A greedy algorithm for $B_h[g]$ sequences

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Abstract

For any positive integers $h \geq 2$ and $g \geq 1$, we present a greedy algorithm that provides an infinite $B_h[g]$ sequence with $a_n \leq 2gn^{h+(h-1)/g}$.

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1. Introduction

Given positive integers $h \geq 2$ and $g \geq 1$, we say that a sequence of integers $A$ is a $B_h[g]$ sequence if the number of representations of any integer $n$ in the form

$$n = a_1 + \cdots + a_h, \quad a_1 \leq \cdots \leq a_h, \quad a_i \in A$$

is bounded by $g$. The $B_h[1]$ sequences are simply called $B_h$ sequences.

A trivial counting argument shows that if $A = \{a_n\}$ is a $B_h[g]$ sequence then $a_n \gg n^h$. On the other hand, the greedy algorithm introduced by Erdős 1 provides an infinite $B_h$ sequence with $a_n \leq 2n^{2h-1}$.

Classic greedy algorithm: Let $a_1 = 1$ and for $n \geq 2$, define $a_n$ as the smallest positive integer, greater than $a_{n-1}$, such that $a_1, \ldots, a_n$ is a $B_h[g]$ sequence.

When $g = 1$, the greedy algorithm defines $a_1 = 1$, $a_2 = 2$ and for $n \geq 3$, defines $a_n$ as the smallest positive integer that is not of the form

$$\frac{1}{k} \left( a_{i_1} + \cdots + a_{i_h} - (a_{i_1'} \cdots + a_{i_{h-k}'}) \right)$$

1 This algorithm has been attributed to Mian and Chowla, but it seems (see [6]) that was Erdős who first used this algorithm.

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for any \(1 \leq i_1, \ldots, i_h, i'_1, \ldots, i'_{h-k} \leq n - 1\) and \(1 \leq k \leq h - 1\). Since there are at most \((n - 1)^{2h-1} + \cdots + (n - 1)^{h+1} \leq (n - 1)^{2h}/(n - 2)\) forbidden elements for \(a_n\), then \(a_n \leq 1 + (n - 1)^{2h}/(n - 2) \leq 2n^{2h-1}\).

It is possible that the classic greedy algorithm may provide a denser sequence when \(g > 1\), but it is not clear how to prove it. For this reason other methods have been used to obtain dense infinite \(B_h[g]\) sequences:

**Theorem A.** Given \(h \geq 2\) and \(g \geq 1\), there exists an infinite \(B_h[g]\) sequence with \(a_n \ll n^{h+\delta}\) with \(\delta = \delta_h(g) \to 0\) when \(g \to \infty\).

Erdős and Renyi [8] proved Theorem A for \(h = 2\) using the probabilistic method. Ruzsa gave the first proof for any \(h \geq 3\) (a sketch of that proof, which consists in an explicit construction, appeared in [7] and a detailed proof in [5]).

The aim of this paper is to describe a distinct greedy algorithm that provides a \(B_h[g]\) sequence that grows slower than all previous known constructions for \(g > 1\). More specifically, Theorem 2.1 gives an easy proof of Theorem A with \(\delta_h(g) = (h - 1)/g\).

In the table below we resume all previous results on this problem for \(g > 1\) expressed in form \(a_n \ll n^{h+\delta}\) and the method used in each case. The probabilistic method, which we denote by PM, has been used in most of the constructions.

<table>
<thead>
<tr>
<th>(\delta_2(g))</th>
<th>(\leq 2/g + o_n(1))</th>
<th>PM [8]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\delta_3(g))</td>
<td>(\leq 1/g + o_n(1))</td>
<td>PM + alteration method [2]</td>
</tr>
<tr>
<td>(\delta_3(g))</td>
<td>(\leq 2/g + \epsilon, \ \epsilon &gt; 0)</td>
<td>PM + combinatorial ingredients [5]</td>
</tr>
<tr>
<td>(\delta_h(g))</td>
<td>(\ll h/(\log g \log \log g))</td>
<td>Explicit construction, Ruzsa [7], [5]</td>
</tr>
<tr>
<td>(\delta_h(g))</td>
<td>(\ll h/g^{1/(h-1)})</td>
<td>PM + Kim-Vu method [9]</td>
</tr>
<tr>
<td>(\delta_h(g))</td>
<td>(\ll 2^h h! 2^g/g)</td>
<td>PM + Sunflower Lemma [5]</td>
</tr>
<tr>
<td>(\delta_h(g))</td>
<td>(\ll (h - 1)/g)</td>
<td>New greedy algorithm, Theorem 2.1</td>
</tr>
</tbody>
</table>

For \(g = 1\) there are special constructions of \(B_h\) sequences with slower growth.

<table>
<thead>
<tr>
<th>(\delta_h(1))</th>
<th>(\leq h - 1)</th>
<th>Classic greedy algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\delta_2(1))</td>
<td>(\leq 1 - \epsilon_n, \ \epsilon_n = \log \log n / \log n)</td>
<td>PM + graph tools [1]</td>
</tr>
<tr>
<td>(\delta_2(1))</td>
<td>(\leq \sqrt{2} - 1 + o_n(1))</td>
<td>Real log method + PM [10]</td>
</tr>
<tr>
<td>(\delta_2(1))</td>
<td>(\leq \sqrt{2} - 1 + o_n(1))</td>
<td>Explicit construction [3]</td>
</tr>
<tr>
<td>(\delta_h(1))</td>
<td>(\leq \sqrt{(h - 1)^2 + 1} - 1 + o_n(1), \ h = 3, 4)</td>
<td>Gaussian arg method + PM [4]</td>
</tr>
<tr>
<td>(\delta_h(1))</td>
<td>(\leq \sqrt{(h - 1)^2 + 1} - 1 + o_n(1), \ h \geq 3)</td>
<td>Discrete log method + PM [3]</td>
</tr>
</tbody>
</table>

2. A new greedy algorithm

We need to introduce the notion of strong \(B_h[g]\) set.

**Definition 1.** We say that \(A_n = \{a_1, \ldots, a_n\}\) is a strong \(B_h[g]\) set if the following conditions are satisfied:
i) $A_n$ is a $B_h[g]$ set.

ii) $|\{x : r_{A_n}(x) \geq s\}| \leq n^{h+(1-s)(h-1)/g}$, for $s = 1, \ldots, g$, where

$$r_{A_n}(x) = |\{(a_{i_1}, \ldots, a_{i_h}) : 1 \leq i_1 \leq \cdots \leq i_h \leq n, \ x = a_{i_1} + \cdots + a_{i_h}\}|.$$

**Theorem 2.1.** Let $a_1 = 1$ and for $n \geq 1$ define $a_{n+1}$ as the smallest positive integer, distinct to $a_1, \ldots, a_n$, such that $a_1, \ldots, a_{n+1}$ is a strong $B_h[g]$ set. The infinite sequence $A = \{a_n\}$ given by this greedy algorithm is a $B_h[g]$ sequence with $a_n \leq 2gn^h+(h-1)/g$.

**Proof.** Let $a_1 = 1$, $a_2 = 2$ and suppose that $A_n = \{a_1, \ldots, a_n\}$ is the strong $B_h[g]$ set given by this algorithm for some $n \geq 2$. We will find an upper bound for the number of forbidden positive integers for $a_{n+1}$. We use the notation $R_s(A_n) = \{|\{x : r_{A_n}(x) \geq s\}|$ to classify the forbidden elements $m$ in the following sets:

i) $F_n = \{m : m \in A_n\}$.

ii) $F_{0,n} = \{m : A_n \cup m$ is not a $B_h[g]$ set $\}$

iii) $F_{s,n} = \{m : R_s(A_n \cup m) > (n+1)^{h+(1-s)(h-1)/g}\}$, $s = 1, \ldots, g$.

Hence $a_{n+1}$ is the smallest positive integer not belonging to $(\bigcup_{s=0}^g F_{s,n}) \cup F_n$ and then the proof of Theorem 2.1 will be completed if we prove that

$$\left| \left( \bigcup_{s=0}^g F_{s,n} \right) \cup F_n \right| \leq 2g(n+1)^{h+(h-1)/g} - 1. \quad (2.1)$$

It is clear that $|F_n| = n$. Next, we find an upper bound for the cardinality of $F_{s,n}$, $s = 0, \ldots, g$.

The elements of $F_{0,n}$ are the positive integers of the form $\frac{1}{k}(x - (a_{i_1} + \cdots + a_{i_{h-k}}))$ for some $1 \leq i_1, \ldots, i_{h-k} \leq n$, $1 \leq k \leq h-1$ and for some $x$ with $r_{A_n}(x) = g$. Thus,

$$|F_{0,n}| \leq (n^{h-1} + \cdots + n + 1)|\{x : r_{A_n}(x) = g\}|$$

$$\leq n^h/(n-1) R_g(A_n)$$

$$\leq 2n^{h-1} n^{(h-1)/g} = 2n^{h+(h-1)/g}.$$

For $s = 1$, note that $R_1(A_n \cup m) \leq (n+1)^h$ for any $m$, so $|F_{1,n}| = 0$.

For $s = 2, \ldots, g$, and for any $m$ we have

$$R_s(A_n \cup m) \leq R_s(A_n) + T_{s,n}(m), \quad (2.2)$$

where

$$T_{s,n}(m) = |\{x : r_{A_n}(x) \geq s-1, x \in km + A_n + \frac{h-k}{h-k} \cdots + A_n$ for some $1 \leq k \leq h\}|.$$
In the case \( k = h \), the expression \( x \in km + A_n + \cdots + A_n \) means \( x = hm \).

We observe that if \( T_{s,n}(m) \leq n^{h-1+(1-s)(h-1)/g} \), using (2.2) and that \( A_n \) is a strong \( B_h[g] \) set, we have

\[
R_s(A_n \cup m) \leq n^{h+(1-s)(h-1)/g} + n^{h-1+(1-s)(h-1)/g} \leq (n+1)^{(h-1)(h-1)/g}
\]

and then \( m \notin F_{s,n} \). Thus,

\[
\sum_m T_{s,n}(m) \geq \sum_{m \in F_{s,n}} T_{s,n}(m) > n^{h-1+(1-s)(h-1)/g} |F_{s,n}|.
\] (2.3)

On the other hand, when we sum \( T_{s,n}(m) \) over all \( m \), each \( x \) with \( r_{A_n}(x) \geq s-1 \) is counted no more than \( |A_n + \cdots + A_n| + \cdots + |A_n| + 1 \leq n^{h-1} + \cdots + n + 1 \) times. Then

\[
\sum_m T_{s,n}(m) \leq (1 + n + \cdots + n^{h-1}) R_{s-1}(A_n) \leq \frac{n^h - 1}{n-1} n^{h+(2-s)(h-1)/g}.
\] (2.4)

Inequalities (2.3) and (2.4) imply

\[
|F_{s,n}| \leq n^{h-1} + (h-1)/g \leq 2n^{h+(h-1)/g}.
\] (2.5)

Taking into account (2.2), the inequalities (2.5) for \( s = 2, \ldots, g \) and the estimate \( |F_n| = n \), we get

\[
\left| \left( \bigcup_{s=0}^{g} F_{s,n} \right) \cup F_n \right| \leq 2n^{h+(h-1)/g} + 2(g-1)n^{h+(h-1)/g} + n = 2gn^{h+(h-1)/g} + n \leq 2g(n+1)^{h+(h-1)/g} - 1,
\]

which, according to (2.1), finishes the proof. \( \square \)


