INFINITE C_4-FREE GRAPHS.

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Abstract. We construct, for any $g \geq 1$, an infinite countable $K_{2,g+1}$-free graph $\mathcal{G}$ having the property that the graphs $\mathcal{G}_n$ induced by the first n vertices have minimum degree $\delta(\mathcal{G}_n) \gg n^{1/2-\epsilon_g}$, where $\epsilon_g = 1/(4g+2)$. Using a more involved argument we construct an infinite $K_{2,2}$-free graph (C_4-free graph) with $\delta(\mathcal{G}_n) \gg n^{\sqrt{2}-1+o(1)}$.

1. Introduction

Given an infinite countable graph $\mathcal{G}(V, \mathcal{E})$, denote by $\mathcal{G}_n(V_n, \mathcal{E}_n)$ the graph spanned by the vertices $V_n = \{1, \ldots, n\}$. Extremal problems on infinite countable graphs have been studied before [7, 8, 10]. We study here an extremal problem on infinite graphs without cycles of length four.

The minimum degree of finite graph $\mathcal{G}(V, \mathcal{E})$ is defined by $\delta(\mathcal{G}) = \min\{\deg v : v \in V\}$. It is not difficult to prove (see Proposition 2.1) that for any n there exists a C_4-free graph $\mathcal{G}$ of order n with $\delta(\mathcal{G}) = (1 + o(1))n^{1/2}$. The analogous result for infinite graphs is more difficult to prove. Indeed, it is not clear if there is an infinite countable C_4-free graph $\mathcal{G}$ with $\delta(\mathcal{G}_n) \gg n^{1/2}$. The starting point in our work are two conjectures concerning this question which, as we will see later, are closely related to some results and conjectures of Erdős on infinite Sidon sequences.

Conjecture 1. *If $\mathcal{G}$ is an infinite countable C_4-free graph, then*

$$\liminf_{n\to\infty} \delta(\mathcal{G}_n)/\sqrt{n} = 0$$

for the sequence of the graphs $\mathcal{G}_n$ spanned by the first n vertices.

The second one goes in the opposite direction.

Conjecture 2. *For any $\epsilon > 0$ there exists an infinite countable C_4-free graph $\mathcal{G}$ with*

$$\delta(\mathcal{G}_n) \gg n^{1/2-\epsilon}.$$

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Our main results can be considered an approach to Conjecture 2.

Theorem 1.1. *For any $g \geq 1$ there is an infinite countable $K_{2,g+1}$-free graph $\mathcal{G}$ with*

$$\delta(\mathcal{G}_n) \gg n^{1/2-\epsilon_g},$$

where $\epsilon_g = 1/(4g+2)$.

We can improve Theorem 1.1 when $g = 1$.

Theorem 1.2. *There is an infinite countable C_4-free graph $\mathcal{G}(V, \mathcal{E})$ with*

$$\delta(\mathcal{G}_n) \gg n^{\sqrt{2}-1+o(1)}.$$

The main obstruction to get an infinite C_4-free graph with $\delta(\mathcal{G}_n) \gg n^{1/2}$ is that the constructions of dense finite C_4-free graphs come from algebraic constructions and theses constructions are too rigid to be extended to an infinite C_4-free graph. A similar problem appears in the analogous problem for Sidon sequences, those sequences of integers having the property that all the differences of two elements of the sequence are distinct. While algebraic constructions provide Sidon sets $A \subset [1, n]$ with $|A| = (1 + o(1))\sqrt{n}$, it is not possible to extend them to get an infinite Sidon sequence with $A(n) \gg \sqrt{n}$, where we denote by $A(n) = |A \cap [1, n]|$ the counting function of the sequence A. Indeed, Erdős [14] proved that

$$\liminf_{n\to\infty} A(n)/\sqrt{n} = 0 \tag{1.1}$$

for any infinite Sidon sequence. Conjecture 1 can be considered as the analogous of (1.1) for infinite C_4-free graphs.

The core of this work is, precisely, the conexion between infinite C_4-free graphs and infinite Sidon sequences. More generally, between infinite $K_{2,g+1}$-free graphs and infinite $B_2^-[g]$ sequences of positive integers, those sequences A such that $d_A(x) \leq g$ for any $x \neq 0$, where

$$d_A(x) \quad = \quad |\{x = a - b : \ a, b \in A\}|.$$

The following theorem shows this conexion.

Theorem 1.3. *Let $g \geq 1$ and A an infinite $B_2^-[g]$ sequence. Then there exists an infinite countable $K_{2,g+1}$-free graph $\mathcal{G}$, with*

$$\delta(\mathcal{G}_n) \geq \min_{x \leq n} A(n + x) - A(x) - 1$$

where $\mathcal{G}_n$ is the graph spanned by the first n vertices.

In [6] we used the probabilistic method to prove the existence of an infinite $B_2^-[g]$ sequence with $A(n) \gg n^{1/2-1/(4g+2)+o(1)}$. The proof was involved and, in addition, that sequences are not suitable to be used in Theorem 1.3. The reason is that we have not control on the lower bound of the counting function of that sequences.

Here we adapt the ideas introduced in [5] to construct, using a greedy algorithm, denser $B_2^-[g]$ sequences that can be used in Theorem 1.3 in an easy way.

Theorem 1.4. *For any positive integer g, there is an infinite $B_2^-[g]$ sequence $A = \{a_n\}$ with*

$$6gn^{2+1/g} \leq a_n \leq 8gn^{2+1/g}.$$

Sidon sequences, which corresponds to the $B_2^-[1]$ sequences, were introduced by Erdős in the thirties. A major problem on infinite Sidon sequence is the construction of Sidon sequences A with counting function $A(n)$ as large as possible. The trivial counting argument shows that $A(n) \ll \sqrt{n}$ but, as we have mentioned above, Erdős also proved that (1.1) holds for any infinite Sidon sequence. In the opposite direction Erdős conjectured that for any $\epsilon > 0$ there is an infinite Sidon sequence with $A(n) \gg n^{1/2-\epsilon}$. Conjecture 2 can be seen as the analogous conjecture in graphs.

The greedy algorithm provided an infinite Sidon sequence with $A(n) \gg n^{1/3}$ and, almost fifty years later, Atjai, Komlos and Szemeredi [1] proved the existence of an infinite Sidon sequence with $A(n) \gg (n \log n)^{1/3}$. Ruzsa [12] proved the existence of a Sidon sequence with $A(n) = n^{\sqrt{2}-1+o(1)}$ and an explicit construction with similar growing was given in [4].

The construction that proves Theorem 1.2 is closely related to the special construction described in [4]. Unfortunately we cannot apply directly Theorem 1.3 because that sequence is quite irregular and we have not control enough on the lower bound of the counting function of the sequence. What we do is to construct the graph directly, but using the ideas behind the construction of that Sidon sequence. The construction, which is quite involved, depends of the fact that the sequence of the prime numbers is a multiplicative Sidon sequence. Some tools of analitic number theory will be used in the construction.

2. A few remarks

1. The usual extremal problems on graphs concern to the maximum number of edges of a graph not containing a given subgraph (for example, the subgraph C_4). A clasic result of Kovari, Sos and Turan [9] says that the maximum number of edges of a C_4-free graph of order n es bounded by $n^{3/2}(1/2 + o(1))$, and then that the minimum degree is bounded by $n^{1/2}(1 + o(1))$.

An easy way to construct an infinite countale C_4-free graph $\mathcal{G}(V, \mathcal{E})$ with $|\mathcal{E}_n| \asymp n^{3/2}$ was communicated to us by Simonovits [13]. Consider the graph $\mathcal{G}(V, \mathcal{E})$ that is the infinite union of independent graphs $\mathcal{G}(k)$, where $\mathcal{G}(k)$ is an extremal C_4-free graph with 2^k vertices. Obviously the graph $\mathcal{G}$ is C_4-free. If $2^{k+1} \leq n < 2^{k+2}$ then $n \in \mathcal{G}(k + 1)$, so

$$|\mathcal{E}_n| \geq |\mathcal{E}(\mathcal{G}(k))| \gg 2^{\frac{3k}{2}} \gg n^{3/2}.$$

It should be noted that, however, the graph above does not satisfies that $\delta_n(\mathcal{G}_n) \gg n^{1/2}$ (indeed $\delta(\mathcal{G}_n) \ll 1$). This is the reason for which extremal problems on infinite C_4-free graphs are more difficult and interesting for the quantity $\delta(\mathcal{G}_n)$ than for $|\mathcal{E}_n|$.

Peng and Timmons [10] have constructed an infinite C_4-free graph with $|\mathcal{E}_n| \geq 0.23n^{3/2}(1+o(1))$ and have proved that $\liminf_{n\to\infty} |\mathcal{E}_n|n^{-3/2} \leq 0.41$ for any infinite C_4-free graph, improving the trivial upper bound $1/2$ coming from the finite cases. Indeed, it implies that $\liminf_{n\to\infty} \delta(\mathcal{G}_n)/\sqrt{n} \leq 0.82$, which is an improving on the trivial upper bound, but is far from Conjecture 1.

2. It is a well known fact that Sidon sets can be used to construct C_4-free graphs. If $A \subset G$ is a Sidon set, the graph $\mathcal{G}(V,\mathcal{E})$ with $V = G$ and

$$\mathcal{E} = \big\{\{x,y\} : \ x \neq y, \ x+y \in A\big\}$$

is a C_4-free graph with minimum degree $\delta(\mathcal{G}) \geq |A|-1$. To see this we observe that if (x,y,u,v) is a C_4 then $x+y=a_1, \ y+u=a_2, \ u+v=a_3, \ v+x=a_4$ for some $a_1,a_2,a_3,a_4 \in A$. Since $(x+y)+(u+v) = (y+u)+(v+x)$ then $a_1+a_3 = a_2+a_4$. Thus $a_1=a_2$ or $a_1=a_4$ and then $x=u$ or $y=v$. On the other hand it is clear that $\deg(x) = |A|-1$ if $2x \in A$ and $\deg(x) = |A|$ otherwise.

3. There are several constructions of C_4-free graphs G of order n with $\delta(G) = (1+o(1))n^{1/2}$ for special sequences of values of n. We have not found in the literature a proof that works for all n but the following probabilistic construction was communicated to us by Alon [2]. Just choose a prime p so that p^2+p+1 is at least n and at most $n+o(n)$ and take the induced subgraph on a random set of n vertices in the usual example on p^2+p+1 vertices (polarity graph of projective plane): an easy probabilistic argument shows that with high probability all degrees will stay $(1+o(1))\sqrt{n}$.

We present also an explicit construction.

Proposition 2.1. *Let θ be a real number having the property that for any x large enough, the interval $[x, x+x^{\theta}]$ contains a prime number. Then*

$$n^{1/2} + O(n^{\theta/2}) \leq \delta(n;C_4) \leq n^{1/2} + 1/2,$$

where $\delta(n,C_4) = \max\{\delta(\mathcal{G}) : \ \mathcal{G}$ has order n and does not contain any $C_4\}$. It is known that we can take $\theta = 0.525$.

Proof. It is known [9] that if $\mathcal{G}(V,\mathcal{E})$ is C_4-free of order n then $|\mathcal{E}| \leq \frac{1}{2}n^{3/2} + \frac{n}{4}$. Thus, $\delta(n;C_4) \leq \delta(\mathcal{G}) \leq \frac{2}{n}\mathrm{ex}(n,C_4) \leq n^{1/2} + 1/2$.

For the lower bound, assume that n is large enough and let p be a prime $p \in [\sqrt{n}, \sqrt{n}+n^{\theta/2}]$. We consider the Sidon set $A = \{(x,x^2) : \ x \in \mathbb{F}_p\} \subset \mathbb{F}_p^2$ and the C_4-free graph $\mathcal{G}(V;\mathcal{E})$ induced by A with $V = \mathbb{F}_p^2$ and

$$\mathcal{E} = \big\{\{(x_1,y_1),(x_2,y_2)\} : \ (x_1,x_2)+(y_1,y_2) \in A, \ (x_1,y_1) \neq (x_2,y_2)\big\}.$$

Define r, s by $p^2 - n = rp + s,\ 0 \le s \le p-1,\ 0 \le r$ and remove the set of vertices

$$V^* = \{(a,b) : 0 \le a \le r-1,\ 0 \le b \le p-1\} \cup \{(r,b) :\ 0 \le b \le s-1\}.$$

Let $\mathcal{G}_0(V_0, \mathcal{E}_0)$ be the graph induced by the vertices $V_0 = V \setminus V^*$. First we observe that $|V_0| = |V| - |V^*| = p^2 - (rp+s) = n$. Consider a vertex $(x,y) \in V_0$. It is easy to check that $\deg_{\mathcal{G}}(x,y) = \begin{cases} |A| - 1 & \text{if } y = 2x^2 \\ |A| & \text{otherwise.} \end{cases}$ Thus

$$\begin{aligned} \deg_{\mathcal{G}_0}(x,y) &= \deg_{\mathcal{G}}(x,y) - |\{(u,v) \in V^* :\ (x,y) + (u,v) \in A\}| \\ &\ge |A| - 1 - |\{(u,v) :\ 0 \le u \le r,\ y + v = (x+u)^2\}| \\ &\ge p - 1 - (r+1) \ge p - 1 - (p - n/p + 1) = n/p - 2 \\ &\ge \frac{n}{\sqrt{n} + n^{\theta/2}} - 2 = \sqrt{n} + O\left(n^{\theta/2}\right). \end{aligned}$$

□

4. The following construction can be considered the finite version of the infinite C_4-free graph in Theorem 1.2.

Theorem 2.1. *Let $\mathbb{F}_q$ be the finite field of q elements. The graph $\mathcal{G}(V, \mathcal{E})$ where $V = \mathbb{F}_q^*$ and $\mathcal{E} = \{\{x,y\} :\ xy \equiv p \pmod q$ for some prime $p \le \sqrt{q}\}$ is a C_4-free graph with $\delta(\mathcal{G}) \sim \frac{\sqrt{q}}{\log \sqrt{q}}$.*

Proof. First we prove that $\mathcal{G}$ is C_4-free. If (x,y,u,v) is a C_4 in $\mathcal{G}$ then there exist primes $p_1, p_2, p_3, p_4 \le \sqrt{q}$ such that $xy \equiv p_1,\ yu \equiv p_2,\ uv \equiv p_3,\ vx \equiv p_4 \pmod q$. It implies that $p_1p_3 \equiv p_2p_4 \pmod q$. Since $1 < p_1p_3,\ p_2p_4 \le q$ we have the equality $p_1p_3 = p_2p_4$ and then $p_1 = p_2$ or $p_1 = p_4$, so $x = u$ or $y = v$.

For the degree condition it is clear that $\delta(\mathcal{G}) \ge \pi(\sqrt{q}) - 1 \sim \frac{\sqrt{q}}{\log \sqrt{q}}$. □

The construction above uses an algebraic part (the finite field $\mathbb{F}_q$) and a non algebraic part, the sequence of the prime numbers, which is common for any q. This construction is not so good as other algebraic constructions with $\delta(\mathcal{G}) \ge \sqrt{q}(1+o(1))$. We loose a logarithm factor but we gain the possibility of combining these constructions for distinct $\mathbb{F}_q$. Roughly, the strategy to construct the graph $\mathcal{G}$ in Theorem 1.2 is to paste the graphs described in Theorem 2.1 for infinite $\mathbb{F}_{q_i}$ using the Chinese remainder theorem. The construction of the graph in Theorem 2.1 for distinct $\mathbb{F}_{q_i}$ and how to paste them are some of the key ideas in the construction of the graph $\mathcal{G}$ in Theorem 1.2.

5. We finish this section with some classic and well known estimates on the distribution of primes that we will use in the proof of Theorem 1.2.

Proposition 2.2. *Let $\pi(x)$ the number of prime numbers less or equal than x and for any positive integers m, a with $(m,a) = 1$ let $\pi(x; m, a)$ denotes the number* *of primes* $p \le x$ *such that* $p \equiv a \pmod q$*. Let also* q_k *the* k*-th odd prime and* $Q_k = \prod_{j=1}^k q_j$*. We have*

i) $\pi(x) \sim \frac{x}{\log x}$
ii) $\pi(x; m, a) \sim \frac{x}{\phi(m)\log x}$ *uniformly for any* $m \ll \log x$*.*
iii) $q_k \sim k \log k$
iv) $\log Q_k \sim k \log k$*.*

3. $B_2^-[g]$ sequences and infinite $K_{2,g+1}$-free graphs.

Theorem 3.1. *Given* $g \ge 1$ *and an infinite* $B_2^-[g]$ *sequence* A*, there exits an infinite* $K_{2,g+1}$*-free graph with*

$$\delta(\mathcal{G}_n) \ge \min_{x \le n} A(n+x) - A(x) - 1.$$

Proof. Consider the graph $\mathcal{G}(V, \mathcal{E})$ where $V = \mathbb{N}$ and

$$\mathcal{E} = \big\{\{i,j\} : \ i \ne j, \ i + j \in A\big\}.$$

We claim that this graph is free of $K_{2,g+1}$. Suppose the contrary and consider a $K_{2,g+1}$ subgraph of $\mathcal{G}$ formed by a bipartite graph of a set V_1 of 2 elements, say a, b and a set V_2 of $g+1$ elements, say $c_1, \dots, c_{g+1}$. Then the integer $a - b$ has at least $g+1$ representations as a difference of two elements of A:

$$a - b = (a + c_1) - (b + c_1) = \cdots = (a + c_{g+1}) - (b + c_{g+1}),$$

which is not allowed because A is a $B_2^-[g]$ sequence.

Concerning to the other condition of the Theorem, let $j, \ 1 \le j \le n$ a vertex of $\mathcal{G}_n$. The neighbours of j are all the integers $i \ne j, \ 1 \le i \le n$ such that $i + j \in A$.

$$\begin{aligned} \deg j &= |\{i : \ 1 \le i \le n, \ i \ne j, \ i \in A - j\}| \\ &= |\{a \in A : \ j < a \le n + j, \ a \ne 2j\}| \ge A(n+j) - A(j) - 1. \end{aligned}$$

□

In view of Theorem 3.1, it is interesting to get $B_2^-[g]$ sequences with counting function as large as possible, but with some control on the lower bound. We adapt the method used in [5] to construct a $B_2^-[g]$ sequence with $A(x) \ll x^{1/2 - 1/(4g-2)}$.

Firstly we have to introduce the notion of strong $B_2^-[g]$ sequence. We say that a set $A_n = \{a_1, \dots, a_n\}$ is a strong $B_2^-[g]$ set if:

i) $6gn^{2+1/g} \le a_n \le 8gn^{2+1/g}$.
ii) $d_{A_n}(x) \le g$ for all integer $x \ne 0$.
iii) $|\{x \ne 0 : \ d_{A_n}(x) \ge s\}| \le 2n^{2-(s-1)/g}, \ s = 1, \dots, g$.

For short we use the notation $D_s(A_n) = |\{x \ne 0 : \ d_{A_n}(x) \ge s\}|$.

Theorem 3.2. *Let $a_1 = 6g$ and for $n \geq 1$ define a_{n+1} as the smallest positive integer, distinct to $a_1, \dots, a_n$, such that $a_1, \dots, a_{n+1}$ is a strong $B_2^-[g]$ set. The infinite sequence $A = \{a_n\}$ given by the greedy algorithm is a $B_2^-[g]$ sequence with $6gn^{2+1/g} \leq a_n \leq 8gn^{2+1/g}$.*

Proof. Let $a_1 = 6g$ and for some $n \geq 2$, let $A_n = \{a_1, \dots, a_n\}$ the strong $B_2^-[g]$ set given by this greedy algorithm. We will find an upper bound for the number of forbiden positive integers for a_{n+1}. We classify the forbidden elements m in the following sets:

i) $F_n = \{m : \ m \in A_n\}$.
ii) $F_{0,n} = \{m : \ A_n \cup m \text{ is not a } B_2^-[g] \text{ set}\}$
iii) $F_{s,n} = \{m : \ D_s(A_n \cup m) > (n+1)^{2-(s-1)/g}\}, \quad s = 1, \dots, g.$

Hence a_{n+1} is the smallest positive integer in the interval

$$I_{n+1} = [6g(n+1)^{2+1/g}, 8g(n+1)^{2+1/g]}]$$

not belonging to $(\bigcup_{s=0}^{g} F_{s,n}) \cup F_n$ and then the proof of Theorem 1.1 will be completed if we prove that

$$\left| \left(\bigcup_{s=0}^{g} F_{s,n} \right) \cup F_n \right| \leq 2g(n+1)^{2+1/g} - 2 \tag{3.1}$$

because at least one integer in I_{n+1} is not forbidden.

It is clear that $|F_n| = n$. Next, we find an upper bound for the cardinality of $F_{s,n}, \ s = 0, \dots, g$.

The elements of $F_{0,n}$ are the positive integers of the form $a_i + x$ or $a_i - x$ for some $1 \leq i \leq n$ and for some $x \neq 0$ with $d_{A_n}(x) = g$. Thus,

$$|F_{0,n}| \ \leq \ 2n|\{x : \ d_{A_n}(x) = g\}| = 2nD_g(A_n) \leq 2n \cdot n^{1+1/g} = 2n^{2+1/g}.$$

For $s = 1$, note that $D_1(A_n \cup m) \leq 2(n+1)^2$ for any m, so $|F_{1,n}| = 0$.

For $s = 2, \dots, g$, and for any m we have

$$D_s(A_n \cup m) \leq D_s(A_n) + T_{s,n}^1(m) + T_{s,n}^2(m), \tag{3.2}$$

where

$$\begin{aligned} T_{s,n}^1(m) &= |\{x : \ d_{A_n}(x) \geq s-1, \ x \in m - A_n\}| \\ T_{s,n}^2(m) &= |\{x : \ d_{A_n}(x) \geq s-1, \ x \in A_n - m\}| \end{aligned}$$

We observe that if $T_{s,n}^1(m) + T_{s,n}^2(m) \leq 2n^{1-(s-1)/g}$, using (3.2) and that A_n is a strong $B_2^-[g]$ set, we have

$$D_s(A_n \cup m) \ \leq \ 2n^{2-(s-1)/g} + 2n^{1-(s-1)/g} \leq 2(n+1)^{2-(s-1)/g}$$

and then $m \notin F_{s,n}$. Thus,

$$\begin{aligned} \sum_m T^1_{s,n}(m) + T^2_{s,n}(m) &\geq \sum_{m\in F_{s,n}} \left(T^1_{s,n}(m) + T^2_{s,n}(m)\right) \\ &> 2n^{1-(s-1)/g}|F_{s,n}|. \end{aligned} \tag{3.3}$$

On the other hand, when we sum $T^1_{s,n}(m)$ over all m (the same happens with $T^2_{sm}(m)$), each x with $d_{A_n}(x) \geq s-1$ is counted n times. Then

$$\sum_m \left(T^1_{s,n}(m) + T^2_{s,n}(m)\right) \leq 2nD_{s-1}(A_n) \leq n \cdot 4n^{2+(2-s)/g}. \tag{3.4}$$

Inequalities (3.3) and (3.4) imply

$$|F_{s,n}| \leq 2n^{2+1/g}. \tag{3.5}$$

Taking into account (3.2), the inequalities (3.5) for $s = 2, \dots, g$ and the estimate $|F_n| = n$, we get

$$\begin{aligned} \left| \left(\bigcup_{s=0}^{g} F_{s,n} \right) \cup F_n \right| &\leq 2n^{2+1/g} + 2(g-1)n^{2+1/g} + n \\ &= 2gn^{2+1/g} + n \leq 2g(n+1)^{2+1/g} - 2, \end{aligned}$$

which, according to (3.1), finishes the proof. □

Proof of Theorem 1.1. We use Theorem 1.3 and the sequences of Theorem 3.1. Firstly, we need a lower and an upper bound for the counting function of those sequences.

Given x, define n such that $8gn^{2+1/g} \leq x < 8g(n+1)^{2+1/g}$. It implies that $A(x) \geq n > (x/(8g))^{1/(2+1/g)} - 1$. Define also m such that $6gm^{2+1/g} \leq x < 6g(m+1)^{2+1/g}$. Thus, $A(x) \leq m \leq (x/(6g))^{1/(2+1/g)}$.

Theorem 1.3 implies that $\delta(\mathcal{G}_n) = A(n+x_0) - A(x_0) - 1$ for some $x_0 \leq n$.

If $x_0 \leq n/2$ we have

$$\begin{aligned} \delta(\mathcal{G}_n) &\geq A(n) - A(n/2) - 1 \\ &\geq (n/(8g))^{1/(2+1/g)} - 1 - (n/(12g))^{1/(2+1/g)} - 1 \\ &\gg n^{1/(2+1/g)}. \end{aligned}$$

If $n/2 < x_0 \leq n$ we have

$$\begin{aligned} \delta(\mathcal{G}_n) &\geq A(3n/2) - A(n) - 1 \\ &\geq (3n/(16g))^{1/(2+1/g)} - 1 - (n/(6g))^{1/(2+1/g)} - 1 \\ &\gg n^{1/(2+1/g)}. \end{aligned}$$

In any case we have that $\delta(\mathcal{G}_n) \gg n^{1/(2+1/g)} = n^{1/2-1/(4g+2)}$. □

We finish this section with a remarkable Sidon sequence found by Pollington and Vanden [11] that we will use later. In the original Theorem, the condition of being multiple of 4 is missing, but it is clear that we can assume that taking C equal to four times the original constant.

Theorem 3.3 (Pollignton-Vanden [11]). *For some $C > 0$ there is an infinite Sidon sequence A whose terms are all multiple of 4, with one element in each interval $[C(m-1)^3, Cm^3),\ m \geq 1$.*

4. A dense infinite C_4-free graph

Let (q_k) the sequence formed by the odd prime numbers and $Q_k = \prod_{j=1}^{k} q_j$. Let $c = \sqrt{2} - 1$ and for any $k \geq 1$ define

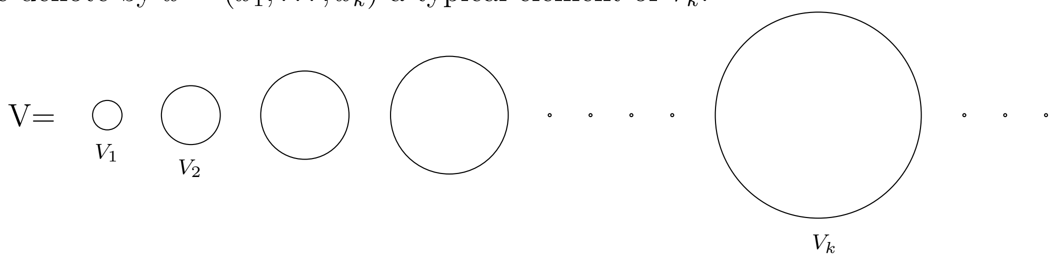
$$P_k = \left\{ p \text{ primes } : \quad \frac{Q_k^c}{k} \leq p < \frac{Q_{k+1}^c}{k+1} \right\}.$$

4.1. The construction of $\mathcal{G}(V, \mathcal{E})$.

- *The vertices.* For any $k \geq 1$ we define

$$V_k = \mathbb{F}_{q_1}^* \times \cdots \times \mathbb{F}_{q_k}^* \quad \text{and} \quad V = \bigcup_{k=1}^{\infty} V_k.$$

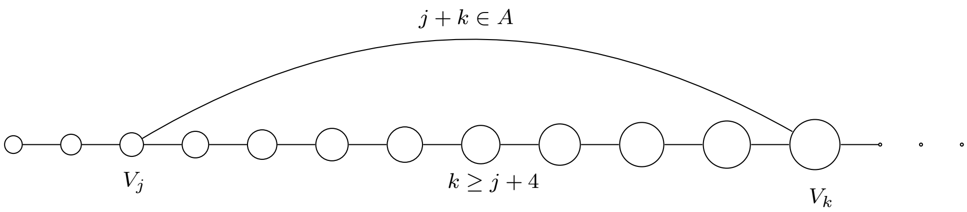
  We denote by $x = (x_1, \dots, x_k)$ a typical element of V_k.



  We order the vertices with the lexicographic order.

- *The edges.* Let A the Sidon sequence given by Theorem 3.3.
  Let $x, y \in V$ and suppose that $x \in V_k$ and $y \in V_j$ with $j \leq k$. Then $x \sim y$ if and only if both conditions a) and b) hold:

  a) i) $k = j + 1$
  or
  ii) $k \geq j + 4$ and $k + j \in A$.



b) There exists a prime $p \in P_k$ such that

$$\begin{aligned} x_i y_i &\equiv p \pmod{q_i}, \ i = 1, \dots, j \\ x_i &\equiv p \pmod{q_i}, \ i = j+1, \dots k. \end{aligned}$$

We will write $x \overset{p}{\sim} y$ to emphasize the role of the prime p.

4.2. **Properties of $\mathcal{G}(V, \mathcal{E})$.**

Proposition 4.1. *If $\mathcal{G}$ contains a C_4, say $x \sim y \sim u \sim v \sim x$, then there is a V_k containing the vertices x, u (but not y or v) or the vertices y, v (but not x or u).*

Proof. Suppose that x, y, u, v belong to distinct V_k, say $x \in V_{k_1}, \ y \in V_{k_2}, \ u \in V_{k_3}, \ v \in V_{k_4}$. We distinguish several cases:

i) There are not consecutive indices.
It implies that $k_1 + k_2, \ k_2 + k_3, \ k_3 + k_4, \ k_4 + k_1 \in A$. Thus, $(k_1 + k_2) + (k_3 + k_4) = (k_2 + k_3) + (k_4 + k_1)$, which is not possible because A is a Sidon sequence.

ii) There are only two consecutives V_k.
For example $k_2 = k_1 + 1$. Then $(k_1 + 1) + k_3, \ k_3 + k_4, \ k_4 + k_1 \in A$. These elements are multiple of 4 but the sum of them is an odd number.

iii) There are two pairs of consecutives V_k, but not three consecutives V_k.
For example $k_2 = k_1 + 1, \ k_4 = k_3 + 1$. Then $(k_1+1)+k_3, \ k_3+(k_4-1) \in A$, but it is not possible because both numbers cannot be multiple of 4.

iv) There are three consecutives V_k.
For example $k_2 = k_1 + 1, \ k_3 = k_1 + 2$. Then $(k_1 + 2) + k_4, \ k_4 + k_1 \in A$. Again it is impossible because all the elements of A must be multiple of 4.

v) There are four consecutives V_k.
For example $k_2 = k_1 + 1, \ k_3 = k_1 + 2, \ k_4 = k_1 + 3$. It is impossible because $|k_4 - k_1| < 4$ and then $v \not\sim x$.

□

Proposition 4.1 implies that at most three distinct V_k are involved in a C_4 in $\mathcal{G}$, say $V_{k_1}, V_{k_2}, V_{k_3}, \ k_1 \leq k_2 \leq k_3$. We say that the cycle is of type $[k_1, k_2, k_3]$.

Proposition 4.2. *Let (x, y, u, v) be a C_4 in $\mathcal{G}(V, \mathcal{E})$ of type $[k_1, k_2, k_3], \ k_1 \leq k_2 \leq k_3$ (excluding the case $k_1 = k_2 = k_3$) and suppose that $x \overset{p_2}{\sim} y \overset{p_2'}{\sim} u \overset{p_3'}{\sim} v \overset{p_3}{\sim} x$. Then*

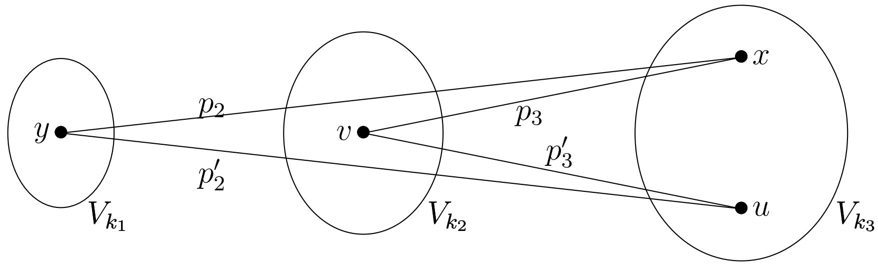
$p_2, p_2' \in P_{k_2},\ p_2 \neq p_2',\ p_3, p_3' \in P_{k_3},\ p_3 \neq p_3'$ *and*

$$\begin{aligned} p_2p_3' \equiv &\quad p_2'p_3 \pmod{Q_{k_2}} \\ p_3' \equiv &\quad p_3 \pmod{Q_{k_3}/Q_{k_2}}. \end{aligned}$$

Furthemore we have $Q_{k_2}^cQ_{k_3}^c \geq Q_{k_2}^c,\ \ Q_{k_3}^c \geq Q_{k_3}/Q_{k_2}$ *and* $Q_{k_2} < Q_{k_3}^{\frac{c}{1-c}}$.

Proof. According Proposition 4.1 we have to consider three cases:

First case: $y \in V_{k_1},\ v \in V_{k_2},\ x, u \in V_{k_3},\ k_1 \leq k_2 < k_3$.



We will prove that this case is not possible. Suppose that there exist $p_2, p_2', p_3, p_3' \in P_{k_3}$ such that

$$\begin{aligned} y_ix_i \equiv &\quad p_2 \pmod{q_i}, &\quad i \leq k_1 &\qquad y_iu_i \equiv \quad p_2' \pmod{q_i}, &\quad i \leq k_1 \\ x_i \equiv &\quad p_2 \pmod{q_i}, &\ k_1 < i \leq k_3 &\qquad u_i \equiv \quad p_2' \pmod{q_i}, &\ k_1 < i \leq k_3 \end{aligned}$$

$$\begin{aligned} v_ix_i \equiv &\quad p_3 \pmod{q_i}, &\quad i \leq k_2 &\qquad v_iu_i \equiv \quad p_3' \pmod{q_i}, &\quad i \leq k_2 \\ x_i \equiv &\quad p_3 \pmod{q_i}, &\ k_2 < i \leq k_3 &\qquad u_i \equiv \quad p_3' \pmod{q_i}, &\ k_2 < i \leq k_3. \end{aligned}$$

Combining the equalities above we have

$$\begin{aligned} y_ix_iv_iu_i \equiv p_2p_3' \equiv p_2'p_3 &\pmod{q_i}, \quad i \leq k_1 \\ x_iv_iu_i \equiv p_2p_3' \equiv p_2'p_3 &\pmod{q_i}, \ k_1 < i \leq k_2 \\ x_iu_i \equiv p_2p_3' \equiv p_2'p_3 &\pmod{q_i}, \ k_2 < i \leq k_3 \end{aligned}$$

The Chinese remainder theorem implies

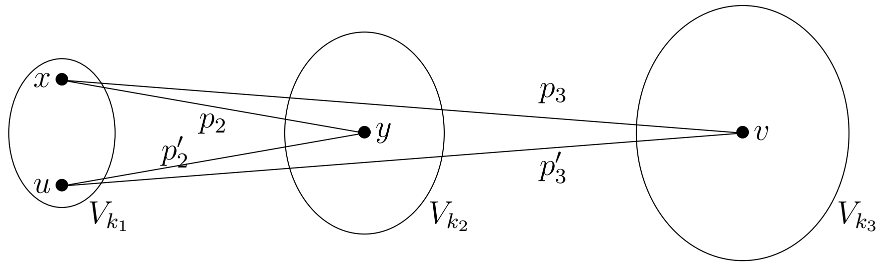
$$p_2p_3' \ \equiv \ p_2'p_3 \pmod{q_1 \cdots q_{k_3}}.$$

Note that if $p_2 = p_2'$ then $x_i \equiv u_i \equiv p_2u_i^{-1} \pmod{q_i},\ 1 \leq i \leq k_1$ and $x_i \equiv u_i \equiv p_2 \pmod{q_i},\ k_1 < i \leq k_3$ and then $x = u$. Thus $p_2 \neq p_2'$. On the other hand $p_2 \neq p_3$ because $P_{k_2} \cap P_{k_3} = \emptyset$, hence $p_2p_3' \neq p_2'p_3$ and we have

$$Q_{k_2}^cQ_{k_3}^c \geq |p_2p_3' - p_2'p_3| \geq Q_{k_3},$$

which is not possible because $Q_{k_2} < Q_{k_3}$ and $c < 1/2$.

Second case: $x, u \in V_{k_1},\ y \in V_{k_2},\ v \in V_{k_3},\ k_1 < k_2 \leq k_3$.



In this case there exist $p_2, p_2' \in P_{k_2}$ and $p_3, p_3' \in P_{k_3}$ such that

$$\begin{aligned} x_iy_i = &\ p_2 \pmod{q_i}, && i \le k_1 & \qquad u_iy_i \equiv &\ p_2' \pmod{q_i}, && i \le k_1 \\ y_i \equiv &\ p_2 \pmod{q_i}, && k_1 < i \le k_2 & y_i \equiv &\ p_2' \pmod{q_i}, && k_1 < i \le k_2 \end{aligned}$$

$$\begin{aligned} x_iv_i \equiv &\ p_3 \pmod{q_i}, && i \le k_1 & \qquad u_iv_i \equiv &\ p_3' \pmod{q_i}, && i \le k_1 \\ v_i \equiv &\ p_3 \pmod{q_i}, && k_1 < i \le k_3 & v_i \equiv &\ p_3' \pmod{q_i}, && k_1 < i \le k_3. \end{aligned}$$

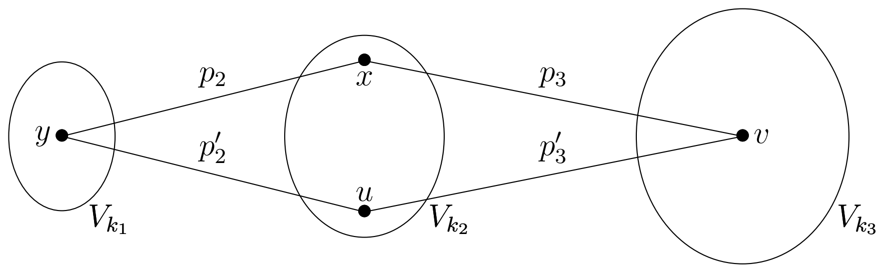
Combining the equalities above we have

$$\begin{aligned} x_iy_iu_iv_i = &\ p_2p_3' \equiv &\ p_2'p_3 \pmod{q_i}, && i \le k_1 \\ y_iv_i \equiv &\ p_2p_3' \equiv &\ p_2'p_3 \pmod{q_i}, && k_1 < i \le k_2 \\ v_i \equiv &\ p_3 \quad \equiv &\ p_3' \pmod{q_i}, && k_1 < i \le k_3. \end{aligned}$$

The Chinese remainder theorem implies

$$\begin{aligned} p_2p_3' &\equiv p_2'p_3 \pmod{q_1 \cdots q_{k_2}} \\ p_3' &\equiv p_3 \pmod{q_{k_2+1} \cdots q_{k_3}}. \end{aligned}$$

Third case: $y \in V_{k_1},\ x, u \in V_{k_2},\ v \in V_{k_3},\ k_1 < k_2 < k_3.$



In this case there exist $p_2, p_2' \in P_{k_2}$ and $p_3, p_3' \in P_{k_3}$ such that

$$\begin{aligned} y_ix_i \equiv &\ p_2 \pmod{q_i}, && i \le k_1 & \qquad y_iu_i \equiv &\ p_2' \pmod{q_i}, && i \le k_1 \\ x_i \equiv &\ p_2 \pmod{q_i}, && k_1 < i \le k_2 & u_i \equiv &\ p_2' \pmod{q_i}, && k_1 < i \le k_2 \end{aligned}$$

$$\begin{aligned} x_iv_i \equiv &\ p_3 \pmod{q_i}, && i \le k_2 & \qquad u_iv_i \equiv &\ p_3' \pmod{q_i}, && i \le k_2 \\ v_i \equiv &\ p_3 \pmod{q_i}, && k_2 < i \le k_3 & v_i \equiv &\ p_3' \pmod{q_i}, && k_2 < i \le k_3. \end{aligned}$$

Combining the equalities above we have

$$\begin{array}{rcll} x_iy_iu_iv_i \equiv & p_2p_3' \equiv & p_2'p_3 & \pmod{q_i}, \qquad i\le k_1\\ x_iu_iv_i \equiv & p_2p_3' \equiv & p_2'p_3 & \pmod{q_i},\ k_1<i\le k_2\\ v_i \equiv & p_3 \equiv & p_3 & \pmod{q_i},\ k_2<i\le k_3\end{array}$$

The Chinese remainder theorem implies

$$\begin{array}{rcll} p_2p_3' & \equiv & p_2'p_3 & \pmod{q_1\cdots q_{k_2}}\\ p_3' & \equiv & p_3 & \pmod{q_{k_2+1}\cdots q_{k_3}}.\end{array}$$

□

4.3. **Bad primes.** For each $k\ge 1$ we define $Bad(P_k)$ as the set formed by the primes $p\in P_k$ such that there exists some j with $Q_j<Q_k^{\frac{c}{1-c}}$ and primes $p'\in P_k,\ r,r'\in P_j,\ pr\neq p'r'$ satisfying the congruences

$$\begin{cases} pr & \equiv p'r' \pmod{Q_j}\\ p & \equiv p' \pmod{Q_k/Q_j}.\end{cases} \tag{4.1}$$

Define $P_k^*=P_k\setminus Bad(P_k)$ and let $\mathcal{G}^*(V,\mathcal{E}^*)$ the graph constructed as the graph $\mathcal{G}(V,\mathcal{E})$ but now $\mathcal{E}^*$ is the set of edges constructed using P_k^* instead P_k.

Corollary 4.1. *The graph $\mathcal{G}^*(V,\mathcal{E}^*)$ is C_4-free.*

Proof. The possible C_4 cycles in the graph satisfy the conditions of Proposition 4.1 but they have been destroyed when we have removed the Bad primes. □

Proposition 4.3. *We have*

$$|P_k|\sim\frac{Q_k^c}{ck^2\log k}\qquad\text{and}\qquad |Bad(P_k)|\ll\frac{Q_k^{\frac{3c-1}{1-c}}}{k^4}.$$

Proof. We use $\frac{Q_{k-1}^c}{k-1}=o\left(\frac{Q_k^c}{k}\right)$ and Lemma 2.2 to obtain

$$|P_k|=\pi\left(\frac{Q_k^c}{k}\right)-\pi\left(\frac{Q_{k-1}^c}{k-1}\right)\sim\pi\left(\frac{Q_k^c}{k}\right)\sim\frac{Q_k^c}{k\log(Q_k^c/k)}\sim\frac{Q_k^c}{ck^2\log k}.$$

The upper bound for $|Bad(P_k)|$ is more involved. If $p_1\in Bad(P_k)$ then, by construction, there exits j with $Q_j<Q_k^{\frac{c}{1-c}}$ and primes $p_1'\in P_k,\ p_2,p_2'\in P_j$ satisfying the congruences (4.1). We can write

$$\begin{aligned} p_1(p_2-p_2') &= p_1p_2-p_1'p_2'+(p_1'-p_1)p_2'\\ &= \frac{p_1p_2-p_1'p_2'}{Q_j}\,Q_j+\frac{(p_1'-p_1)p_2'}{Q_k/Q_j}\,Q_k/Q_j.\end{aligned}$$

The congruences (4.1) imply that $s_1=\frac{p_1p_2-p_1'p_2'}{Q_j}$ and $s_2=\frac{(p_1'-p_1)p_2'}{Q_k/Q_j}$ are nonzero integers satisfying

$$|s_1|=\frac{|p_1p_2-p_1'p_2'|}{Q_1}\le \frac{Q_j^cQ_k^c}{jkQ_j},\qquad |s_2|=\frac{|(p_1'-p_1)p_2'|}{Q_k/Q_j}\le \frac{Q_j^cQ_k^c}{jkQ_k/Q_j}.$$

Thus, if $p_1\in Bad(P_k)$ then p_1 must be a divisor prime of some integer $s\neq 0$ of some set

$$S_{j,k}=\left\{s=s_1Q_j+s_2Q_k/Q_j:\ 1\le |s_1|\le \frac{Q_j^cQ_k^c}{jkQ_j},\quad 1\le |s_2|\le \frac{Q_j^cQ_k^c}{jkQ_k/Q_j}\right\}$$

with $Q_j<Q_k^{\frac{c}{1-c}}$. If we write $\omega_k(s)$ to denote the number of primes $p_1\in P_k$ dividing s we have that

$$|Bad(P_k)|\le \sum_{\substack{j\\ Q_j<Q_k^{c/(1-c)}}}\ \sum_{s\in S_{j,k}}\omega_k(s).$$

We claim that $\omega_k(s)\le 1$ for k large enough. We observe that if some $s\in S_{j,k}$ has two distinct primes divisors $p,p'\in\mathcal{P}_k$ then we would have that

$$Q_{k-1}^{2c}<pp'\le |s|\le 2\cdot Q_j^cQ_k^c$$

and then that $Q_{k-1}^{2c}<2Q_k^{\frac{c^2}{1-c}}Q_k^c=2Q_{k-1}^{\frac{c}{1-c}}q_k^{\frac{c}{1-c}}$, which implies that $Q_{k-1}^{\frac{c(1-2c)}{1-c}}\le 2q_k^{\frac{c}{1-c}}$ and it is not possible for k large because $c<1/2$ and $q_k=o(Q_{k-1})$.

Then we have that

$$|Bad(P_k)|\le \sum_{\substack{j\\ Q_j<Q_k^{c/(1-c)}}}|S_{j,k}|\ll \sum_{\substack{j\\ Q_j<Q_k^{c/(1-c)}}}\frac{Q_j^{2c}Q_k^{2c}}{j^2k^2Q_k}\ll \frac{Q_k^{\frac{3c-1}{1-c}}}{k^4}.$$

□

4.4. **The minimun degree fo $\mathcal{G}_n$.**

Proposition 4.4. *If a vertex in V_k is labeled with n then*

$$\phi(Q_1)+\cdots+\phi(Q_{k-1})<n\le \phi(Q_1)+\cdots+\phi(Q_k)$$

where ϕ is the Euler totient function. In particular we have that $k\sim \log n/\log\log n$ and $Q_k=n^{1+o(1)}$.

Proof. If a vertex is labeled with $n\in V_k$ then

$$|V_1|+\cdots+|V_{k-1}|<n\le |V_1|+\cdots+|V_k|$$

and clearly $|V_j|=\prod_{i=1}^j|\mathbb{F}_{q_i}^*|=\prod_{i=1}^j(q_i-1)=\phi(Q_j)$. Thus we have

$$\frac{\phi(Q_k)}{q_k-1}=\phi(Q_{k-1})\le \phi(Q_1)+\cdots+\phi(Q_{k-1})\le n\le \phi(Q_1)+\cdots+\phi(Q_k)\le kQ_k$$

The well known estimate $\phi(m) = m^{1+o(1)}$ implies that $n = Q_k^{1+o(1)}$ and Proposition 2.2 imply the last part of the Proposition. □

Proposition 4.5. $\delta(\mathcal{G}_n) \geq n^{\sqrt{2}-1+o(1)}$.

Proof. We have to prove that for n large enough then $\deg_{G_n}(y) = n^{\sqrt{2}-1+o(1)}$ for any vertex y labeled with $m \leq n$. Suppose that $n \in V_k$ and that $y \in V_j$ for some $j \leq k$. Let m the integer such that $Cm^3 \leq j+k-1 < C(m+1)^3$. We distinguish two cases:

Case $2j + 4 > C(m-1)^3$. In this case we will use that

$$\deg_{\mathcal{G}_n}(y) \geq \deg_{\mathcal{G}_n}(y; V_{j-1}) = |\{x \in V_{j-1} : \ \{x,y\} \in \mathcal{E}\}|. \tag{4.2}$$

Suppose that

$$y = (y_1, \ldots, y_j) \in V_j.$$

First we observe that for any prime $p \in P_j$ such that $p \equiv y_j \pmod{q_j}$ the vertex

$$x = (x_1, \ldots, x_{j-1}) = (y_1^{-1}p, \ldots, y_{j-1}^{-1}p) \in V_{j-1}$$

is a neighbor of y. Indeed we have that

$$\begin{aligned} x_i y_i &\equiv p \pmod{q_i} \text{ for any } i \leq j-1 \\ y_j &= p \pmod{q_j}. \end{aligned}$$

Observe that each prime p provides a distinct neighbor x of y. If $x \overset{p}{\sim} y$ and $x \overset{p'}{\sim} y$ then $p \equiv p' \pmod{Q_j}$ and $Q_j^c/j \geq |p-p'| \geq Q_j$, which is not possible.

We have

$$\deg_{G_n}(y, V_{j-1}) \ \geq \ |P_j(q_j, y_j)| - |Bad(P_j)| \tag{4.3}$$

where

$$P_j(m,l) = \{p \in P_j : \ p \equiv l \pmod{m}\}.$$

Thus we have that

$$|P_j(q_j, y_j)| \ = \ \pi(Q_j^c/j; q_j, y_j) - \pi(Q_{j-1}^c/(j-1); q_j, y_j)$$

Since $q_j \ll \log(Q_j^c/j)$ we can apply Lemma 2.2 to get

$$|P_j(q_j, y_j)| \ = \ (1+o(1))\frac{Q_j/j}{(q_j-1)\log(Q_j^c/j))} - (1+o(1))\frac{Q_{j-1}/(j-1)}{(q_{j-1}-1)\log(Q_{j-1}^c/(j-1))}.$$

Taking into account that $Q_{j-1} = o(Q_j)$ and using Lemma 2.2 we have

$$|P_j(q_j, y_j)| \sim \frac{Q_j^c/j}{(q_j-1)\log(Q_j^c/j)} \sim \frac{Q_j^c}{cj^3\log^2 j}. \tag{4.4}$$

Since $\frac{3c-1}{1-c} = c$ for $c = \sqrt{2}-1$, Proposition 4.3 implies that

$$|Bad(P_j)| \ll Q_j^c/j^4 = o\left(|P_j(q_j, y_j)|\right). \tag{4.5}$$

Easily (4.2), (4.3) and (4.5) imply

$$\deg_{G_n}(y) \geq (1+o(1))\frac{Q_j^c}{cj^3 \log^2 j}.$$

Since $Q_k = n^{1+o(1)}$ (see Proposition 4.4) we have

$$\deg_{G_n}(y) \geq \frac{n^{1+o(1)}}{cj^3 \log^2 j (Q_k/Q_j)^c}.$$

Note that the condition $Cm^3 \leq j+k-1 < C(m+1)^3$ imply that

$$\begin{aligned} k-j &= j+k-1-(2j-1) < C(m+1)^3 - (2j-1) \\ &< C(m+1)^3 - (2C(m-1)^3 - 5) \ll m^2 \ll k^{2/3}. \end{aligned}$$

We observe that $(Q_k/Q_j) \leq q_k^{k-j}$. Using that $k \sim \log n / \log\log n$ we have that

$$\log(cj^3 \log^2 j (Q_k/Q_j)^c) \ll \log k + (k-j)\log q_k \ll k^{2/3}\log k = o(\log n).$$

Thus $cj^3 \log^2 j (Q_k/Q_j)^c \leq n^{o(1)}$ and then $\deg_{\mathcal{G}_n}(y) \gg n^{c+o(1)}$.

Case $2j+4 \leq C(m-1)^3$. In this case we consider the element $a \in A$ in the interval $[C(m-1)^3, Cm^3)$ and we take $l = a-j$. Note that $|l-j| = |a-2j| \geq C(m-1)^3 - 2j \geq 4$ and that $l < Cm^3 - j \leq k-1$. Hence we have

$$\deg_{G_n}(y) \geq \deg_{G_n}(y, V_l).$$

Note that for any $p \in P_l$, the element $x = (x_1, \dots, x_l) \in V_l$ with

$$\begin{aligned} x_i &\equiv y_i^{-1}p \pmod{q_i}, \ 1 \leq i \leq j \\ x_i &\equiv p \pmod{q_i}, \ j < i \leq l \end{aligned}$$

is a neighbor of y. Since distinct primes give distinct neighbors of y we have

$$\deg_{G_n}(y, V_l) \geq |P_l^*| = |P_l| - |Bad(P_l)|.$$

The Prime Number Theorem implies that

$$|P_l| \sim \frac{Q_l^c}{cl^2 \log l}.$$

Since $\frac{3c-1}{1-c} = c$ for $c = \sqrt{2}-1$, Proposition 4.3 implies that $|Bad(P_l)| \ll Q_l^c/l^4$ and we conclude that

$$\deg_{G_n}(y) \geq \frac{Q_l^c}{cl^2 \log l}(1+o(1)).$$

Since $Q_k = n^{1+o(1)}$ (see Proposition 4.4) we have

$$\deg_{G_n}(y) \geq \frac{n^{c+o(1)}}{cl^2 \log l (Q_k/Q_l)^c}.$$

We observe that $(Q_k/Q_l) \leq q_k^{k-l}$ and note that

$$\begin{aligned} k-l = k-(a-j) = k+k-a \leq k+j-C(m-1)^3 \\ < C(m+1)^3 + 1 - C(m-1)^3 \ll m^2 \ll k^{2/3}. \end{aligned}$$

Using also that $k \sim \log n/\log\log n$ we have that

$$\log(cl^2 \log l (Q_k/Q_l)^c) \ll \log k + (k-l)\log q_k \ll k^{2/3}\log k = o(\log n),$$

which implies that

$$\deg_{G_n}(y) \geq n^{c+o(1)}.$$

□

Theorem 1.2 is consequence of Corollary 4.1 and Proposition 4.5.

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