A greedy algorithm for $B_h[g]$ sequences

Javier Cilleruelo

Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM) and
Departamento de Matemáticas
Universidad Autónoma de Madrid
28049 Madrid, Spain

Abstract

For any positive integers $h \geq 2$ and $g \geq 1$, we present a greedy algorithm that provides an infinite $B_h[g]$ sequence with $a_n \leq 2gn^{h+(h-1)/g}$.

Keywords: Sidon sets, greedy algorithm
2000 MSC: Primary 11B83.

1. Introduction

Given positive integers $h \geq 2$ and $g \geq 1$, we say that a sequence of integers A is a $B_h[g]$ sequence if the number of representations of any integer n in the form

$$n = a_1 + \cdots + a_h, \quad a_1 \leq \cdots \leq a_h, \quad a_i \in A$$

is bounded by g. The $B_h[1]$ sequences are simply called B_h sequences.

A trivial counting argument shows that if $A = \{a_n\}$ is a $B_h[g]$ sequence then $a_n \gg n^h$. On the other hand, the greedy algorithm introduced by Erdős [1] provides an infinite B_h sequence with $a_n \leq 2n^{2h-1}$.

Classic greedy algorithm: *Let $a_1 = 1$ and for $n \geq 2$, define a_n as the smallest positive integer, greater than a_{n-1}, such that $a_1, \dots, a_n$ is a $B_h[g]$ sequence.*

When $g = 1$, the greedy algorithm defines $a_1 = 1,\ a_2 = 2$ and for $n \geq 3$, defines a_n as the smallest positive integer that is not of the form

$$\frac{1}{k}\big(a_{i_1} + \cdots + a_{i_h} - (a_{i'_1} \cdots + a_{i'_{h-k}})\big)$$

Email address: `franciscojavier.cilleruelo@uam.es` (Javier Cilleruelo)
URL: `http://www.uam.es/personal_pdi/ciencias/cillerue/` (Javier Cilleruelo)

[1] This algorithm has been atributed to Mian and Chowla, but it seems (see [6]) that was Erdős who first used this algorithm.

for any $1 \le i_1, \dots, i_h, i'_1, \dots, i'_{h-k} \le n-1$ and $1 \le k \le h-1$. Since there are at most $(n-1)^{2h-1} + \cdots + (n-1)^{h+1} \le (n-1)^{2h}/(n-2)$ forbidden elements for a_n, then $a_n \le 1 + (n-1)^{2h}/(n-2) \le 2n^{2h-1}$.

It is possible that the classic greedy algorithm may provide a denser sequence when $g > 1$, but it is not clear how to prove it. For this reason other methods have been used to obtain dense infinite $B_h[g]$ sequences:

Theorem A. *Given $h \ge 2$ and $g \ge 1$, there exists an infinite $B_h[g]$ sequence with $a_n \ll n^{h+\delta}$ with $\delta = \delta_h(g) \to 0$ when $g \to \infty$.*

Erdős and Renyi [8] proved Theorem A for $h = 2$ using the probabilistic method. Ruzsa gave the first proof for any $h \ge 3$ (a sketch of that proof, which consists in an explicit construction, appeared in [7] and a detailed proof in [5]).

The aim of this paper is to describe a distinct greedy algorithm that provides a $B_h[g]$ sequence that grows slower than all previous known constructions for $g > 1$. More specifically, Theorem 2.1 gives an easy proof of Theorem A with $\delta_h(g) = (h-1)/g$.

In the table below we resume all previous results on this problem for $g > 1$ expressed in form $a_n \ll n^{h+\delta_h(g)}$ and the method used in each case. The probabilistic method, which we denote by PM, has been used in most of the constructions.

$\delta_2(g)$	$\le 2/g + o_n(1)$	PM [8]
$\delta_2(g)$	$\le 1/g + o_n(1)$	PM + alteration method [2]
$\delta_3(g)$	$\le 2/g + \epsilon, \quad \epsilon > 0$	PM+ combinatorial ingredients [5]
$\delta_h(g)$	$\ll_h 1/(\log g \log\log g)$	Explicit construction, Ruzsa [7],[5]
$\delta_h(g)$	$\ll_h 1/g^{1/(h-1)}$	PM+ Kim-Vu method [9]
$\delta_h(g)$	$\ll 2^h h(h!)^2/g$	PM + Sunflower Lemma [5]
$\delta_h(g)$	$\le (h-1)/g$	New greedy algorithm, Theorem 2.1

For $g = 1$ there are special constructions of B_h sequences with slower growth.

$\delta_h(1)$	$\le h-1$	Classic greedy algorithm
$\delta_2(1)$	$\le 1-\epsilon_n, \quad \epsilon_n = \log\log n/\log n$	PM + graph tools [1]
$\delta_2(1)$	$\le \sqrt{2} - 1 + o_n(1)$	Real log method + PM [10]
$\delta_2(1)$	$\le \sqrt{2} - 1 + o_n(1)$	Explicit construction [3]
$\delta_h(1)$	$\le \sqrt{(h-1)^2+1} - 1 + o_n(1),\ h = 3, 4$	Gaussian arg method + PM [4]
$\delta_h(1)$	$\le \sqrt{(h-1)^2+1} - 1 + o_n(1),\ h \ge 3$	Discrete log method + PM [3]

2. A new greedy algorithm

We need to introduce the notion of *strong* $B_h[g]$ set.

Definition 1. *We say that $A_n = \{a_1, \dots, a_n\}$ is a strong $B_h[g]$ set if the following conditions are satisfied:*

i) A_n *is a* $B_h[g]$ *set.*

ii) $|\{x : r_{A_n}(x) \geq s\}| \leq n^{h+(1-s)(h-1)/g}$, *for* $s = 1, \dots, g$, *where*

$$r_{A_n}(x) = |\{(a_{i_1}, \dots a_{i_h}) : \quad 1 \leq i_1 \leq \dots \leq i_h \leq n, \quad x = a_{i_1} + \dots + a_{i_h}\}|.$$

Theorem 2.1. *Let* $a_1 = 1$ *and for* $n \geq 1$ *define* a_{n+1} *as the smallest positive integer, distinct to* $a_1, \dots, a_n$*, such that* $a_1, \dots, a_{n+1}$ *is a strong* $B_h[g]$ *set. The infinite sequence* $A = \{a_n\}$ *given by this greedy algorithm is a* $B_h[g]$ *sequence with* $a_n \leq 2gn^{h+(h-1)/g}$.

Proof. Let $a_1 = 1$, $a_2 = 2$ and suppose that $A_n = \{a_1, \dots, a_n\}$ is the strong $B_h[g]$ set given by this algorithm for some $n \geq 2$. We will find an upper bound for the number of forbiden positive integers for a_{n+1}. We use the notation $R_s(A_n) = |\{x : r_{A_n}(x) \geq s\}|$ to classify the forbidden elements m in the following sets:

i) $F_n = \{m : \ m \in A_n\}$.

ii) $F_{0,n} = \{m : \ A_n \cup m$ is not a $B_h[g]$ set$\}$

iii) $F_{s,n} = \{m : \ R_s(A_n \cup m) > (n+1)^{h+(1-s)(h-1)/g}\}, \quad s = 1, \dots, g.$

Hence a_{n+1} is the smallest positive integer not belonging to $(\bigcup_{s=0}^{g} F_{s,n}) \cup F_n$ and then the proof of Theorem 2.1 will be completed if we prove that

$$\left| \left(\bigcup_{s=0}^{g} F_{s,n} \right) \cup F_n \right| \leq 2g(n+1)^{h+(h-1)/g} - 1. \tag{2.1}$$

It is clear that $|F_n| = n$. Next, we find an upper bound for the cardinality of $F_{s,n}, \ s = 0, \dots, g$.

The elements of $F_{0,n}$ are the positive integers of the form $\frac{1}{k}\left(x - (a_{i_1} + \dots + a_{i_{h-k}})\right)$ for some $1 \leq i_1, \dots, i_{h-k} \leq n, \ 1 \leq k \leq h-1$ and for some x with $r_{A_n}(x) = g$. Thus,

$$\begin{aligned} |F_{0,n}| &\leq (n^{h-1} + \dots + n + 1)|\{x : \ r_{A_n}(x) = g\}| \\ &\leq n^h/(n-1) \ R_g(A_n) \\ &\leq 2n^{h-1}n^{1+(h-1)/g} = 2n^{h+(h-1)/g}. \end{aligned}$$

For $s = 1$, note that $R_1(A_n \cup m) \leq (n+1)^h$ for any m, so $|F_{1,n}| = 0$.

For $s = 2, \dots, g$, and for any m we have

$$R_s(A_n \cup m) \leq R_s(A_n) + T_{s,n}(m), \tag{2.2}$$

where

$$T_{s,n}(m) = |\left\{x : \ r_{A_n}(x) \geq s-1, \ x \in km + A_n + \overset{h-k}{\cdots} + A_n \text{ for some } 1 \leq k \leq h\right\}|.$$

In the case $k = h$, the expression $x \in km + A_n + \overset{h-k}{\cdots} + A_n$ means $x = hm$.

We observe that if $T_{s,n}(m) \le n^{h-1+(1-s)(h-1)/g}$, using (2.2) and that A_n is a strong $B_h[g]$ set, we have

$$\begin{aligned} R_s(A_n \cup m) &\le n^{h+(1-s)(h-1)/g} + n^{h-1+(1-s)(h-1)/g} \\ &\le (n+1)^{h+(1-s)(h-1)/g} \end{aligned}$$

and then $m \notin F_{s,n}$. Thus,

$$\sum_m T_{s,n}(m) \quad \ge \sum_{m \in F_{s,n}} T_{s,n}(m) > n^{h-1+(1-s)(h-1)/g} |F_{s,n}|. \qquad (2.3)$$

On the other hand, when we sum $T_{s,n}(m)$ over all m, each x with $r_{A_n}(x) \ge s-1$ is counted no more than $|A_n + \overset{h-1}{\cdots} + A_n| + \cdots + |A_n| + 1 \le n^{h-1} + \cdots + n + 1$ times. Then

$$\begin{aligned} \sum_m T_{s,n}(m) &\le (1 + n + \cdots + n^{h-1}) R_{s-1}(A_n) \qquad (2.4) \\ &\le \frac{n^h - 1}{n-1} n^{h+(2-s)(h-1)/g}. \end{aligned}$$

Inequalities (2.3) and (2.4) imply

$$|F_{s,n}| \le \frac{n^h - 1}{n-1} n^{1+(h-1)/g} \le 2n^{h+(h-1)/g}. \qquad (2.5)$$

Taking into account (2.2), the inequalities (2.5) for $s = 2, \dots, g$ and the estimate $|F_n| = n$, we get

$$\begin{aligned} \left| \left(\bigcup_{s=0}^{g} F_{s,n} \right) \cup F_n \right| &\le 2n^{h+(h-1)/g} + 2(g-1)n^{h+(h-1)/g} + n \\ &= 2gn^{h+(h-1)/g} + n \le 2g(n+1)^{h+(h-1)/g} - 1, \end{aligned}$$

which, according to (2.1), finishes the proof. □

[1] M. Ajtai, J. Komlós, and E. Szemerédi, *A dense infinite Sidon sequence*, European J. Combin. 2 (1981), 1–11.

[2] J. Cilleruelo, *Probabilistic constructions of $B_2[g]$ sequences*, Acta Mathematica Sinica 26 (2010), no. 7, 1309–1314.

[3] J. Cilleruelo, *Infinite Sidon sequences*, Advances in Mathematics 255 (2014), 474–486.

[4] J. Cilleruelo and R. Tesoro, *Dense infinite B_h sequences*, Publicacions Matematiques, vol 59, n1 (2015).

[5] J. Cilleruelo, S. Kiss, I. Ruzsa and C. Vinuesa, *Generalization of a theorem of Erdos and Renyi on Sidon sets*, Random Structures and Algorithms, vol 37, n4 (2010)

[6] P. Erdős, *Solved and unsolved problems in combinatorics and combinatorial number theory* Congressus Numeratium, Vol . 32 (1981), pp . 49–62.

[7] P. Erdős and R. Freud, *On Sidon sequences and related problems*, Mat. Lapok 1 (1991), 1–44.

[8] P. Erdős and A.Renyi, *Additive properties of random sequences of positive integers.* Acta Arithmetica. 6 (1960) 83–110.

[9] J.H. Kim and V. Vu, *Concentration of multivariante Polynomials and its applications* Combinatorica 20 (3) (2000) 417–434.

[10] I. Ruzsa, *An infinite Sidon sequence.* J. Number Theory 68 (1998), no. 1, 63–71.