g(h,\varepsilon)$ such that, for any function $f : \mathbb{Z} \to \mathbf{N}$ satisfying $\liminf_{|n|\to\infty} f(n) \geq g$, there exists a sequence $\mathcal{A}$ of integers such that*

$$r_{\mathcal{A},h}(n) = f(n) \quad \text{for all} \quad n \in \mathbb{Z} \qquad \text{and} \qquad \mathcal{A}(x) \gg x^{\frac{1}{h}-\varepsilon}.$$

The construction in [8] for the set $\mathcal{A}$ satisfying the growth condition (1) was based on the greedy algorithm. In this paper we construct the set $\mathcal{A}$ by adjoining a very sparse sequence $\{u_k\}$ to a suitable $B_h[g]$ sequence $\mathcal{B}$. This idea was used in [3], but in a simpler way, to construct dense *perfect difference sets*, which are sets such that every nonzero integer has a unique representation as a difference of two elements of $\mathcal{A}$. The proof of the main theorem in [3] can be adapted easily to our problem in the simplest case $h = 2$.

Theorem 1.2. *Let $f : \mathbb{Z} \to \mathbf{N}$ be a function such that $\liminf_{|n|\to\infty} f(n) \geq g$, and let $\mathcal{B}$ be a $B_2[g]$ sequence. Then there exists a sequence of integers $\mathcal{A}$ such that*

$$r_{\mathcal{A},2}(n) = f(n) \quad \text{for all} \quad n \in \mathbb{Z} \qquad \text{and} \qquad \mathcal{A}(x) \gg \mathcal{B}(x/3).$$

We omit the proof because it is very close to the proof of the main theorem in [3]. Unfortunately, that proof cannot be adapted to the case $h \geq 3$. We need another definition of a "suitable" $B_h[g]$ set. In section §2 we shall show how to modify a $B_h[g]$ sequence $\mathcal{B}$ so that it becomes "suitable." We do this by applying the "Inserting Zeros Transformation" to an arbitrary $B_h[g]$ set. This is the main ingredient in the proof of Theorem 1.1.

1.2. **Related results.** Chen [1] has proved that for any $\epsilon > 0$ there exists a unique-representation basis $\mathcal{A}$ (that is, a set $\mathcal{A}$ with $r_{\mathcal{A},2}(k) = 1$ for all $k \in \mathbb{Z}$) such that $\limsup_{x\to\infty} \mathcal{A}(x)/x^{1/2-\epsilon} > 1$. J. Lee [6] has improved this result by proving that for any increasing function ω tending to infinity there exists a unique-representation basis $\mathcal{A}$ such that $\limsup_{x\to\infty} \mathcal{A}(x)\omega(x)/\sqrt{x} > 0$.

Theorem 1.2 and the classical constructions of Erdős [11] and Krückeberg [5] of infinite Sidon sets $\mathcal{B}$ such that $\limsup_{x\to\infty} \mathcal{B}(x)/\sqrt{x} > 0$ provide a unique-representation basis $\mathcal{A}$ such that $\limsup_{x\to\infty} \mathcal{A}(x)/\sqrt{x} > 0$. Indeed, we can easily adapt the proof of Theorem 1.3 in [3] to the case of the additive representation function $r(n)$ (instead of the subtractive representation function $d(n) = \#\{n = a - a',\ a, a' \in \mathcal{A}\}$).

Theorem 1.3. *There exists a unique-representation basis $\mathcal{A}$ such that*

$$\limsup_{x\to\infty} \frac{\mathcal{A}(x)}{\sqrt{x}} \geq \frac{1}{\sqrt{2}}.$$

Again we omit the proof because it is very close to the proof of Theorem 1.3 in [3].

Theorem 1.3 answers affirmatively the first open problem in [1]. Note also that if $\mathcal{A}$ is an infinite Sidon set of integers, then the set

$$\mathcal{A}' = \{4a : a \geq 0\} \cup \{-4a + 1 : a < 0\}$$

is also a Sidon set and, in this case, $\liminf |\mathcal{A} \cap (-x, x)|/\sqrt{x} = \liminf \mathcal{A}'(4x)/\sqrt{x}$. A well known result of Erdős states that $\liminf \mathcal{B}(x)/\sqrt{x} = 0$ for any Sidon set $\mathcal{B}$. Then the above limit is zero, so it answers negatively the second open problem in [1].

To obtain a similar result for $h \geq 3$, although weaker, we can use that if $\mathcal{B}_1, \mathcal{B}_2$ are B_h sequences and $\mathcal{B}_1 \subset [1, n)$ then the set $\mathcal{B}_1 \cup (hn * \mathcal{B}_2)$ is also a B_h sequence. Here we use the notation $t * \mathcal{B} = \{tb,\ b \in \mathcal{B}\}$.

Using this fact it is easy to prove that for any function ω tending to infinity there exists a B_h sequence $\mathcal{A}$ such that

$$\limsup_{x\to\infty} \frac{\mathcal{A}(x)\omega(x)}{x^{1/h}} > 1. \tag{4}$$

We can construct the sequence $\mathcal{A}$ as follows: Let $s_1, \ldots, s_k, \ldots$ be an infinite sequence of positive integers such that $\omega(s_k) > (hs_{k-1})^{1/h}$ and consider, for each k, a B_h sequence $\mathcal{B}_k \subset [1, s_k/(hs_{k-1}))$ with $|\mathcal{B}_k| \gg (s_k/(hs_{k-1}))^{1/h}$. Easily we can check that the set $\mathcal{A} = \cup_k (hx_{k-1}) * \mathcal{B}_k$ is a B_h sequence and satisfies $\mathcal{A}(s_k) \gg (s_k)^{1/h}/\omega(s_k)$.

The construction above and Theorem 1.1 yield the following corollary, which extends the main theorem in [1] in several ways.

Corollary 3. *Let $f : \mathbb{Z} \to \mathbf{N}$ any function such that $\liminf_{|n|\to\infty} f(n) \geq 1$. For any increasing function ω tending to infinity there exists a set $\mathcal{A}$ such that $r_{\mathcal{A},h}(n) = f(n)$ for all integers n, and*

$$\limsup_{x\to\infty} \frac{\mathcal{A}(x)\omega(x)}{x^{1/h}} > 0.$$

2. Proof of Theorem 1.1

The main ideas of the proof are the following:

1) We start with a B_h set of integers $\mathcal{A}_0$ such that binary expansion of its elements have blocks of zeros at fixed places. We can obtain a sequence of this form by applying the Insert Zeros Transformation described below to a B_h sequence given.

2) We consider also a special sequence (u_k) such that the sequence (z_k), defined by $z_k = (h-1)u_{2k-1} + u_{2k}$, takes all the integer values infinitely often.

3) For $k \geq 1$, we define $\mathcal{A}_k = \mathcal{A}_{k-1}$ if $r_{\mathcal{A}_{k-1},h}(z_k) = f(z_k)$ and $\mathcal{A}_k = \mathcal{A}_{k-1} \cup \{u_{2k-1}, u_{2k}\}$ if $r_{\mathcal{A}_{k-1},h}(z_k) < f(z_k)$. Then $r_{\mathcal{A}_k,h}(z_k) \geq r_{\mathcal{A}_{k-1},h}(z_k) + 1$ and, since the sequence (z_k) takes all the integre values infinitely often, the sequence $\mathcal{A} = \cup_{k=0}^{\infty} \mathcal{A}_k$ satisfies $r_{\mathcal{A},h}(n) \geq f(n)$ for any integer n.

4) The difficult part of the proof is to prove that when we add $\{u_{2k-1}, u_{2k}\}$ to $\mathcal{A}_{k-1}$ to obtain a new representation of z_k, we don't obtain a new representation of other integer m for which we had $r_{A_{k-1},h}(m) = f(m)$. To avoid this problem we have chosen the sequence u_k with the additional property that the one's digits of the binary expansion of its elements lie just on the places where the the elements of $\mathcal{A}_0$ have blocks of zeros.

Roughly speaking, the Insert Zeros Transformation allow us to work with integers as they were vectors where the distinct components are separated by blocks of zeros at fixed places.

5) When we apply the Insert Zeros Transformation to the elements of a sequence we obtain a less dense sequence. But if the places where the blocks are located are very sparse we don't lose too much density. To concrete this we will choose the sequence $l_0 < l_1 < \cdots$ associated to the Insert Zeros Transformation according to the function $\epsilon(x)$.

2.1. The Inserting Zeros Transformation.

Given any infinite increasing sequence of nonnegative integers $0 = l_0 < l_1 < \cdots < l_k < \cdots$ we can write any positive number n in an unique way the form

$$n = n_0 2^{l_0} + n_1 2^{l_1} + n_2 2^{l_2} + \cdots + n_k 2^{l_k} + \cdots \tag{5}$$

with $0 \leq n_i < 2^{l_{i+1}-l_i}$.

Definition 2.1. For any positive integer r we define the Inserting Zeros Transformation of order r associated to an increasing sequence of non negative integers $0 = l_0 < l_1 < \cdots$ as the function

$$t_r(n) = n_0 2^{l_0+2r} + n_1 2^{l_1+4r} + n_2 2^{l_2+6r} + \cdots + n_k 2^{l_k+2(k+1)r} + \cdots \tag{6}$$

where the n_i as defined as in (5).

We observe that $t_r(n)$ is the result of inserting strings of zeros of length $2r$ in the binary expression of n at places $l_i,\ i \geq 0$.

For short we will write

$$m_k = 2^{l_k+2(k+1)r}. \tag{7}$$

With this notation (6) can be written as

$$t_r(n) = n_0 m_0 + n_1 m_1 + n_2 m_2 + \cdots + n_k m_k + \cdots \tag{8}$$

The following seminorm will be useful to prove some lemmas.

Definition 2.2. For all integers $m \geq 2$ and $x \in \mathbb{Z}$ we define
$$\|x\|_m = \min\{|y|,\ x \equiv y \pmod{m}\}.$$

Note that $\|x_1 + x_2\|_m \leq \|x_1\|_m + \|x_2\|_m$ for all integers x_1 and x_2. Through the proof we will prove that some equalities $a = b$ can not hold by proving that $\|a\|_m \neq \|b\|_m$ for some m.

The Inserting Zeros Transformation of order r has some important properties which we resume in Lemma 2.1 and Lemma 2.2.

Lemma 2.1. *For any $n \geq 1$ and $k \geq 1$ we have $\|t_r(n)\|_{m_k} < m_k 2^{-2r}$.*

Proof. As a consequence of (8) and since $m_k \mid m_j$ for all $j > k$, we have
$$t_r(n) \equiv c \pmod{m_k}$$
where
$$c = n_0 m_0 + n_1 m_1 + n_2 m_2 + \cdots + n_{k-1} m_{k-1}$$
for some $n_i,\ 0 \leq n_i \leq 2^{l_{i+1}-l_i} - 1$. Thus we have
$$\begin{aligned} 0 \leq c &\leq (2^{l_1-l_0} - 1)m_0 + (2^{l_2-l_1} - 1)m_1 + \cdots + (2^{l_k-l_{k-1}} - 1)m_{k-1} \\ &= 2^{l_1-l_0} m_0 + 2^{l_2-l_1} m_1 + \cdots 2^{l_k-l_{k-1}} m_{k-1} - (m_0 + m_1 + \cdots + m_{k-1}) \\ &= (m_1 + m_2 + \cdots + m_k)2^{-2r} - (m_0 + m_1 + \cdots + m_{k-1}) \\ &< m_k 2^{-2r}. \end{aligned}$$
□

Lemma 2.2. *Let $\mathcal{B}$ a $B_h[g]$ sequence and $h \leq 2^{2r}$. Then the set $t_r(\mathcal{B}) = \{t_r(b) :\ b \in \mathcal{B}\}$ is also a $B_h[g]$ sequence.*

Proof. It is enough to prove that if $t_r(b_1) + \cdots + t_r(b_h) = t_r(b_1') + \cdots + t_r(b_h')$ then $b_1 + \cdots + b_h = b_1' + \cdots + b_h'$.

Let $b_i = n_{0i} 2^{l_0} + \cdots + n_{ki} 2^{l_k} + \cdots$ with $n_{ki} < 2^{l_k - l_{k-1}}$. By (8) we have
$$t_r(b_i) = n_{0i} m_0 + \cdots + n_{ki} m_k + \cdots$$
The assumption $t_r(b_1) + \cdots + t_r(b_h) = t_r(b_1') + \cdots + t_r(b_h')$ implies
$$\begin{aligned} &(n_{01} + \cdots + n_{0h})m_0 + \cdots (n_{k1} + \cdots n_{kh})m_k + \cdots \\ &= (n_{01}' + \cdots + n_{0h}')m_0 + \cdots (n_{k1}' + \cdots n_{kh}')m_k + \cdots \end{aligned}$$
The inequality $h \leq 2^{2r}$ gives
$$n_{k1} + \cdots + n_{kh} < h 2^{l_{k+1}-l_k} \leq 2^{l_{k+1}-l_k+2r} = m_{k+1}/m_k.$$
Since there is only a way to write an integer as
$$x_0 m_0 + \cdots + x_k m_k + \cdots$$
with $x_k < m_{k+1}/m_k$ we conclude that
$$n_{k1} + \cdots + n_{kh} = n_{k1}' + \cdots + n_{kh}'$$
for all k. Thus
$$\begin{aligned} b_1 + \cdots + b_h &= (n_{01} + \cdots + n_{0h})2^{l_0} + \cdots (n_{k1} + \cdots + n_{kh})2^{l_k} + \cdots \\ &= (n_{01}' + \cdots + n_{0h}')2^{l_0} + \cdots (n_{k1}' + \cdots + n_{kh}')2^{l_k} + \cdots = b_1' + \cdots + b_h'. \end{aligned}$$
□

2.2. **Construction of the sequence $\mathcal{A}$.** The condition on f in Theorem 1.1 implies that there exists n_0 such that $f(n) \geq g$, for any $n \geq n_0$.

Let $\mathcal{B}$ be a $B_h[g]$ sequence and fix r satisfying

$$h^2 < 2^{r-1} \text{ and } n_0 \leq 2^r. \tag{9}$$

The starting point will be the sequence $\mathcal{A}_0 = t_r(\mathcal{B})$ which is also a $B_h[g]$ sequence by Lemma 2.2.

Consider the sequence $(z_j)_{j=1}^{\infty}$ defined by

$$z_j = j - [\sqrt{j}]([\sqrt{j}] + 1). \tag{10}$$

This sequence takes of all the integers values infinitely many times each integer.

Also we consider the sequence $(u_i)_{i=1}^{\infty}$ defined by

$$\begin{cases} u_{2k-1} & = & -m_k 2^{-r} \\ u_{2k} & = & z_k + (h-1)m_k 2^{-r}. \end{cases} \tag{11}$$

where

$$m_k = 2^{l_k + 2(k+1)r} \tag{12}$$

and $0 = l_0 < l_1 < \cdots$ is a given sequence of non negative integers.

For $k \geq 1$, we define

$$\mathcal{A}_k = \begin{cases} \mathcal{A}_{k-1} \cup \{u_{2k-1}, u_{2k}\} & \text{if } r_{\mathcal{A}_{k-1},h}(z_k) < f(z_k) \\ \mathcal{A}_{k-1} & \text{otherwise.} \end{cases} \tag{13}$$

We shall prove that the set

$$\mathcal{A} = \bigcup_{k=0}^{\infty} \mathcal{A}_k \tag{14}$$

satisfies $r_{\mathcal{A},h}(n) = f(n)$ for all integers n as consequence of Propositions 2.3 and 2.4 below.

Proposition 2.3. The sequence $\mathcal{A}$ defined in (14) satisfies $r_{\mathcal{A},h}(n) \geq f(n)$ for all integers n.

Proof. Since

$$\underbrace{u_{2k-1} + \cdots + u_{2k-1}}_{h-1} + u_{2k} = z_k$$

it follows that if $r_{\mathcal{A}_{k-1},h}(z_k) < f(z_k)$, then $\mathcal{A}_k = \mathcal{A}_{k-1} \cup \{u_{2k-1}, u_{2k}\}$ and

$$r_{\mathcal{A}_k,h}(z_k) \geq r_{\mathcal{A}_{k-1},h}(z_k) + 1.$$

Since the sequence (z_k) takes all the integers infinitely many times, then $r_{\mathcal{A}_k,h}(n) \geq f(n)$ for some k (if $f(n) < \infty$) or $\lim_{k\to\infty} r_{\mathcal{A}_k,h}(n) = \infty$ (if $f(n) = \infty$). $\square$

2.3. **Technical lemmas.**

Lemma 2.3. *For $k \geq 1$ and r, z_k, m_k, $\mathcal{A}_k$ defined as in* (9), (10), (12) *and* (13) *we have*

i) $|z_j| \leq m_k 2^{-2r}$ *for any $j \leq k$.*
ii) $\|a_i\|_{m_k} < m_k 2^{-2r}$ *for all $a_i \in \mathcal{A}_{k-1}$.*

Proof. i) $|z_j| = |j - [\sqrt{j}]([\sqrt{j}] + 1)| \leq \sqrt{j} + 1 \leq 2^j \leq 2^{l_j} \leq 2^{l_k} = m_k 2^{-2(k+1)r} \leq m_k 2^{-2r}$.

ii) It is clear when $a_i \in \mathcal{A}_0$ due to Lemma 2.1. If $a_i \notin \mathcal{A}_0$ we have that $a_i = u_j$ for some $j \leq 2k-2$. Thus, $a_i = -m_j 2^{-r}$ or $a_i = z_j + (h-1)m_j 2^{-r}$ for some $j \leq k-1$. Using part i) of this lemma, the first condition in (9) and the inequality $m_{k-1} \leq m_k 2^{-2r}$ we get

$$\|a_i\|_{m_k} \leq |a_i| \leq |z_j| + (h-1)m_{j-1}2^{-r} \leq hm_{k-1}2^{-r} \leq hm_k 2^{-3r} < m_k 2^{-2r}.$$

□

Let $k \geq 1$: For any s, t non negative integers with $s + t \leq h$ we define

$$\mathcal{A}_k^{(s,t)} = (h - s - t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}, \tag{15}$$

where we have used the notation $rS = \{s_1 + \cdots + s_r : \; s_i \in S\}$.

Lemma 2.4. *The sets $\mathcal{A}_k^{(s,t)}$ are pairwise disjoint, except possibly the sets $\mathcal{A}_k^{(0,0)}$ and $\mathcal{A}_k^{(h-1,1)}$.*

Proof. Suppose that

$$a_1 + \cdots + a_{h-s-t} + su_{2k-1} + tu_{2k} = a'_1 + \cdots + a'_{h-s'-t'} + s'u_{2k-1} + t'u_{2k}$$

with

$$(s,t) \neq (s', t') \text{ and } \{(s,t),(s',t')\} \neq \{(0,0),(h-1,1)\}. \tag{16}$$

Using that $u_{2k} = z_k + (h-1)u_{2k-1}$ and that $u_{2k-1} = -m_k 2^{-r}$ we have

$$a_1 + \cdots + a_{h-s-t} - (a'_1 + \cdots + a'_{h-s'-t'}) + (t - t')z_k = ((t'-t)(h-1) + s - s')m_k 2^{-r}. \tag{17}$$

We will prove that equality (17) can not hold by proving that the two sides have distinct seminorm $\|\cdot\|_{m_k}$.

The conditions (16) and the inequality $s + t \leq h$ imply that

$$1 \leq |(t'-t)(h-1) + s - s'| \leq h^2.$$

Thus,

$$|((t'-t)(h-1) + s - s')m_k 2^{-r}| \leq h^2 2^{-r} m_k < m_k/2.$$

Now, we observe that if $|x| < m/2$ then $\|x\|_m = |x|$. Therefore,

$$\|((t'-t)(h-1) + s - s')m_k 2^{-r}\|_{m_k} = |((t'-t)(h-1) + s - s')m_k 2^{-r}| \geq m_k 2^{-r}. \tag{18}$$

On the other hand, for the left side of (17), we use Lemma 2.3 to obtain

$$\begin{aligned} \|a_1 + \cdots + a_{h-s-t} - &(a'_1 + \cdots + a'_{h-s'-t'}) + (t-t')z_k\|_{m_k} \\ &\leq \sum_{i=1}^{h-s-t} \|a_i\|_{m_k} + \sum_{i=1}^{h-s'-t'} \|a'_i\|_{m_k} + |t-t'||z_k| \\ &\leq (2h - s - s' - t - t')m_k 2^{-2r} + |t-t'|m_k 2^{-2r} \\ &\leq 2hm_k 2^{-2r} \\ &< m_k 2^{-r}. \end{aligned} \tag{19}$$

□

Lemma 2.5. *If $n \in \mathcal{A}_k^{(s,t)}$ for some $k \geq 1$ and $(s,t) \notin \{(0,0),(h-1,1)\}$, then $|n| > n_0$.*

Proof. If $n \in \mathcal{A}_k^{(s,t)}$ then

$$\begin{aligned} n &= a_1 + \cdots + a_{h-s-t} + su_{2k-1} + tu_{2k} \\ &= a_1 + \cdots + a_{h-s-t} + (t(h-1) - s)m_k 2^{-r} + tz_k. \end{aligned}$$

Thus,

$$\begin{aligned} |n| &\geq \|n\|_{m_k} \\ &= \|a_1 + \cdots + a_{h-s-t} + (t(h-1) - s)m_k 2^{-r} + tz_k\|_{m_k} \\ &\geq \|(t(h-1) - s)m_k 2^{-r}\|_{m_k} - \|a_1 + \cdots + a_{h-s-t} + tz_k\|_{m_k}. \end{aligned}$$

The conditions on (s,t) imply that $1 \leq |t(h-1)-s| \leq h^2 \leq 2^{r-1}$. Thus

$$\|(t(h-1)-s)m_k 2^{-r}\|_{m_k} = |(t(h-1)-s)m_k 2^{-r}|_{m_k} \geq m_k 2^{-r}. \tag{20}$$

On the other hand, Lemma 2.3 implies that

$$\|a_1+\cdots+a_{h-s-t}+tz_k\|_{m_k} \leq (h-s-t)m_k 2^{-2r}+tm_k 2^{-2r} < hm_k 2^{-2r} \leq m_k 2^{-r-1}. \tag{21}$$

Then we have

$$|n| > m_k 2^{-r-1} \geq 2^{3r} > n_0.$$

□

Lemma 2.6. *For any $k \geq 0$, for any $h' < h$ and for any integer m we have that*

$$r_{\mathcal{A}_k,h'}(m) \leq g.$$

Proof. By induction on k. Lemma 2.2 implies that $\mathcal{A}_0$ is a $B_h[g]$-sequences. In particular, $\mathcal{A}_0$ is a $B_{h'}[g]$ sequence for $h' < h$. Then $r_{\mathcal{A}_0,h'}(m) \leq g$ for any integer m.

Suppose that it is true that for any $h' < h$, and for any integer m we have that $r_{\mathcal{A}_{k-1},h'}(m) \leq g$.

Consider $m \in h'\mathcal{A}_k$.

- Suppose $m \notin (h'-s-t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}$ for any $(s,t) \neq (0,0)$. Then $r_{\mathcal{A}_k,h'}(m) = r_{\mathcal{A}_{k-1},h'}(m) \leq g$ by the induction hypothesis.
- Suppose that $m \in (h'-s-t)\mathcal{A}_{k-1}+su_{2k-1}+tu_{2k}$ for some $(s,t) \neq (0,0)$. Consider an element $a \in \mathcal{A}_0$. Then

$$m+(h-h')a \in \mathcal{A}_k^{(s,t)} \in (h-s-t)\mathcal{A}_{k-1}+su_{2k-1}+tu_{2k}.$$

Since $(s,t) \neq (h-1,1)$ (because $h' < h$) we can apply lemma 2.4 and we have

$$r_{\mathcal{A}_k,h'}(m) \leq r_{\mathcal{A}_k,h}(m+(h-h')a) = r_{\mathcal{A}_{k-1},h-s-t}(m+(h-h')a - su_{2k-1} - tu_{2k}).$$

We can the apply the induction hypothesis because $h-s-t < h$.

□

2.4. **End of the proof.**

Proposition 2.4. The sequence $\mathcal{A}$ defined in (14) satisfies $r_{\mathcal{A},h}(n) \leq f(n)$ for all integers n.

Proof. Next we show that, for every integer k, the sequence $\mathcal{A}_k$ satisfies $r_{\mathcal{A}_k,h}(n) \leq f(n)$ for all n. The proof is by induction on k. To check it for $k=0$ we observe that if $a \in \mathcal{A}_0$ then $a \geq 2^{2r} \geq n_0$. If $n < n_0$ then $r_{\mathcal{A}_0,h}(n) = 0 \leq f(n)$. On the other hand, if $n \geq n_0$ then $r_{\mathcal{A}_0,h}(n) \leq g$ because $\mathcal{A}_0$ is a $B_h[g]$ sequence (see Lemma 2.2) and then $r_{\mathcal{A}_0,h}(n) \leq f(n)$ because $f(n) \geq g$ for $n \geq n_0$.

Now, suppose that it is true for $k-1$. In particular $r_{\mathcal{A}_{k-1},h}(z_k) \leq f(z_k)$. If $r_{\mathcal{A}_{k-1},h}(z_k) = f(z_k)$ there is nothing to prove because in that case $\mathcal{A}_k = \mathcal{A}_{k-1}$. But if $r_{\mathcal{A}_{k-1},h}(z_k) \leq f(z_k)-1$, then $\mathcal{A}_k = \mathcal{A}_{k-1} \cup \{u_{2k-1}, u_{2k}\}$. We will assume this until the end of the proof.

If $n \notin h\mathcal{A}_k$ then $r_{\mathcal{A}_k,h}(n) = 0 \leq f(n)$.

If $n \in h\mathcal{A}_k$, since $\mathcal{A}_k = \mathcal{A}_{k-1} \cup \{u_{2k-1}, u_{2k}\}$ we can write

$$h\mathcal{A}_k = \bigcup_{\substack{s,t=0 \\ s+t\leq h}}^{h} \left((h-s-t)\mathcal{A}_{k-1}+su_{2k-1}+tu_{2k}\right).$$

Thus,

$$n = a_1 + \cdots + a_{h-s-t} + su_{2k-1} + tu_{2k} \tag{22}$$

for some s, t, satisfying $0 \le s, t,\ s + t \le h$ and for some $a_1, \ldots, a_{h-s-t} \in \mathcal{A}_{k-1}$.

For short we write $r_{s,t}(n)$ for the number of solutions of (22).

- If $n \in (h-s-t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}$ for some $(s,t) \notin \{(0,0),(h-1,1)\}$ then, due to Lemma 2.4, we have that $r_{\mathcal{A}_k,h}(n) = r_{s,t}(n)$.
  - For $|n| \le n_0$ we have that $r_{s,t}(n) = 0 \le f(n)$ (due to Lemma 2.5).
  - For $|n| > n_0$ we apply Lemma 2.6 in the first inequality below with $h' = h - s - t$ and $m = n - su_{2k-1} - tu_{2k}$,

$$r_{s,t}(n) = r_{\mathcal{A}_{k-1},h-s-t}(n - su_{2k-1} - tu_{2k}) \le g \le f(n)$$

- If $n \in (h-s-t)\mathcal{A}_{k-1} + su_{2k-1} + tu_{2k}$ for some $(s,t) \in \{(0,0),\ (h-1,1)\}$, then $r_{\mathcal{A}_k,h}(n) = r_{0,0}(n) + r_{h-1,1}(n)$. Notice that $r_{0,0}(n) = r_{\mathcal{A}_{k-1},h}(n)$ and that $r_{h-1,1}(n) = 1$ if $n = z_k$ and $r_{h-1,1}(n) = 0$ otherwise.
  - If $n \ne z_k$, then $r_{\mathcal{A}_k,h}(n) = r_{A_{k-1},h}(n) \le f(n)$ by the induction hypothesis.
  - If $n = z_k$, then $r_{\mathcal{A}_k,h}(n) = r_{\mathcal{A}_{k-1},h}(z_k) + r_{h-1,1}(z_k) \le (f(z_k)-1)+1 = f(n)$.

□

Propositions 2.3 and 2.4 proves that $r_{\mathcal{A},h} = f$. To finish the proof of Theorem 1.1 we have to prove that $\mathcal{A}(x) \gg \mathcal{B}(x\epsilon(x))$ for a suitable sequence $0 = l_0 < l_1 < \cdots$.

Recall that $0 = l_0 < l_1 < \cdots$ is a strictly increasing sequence. Let $\mathbb{R}_{\ge 0} = \{x \in \mathbb{R} : x \ge 0\}$. We extend this sequence to a strictly increasing function $l : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$. (For example, define $l(x) = l(k+1)(x-k) + l(k)(k+1-x)$ for $k \le x \le k+1$.)

We have

$$\mathcal{A}(x) \ge \mathcal{A}_0(x).$$

Thus, to find a lower bound for $\mathcal{A}(x)$ it suffices to find a lower bound for the density of $\mathcal{A}_0(x)$.

Lemma 2.7. $\mathcal{A}_0(x) \ge \mathcal{B}(x2^{-2(l^{-1}(\log_2 x)+2)r})$.

Proof. Let b be a positive integer such that

$$b \le x2^{-2(l^{-1}(\log_2 x)+2)r}.$$

Let k be such that $2^{l(k)} \le b < 2^{l(k+1)}$. In particular, $k \le l^{-1}(\log_2 b) \le l^{-1}(\log_2 x)$.

Thus,

$$\begin{aligned} t_r(b) = n_0m_0 + \cdots + n_km_k < m_{k+1} &= 2^{l(k+1)+2(k+2)r} \\ &< b2^{2(k+2)r} \\ &< b2^{2(l^{-1}(\log_2 x)+2)r} \le x. \end{aligned}$$

□

Recall that $\epsilon(x)$ is a decreasing positive function defined on $[1,\infty)$ such that $\lim_{x\to\infty} \epsilon(x) = 0$. Lemma 2.7 completes the proof of Theorem 1.1 by choosing a function $l(x)$ satisfying the inequality

$$2^{-2(l^{-1}(\log_2 x)+2)r} \ge \epsilon(x)$$

for x large enough.

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