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ON A QUESTION OF SÁRKOZY AND SÓS FOR BILINEAR FORMS

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Abstract

We prove that if $2 \leq k_1 \leq k_2$, then there is no infinite sequence $\mathcal{A}$ of positive integers such that the representation function $r(n) = \#\{(a,a') : \ n = k_1a + k_2a', \ a,a' \in \mathcal{A}\}$ is constant for n large enough. This result completes previous work of Dirac and Moser for the special case $k_1 = 1$ and answers a question posed by Sárkozy and Sós.

1. *Introduction*

Given an infinite sequence of positive integers $\mathcal{A}$, the representation functions $r(n)$ and $R(n)$ are defined as the number of solutions of the equations

$$n = a + a', \ a, a' \in \mathcal{A}$$
$$n = a + a', \ a, a' \in \mathcal{A}, \ a \leq a',$$

respectively.

It is obvious that $r(n)$ is odd when $n = 2a, \ a \in \mathcal{A}$, and even otherwise. So it is not possible for $r(n)$ to be constant for n large enough. This asymmetry disappears in $R(n)$ but Dirac [**1**] gave a beautiful argument that also proves that $R(n)$ cannot be constant for n large enough.

For $k \geq 2$, Moser [**3**] considered the representation function

$$r(n) = \#\{(a,a'), \ n = a + ka', \ a,a' \in \mathcal{A}\}.$$

Surprisingly he constructed a sequence $\mathcal{A}$ such that $r(n) = 1$ for all $n \geq 0$.

Sárkozy and Sós [**4**] asked for which (k_1,k_2) the representation function

$$r_{k_1,k_2}(n,\mathcal{A})) = \#\{(a,a') : \ n = k_1a + k_2a', \ a,a' \in \mathcal{A}\}.$$

can be constant for n large enough. We answer this question by showing that the only cases with affirmative answer are those considered by Moser.

Theorem 1.1. *Let $k_1, k_2, \ 2 \leq k_1 \leq k_2$. Then there is no infinite sequence of positive integers $\mathcal{A}$ such that*

$$r_{k_1,k_2}(n,\mathcal{A}) = \#\{(a,a') : \ n = k_1a + k_2a', \ a,a' \in \mathcal{A}\}$$

is constant for n large enough.

The question posed in [**4**] actually concerns general linear forms $k_1x_1 + \cdots + k_hx_h, \ h \geq 2$. The same arguments we use for the case $h = 2$ can be extended to the general case when the $k_i's$ are pairwise coprimes but they are best illustrated in the situation presented in this paper. The more general case with arbitrary coefficients requires a different approach and it is considered in a forthcoming paper.

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2. *Translation of the problem into generating functions: Dirac's and Moser's arguments*

As Dirac and Moser did, we use the language of generating functions: to every set $\mathcal{A}$ of nonnegative integers, we write the formal power series $f_{\mathcal{A}}(z)$ defined as

$$f_{\mathcal{A}}(z) := f(z) = \sum_{a\in\mathcal{A}} z^a.$$

This formal power series is called the *generating function associated to* $\mathcal{A}$.

For every set $\mathcal{A}$, the corresponding generating function defines an analytic function around $z = 0$. This analytic function is a polynomial if $\mathcal{A}$ is finite and has a singularity at $z = 1$ if $|\mathcal{A}|$ is infinite. In fact, if $|\mathcal{A}|$ is infinite, then the Taylor expansion around $z = 0$ defined by the formal power series has radius of convergence $r = 1$.

We proceed to translate the general problem into the language of generating functions. The fundamental equation we use is:

$$f(z^{k_1})f(z^{k_2}) = \sum_{a,a'\in\mathcal{A}} z^{k_1a+k_2a'} = \sum_{n=0}^{\infty} r_{k_1,k_2}(n,\mathcal{A})z^n. \tag{2.1}$$

2.1. *Dirac's argument*

We observe that for the functions $r(n)$ and $R(n)$ we have the relation $r(n) = 2R(n) - \delta(n)$, where $\delta(n) = 1$ if $n = 2a$ for some $a \in \mathcal{A}$ and 0 otherwise. By (2.1) we obtain

$$f^2(z) = \sum_{n=0}^{\infty} r(n)z^n = 2\sum_{n=0}^{\infty} R(n)z^n - \sum_{a\in\mathcal{A}} z^{2a},$$

which can be written in the form

$$f^2(z) + f(z^2) = 2\sum_{n=0}^{\infty} R(n)z^n.$$

Dirac proved that $R(n)$ cannot be a constant c for $n \geq n_0$ with an easy but clever argument: suppose that $R(n) = c$ for $n \geq n_0$. Then

$$f^2(z) + f(z^2) = Q(z) + 2c\frac{z^{n_0+1}}{1-z} = \frac{P(z)}{1-z},$$

where $P(z)$ is a polynomial of finite degree with $P(1) \neq 0$. Then we obtain a contradiction by taking the limit for $z \to -1$ in both sides of the equation: the left hand side of the equality diverges, but the right hand side has a finite limit.

2.2. *Moser's argument*

Moser [**3**] studied the case $k_1 = 1$, $k_2 \geq 2$. He wondered if for these cases there exists an infinite sequence of nonnegative integers such that $r_{1,k}(n,\mathcal{A}) = 1$ for all $n \geq 0$. If this is the case, equation (2.1) implies

$$f(z)f(z^k) = \sum_{n\geq 0} z^n = \frac{1}{1-z}.$$

If we make the change of variables $z := z^k$ we get

$$f(z^k)f(z^{k^2}) = \frac{1}{1-z^k}.$$

Dividing the initial equation by this one we obtain

$$f(z) = \frac{1-z^k}{1-z} f(z^{k^2}) = (1+z+z^2+\cdots+z^{k-1}) f(z^{k^2}).$$

By iterating we get the relation

$$f(z) = \prod_{j=0}^{\infty} (1 + z^{(k^2)^j} + z^{2(k^2)^j} + \cdots + z^{(k-1)(k^2)^j}).$$

This product defines an analytic function at the origin, which can be written using its series expansion around $z = 0$. Moreover, by the unique k^2–adic representation of an integer, the Taylor's coefficients of $f(z)$ are either 0 or 1. So this function $f(z)$ defines a set $\mathcal{A}$ which satisfies our assumptions.

More precisely, the set $\mathcal{A}$ is the set of all nonnegative integers such that all its digits in its k^2–adic expansion are smaller than k.

2.3. *The general case*

We want to know if, given $k_1, k_2,\ 1 \le k_1 \le k_2$, there exists an infinite sequence of non negative integers $\mathcal{A}$ and a value (say n_0) such that $r_{k_1,k_2}(n, \mathcal{A})$ is a positive constant c for $n \ge n_0$. Since the cases $k_1 = 1$ have been considered by Dirac ($k_2 = 1$ with negative answer in both ordered and unordered representations) and Moser ($k_2 \ge 2$ with affirmative answer) we may assume that $2 \le k_1 \le k_2$. We may also assume that $\gcd(k_1, k_2) = 1$, since otherwise we have $r_{k_1,k_2}(n, \mathcal{A}) = 0$ for all $n \not\equiv 0 \pmod{\gcd(k_1, k_2)}$.

If such a sequence $\mathcal{A}$ exists, then by (2.1) we have

$$f(z^{k_1}) f(z^{k_2}) = \sum_{n=0}^{n_0-1} a_n z^n + \sum_{n=n_0}^{\infty} c z^n = Q(z) + \frac{c z^{n_0+1}}{1-z} = \frac{P(z)}{1-z},$$

where $Q(z), P(z)$ are polynomials in $\mathbb{Z}[z]$ with $P(1) \ne 0$. This last relation is equivalent to the condition $c \ne 0$.

For convenience, we take the square of the previous equation. By writing $F(z) = f^2(z)$, we want to show that there is no function $F(z)$, analytic in the disc $|z| < 1$, such that

$$F(z^{k_1}) F(z^{k_2}) = \frac{P^2(z)}{(1-z)^2}.$$

Theorem 1.1 will be a consequence of a more general theorem:

Theorem 2.1. *For any integers $k_1, k_2,\ 2 \le k_1 < k_2$ with $\gcd(k_1, k_2) = 1$ and any polynomial $P(z) \in \mathbb{Z}[z]$ with $P(1) \ne 0$, there is no function $F(z)$, analytic in the disc $|z| < 1$, satisfying*

$$F(z^{k_1}) F(z^{k_2}) = \frac{P^2(z)}{(1-z)^2}. \tag{2.2}$$

In what follows we concentrate in the proof of Theorem 2.1.

3. *Algebraic preliminaries and notation used*

In our work we will use cyclotomic polynomials.

Recall that the *cyclotomic polynomial* of order n is defined by the relation

$$\Phi_n(z) = \prod_{\xi \in \phi_n} (z - \xi) \in \mathbb{Z}[z],$$

where ϕ_n denotes the set of primitive roots of order n,

$$\phi_n = \{\xi \in \mathbb{C} : \xi^k = 1 \text{ if and only if } k \equiv 0 \pmod{n}\}.$$

Many properties of these polynomials are well–known. The most important one for our present purposes is that they are irreducible over $\mathbb{Z}[z]$. As a consequence, it is known that if a polynomial $P(z) \in \mathbb{Z}[z]$ vanishes at a primitive root of order n then there exists a positive integer s such that

$$P(z) = \Phi_n^s(z)Q(z)$$

where $Q(z) \in \mathbb{Z}[z]$ and $Q(\xi) \neq 0$ for all primitive roots ξ of order n.

Given $k_1, k_2,\ 2 \leq k_1 < k_2$ we write ξ_{j_1,j_2} to denote a primitive root of order $k_1^{j_1}k_2^{j_2},\ 0 \leq j_1, j_2$, and we let s_{j_1,j_2} denote the nonnegative integer such that

$$P(z) = \Phi^{s_{j_1,j_2}}_{k_1^{j_1}k_2^{j_2}}(z)P_{j_1,j_2}(z)$$

with $P_{j_1,j_2}(z) \in \mathbb{Z}[z]$ and $P_{j_1,j_2}(\xi) \neq 0$ for all $\xi \in \phi_{k_1^{j_1}k_2^{j_2}}$.

It is not true in general that for an analytic function $F(z)$ with integer coefficients there exists r_{j_1,j_2} such that

$$F(z) = \Phi^{r_{j_1,j_2}}_{k_1^{j_1}k_2^{j_2}}(z)F_{j_1,j_2}(z)$$

with $\lim_{z\to\xi} F_{j_1,j_2}(z) \neq 0, \infty$ for any $\xi \in \phi_{j_1,j_2}$. However we will prove that there is such a factorization if $F(z)$ satisfies (2.2).

4. *Proof of theorem 2.1*

The proof of Theorem 2.1 is a consequence of the two propositions below:

Proposition 4.1. *With the notation used in Section 3, $r_{j,0}$ and $r_{0,j}$ are well defined for any $j \geq 0$ and they verify the following recurrence relations*

$$r_{j+1,0} = 2s_{j+1,0} - r_{j,0} \qquad \textit{(horizontal recurrence)}$$
$$r_{0,j+1} = 2s_{0,j+1} - r_{0,j} \qquad \textit{(vertical recurrence)}$$

for any $j \geq 0$, and initial condition $r_{0,0} = -1$.

Proposition 4.2. *With the notation used in Section 3, r_{j_1,j_2} are well defined for any $j_1, j_2 \geq 0$ and these numbers verify the recurrence relation*

$$r_{j_1-1,j_2} + r_{j_1,j_2-1} = 2s_{j_1,j_2} \qquad \textit{(diagonal recurrence)}$$

for $j_1 \geq 1,\ j_2 \geq 1$.

Before proving the above propositions we show how Theorem 1.1 can be deduced from them.

Since $r_{0,0} = -1$ is an odd number, from the two propositions above we see that all values r_{j_1,j_2} are odd numbers.

As $P(z)$ is a polynomial, it is clear that $s_{j_1,j_2} = 0$ when $j_1 + j_2 \geq j$ for a suitable j.

Using proposition 4.1 and for this value of j, we obtain the relations:

$$r_{j+1,0} = -r_{j,0} \qquad \text{and}$$
$$r_{0,j+1} = -r_{0,j}.$$

From proposition 4.2 we get that $r_{j+1,0} = -r_{j,1} = r_{j-1,2} = \cdots$, so

$$r_{j+1,0} = (-1)^{j+1}r_{0,j+1} \qquad \text{and}$$
$$r_{j,0} = (-1)^j r_{0,j}.$$

The above relations are possible if and only if $r_{j,0} = r_{j+1,0} = r_{0,j} = r_{0,j+1} = 0$, giving a contradiction. This proves Theorem 2.1 and Theorem 1.1.

We next prove the two propositions. In what follows all the limits considered are taken along a radius.

Proof of proposition 4.1. We deal only with the horizontal recurrence. The proof for the vertical recurrence is similar.

We write $r_j = r_{j,0}$, $s_j = s_{j,0}$ and $\xi_j = \xi_{j,0}$ for simplicity. We shall prove by induction that all the r_j's are well defined.

By writing $F(z) = F_0(z)/(z-1)$ in (2.2) we have

$$F_0(z^{k_1})F_0(z^{k_2}) = P^2(z)(1+z+\cdots z^{k_1-1})(1+z+\cdots z^{k_2-1}).$$

For $z \to 1$ the left hand side of the above equation clearly goes to $F_0^2(1)$ and the right hand side is neither 0 nor ∞. Since $\Phi_1(z) = z-1$ we obtain that $r_0 = -1$.

Assume now that r_j is well defined. Then

$$F(z) = \Phi_{k_1^j}^{r_j}(z)F_j(z), \tag{4.1}$$

where $F_j(\xi) \neq 0, \infty$ for all $\xi \in \phi_{k_1^j}$. Now we write

$$F(z) = \Phi_{k_1^{j+1}}^{2s_{j+1}-r_j}(z)F_{j+1}(z). \tag{4.2}$$

In what follows, we prove that $\lim_{z\to\xi_{j+1}} F_{j+1}(z) \notin \{0,\infty\}$ for any $\xi \in \phi_{k_1^{j+1}}$. This will show that r_{j+1} exists and that $r_{j+1} = 2s_{j+1} - r_j$.

We use (4.1) in $F(z^{k_1})$ and (4.2) in $F(z^{k_2})$ to write the equation (2.2) in the form

$$F_{j+1}(z^{k_2}) = \frac{P_{j+1}^2(z)\Phi_{k_1^{j+1}}^{2s_{j+1}}(z)}{(1-z)^2\Phi_{k_1^{j+1}}^{2s_{j+1}-r_j}(z^{k_2})F_j(z^{k_1})\Phi_{k_1^j}^{r_j}(z^{k_1})}.$$

By making the substitution $z = \xi_{j+1}\omega$ we have

$$\begin{aligned} F_{j+1}(\xi_{j+1}^{k_2}\omega^{k_2}) = &\frac{P_{j+1}^2(\xi_{j+1}\omega)}{(1-\xi_{j+1}\omega)^2}\cdot\frac{1}{F_j(\xi_{j+1}^{k_1}\omega^{k_1})} \\ &\cdot\frac{\Phi_{k_1^{j+1}}^{2s_{j+1}}(\xi_{j+1}\omega)}{\Phi_{k_1^{j+1}}^{2s_{j+1}-r_j}(\xi_{j+1}^{k_2}\omega^{k_2})\Phi_{k_1^j}^{r_j}(\xi_{j+1}^{k_1}\omega^{k_1})}. \end{aligned} \tag{4.3}$$

We let $\omega \to 1$ and we observe that all the primitive roots of order k_1^{j+1} can be written in the form $\xi_{j+1}^{k_2}$ for a suitable ξ_{j+1} since $\gcd(k_2, k_1) = 1$.

To conclude, we show that the limit is neither 0 nor ∞. In fact, this is the case for each of the three factors on the right hand side of (4.3).

It is clear that $P_{j+1}(\xi_{j+1}) \neq 0$ by definition.

Since $\xi_{j+1}^{k_1} \in \phi_{k_1^j}$, we use the induction hypothesis to conclude that the limit of the second factor is neither 0 nor ∞.

To study the third factor when $\omega \to 1$ it suffices to analyze the factors in the cyclotomic polynomials which vanish at $\omega = 1$. It should be noticed that $\xi_{j+1}^{k_1} \in \phi_{k_1^j}$ and $\xi_{j+1}^{k_2} \in \phi_{k_1^{j+1}}$. The contribution of these factors is

$$\begin{aligned} &\frac{(\xi_{j+1}\omega - \xi_{j+1})^{2s_{j+1}}}{(\xi_{j+1}^{k_2}\omega^{k_2} - \xi_{j+1}^{k_2})^{2s_{j+1}-r_j}(\xi_{j+1}^{k_1}w^{k_1} - \xi_{j+1}^{k_1})^{r_j}} \\ = &\frac{\xi_{j+1}^{2s_{j+1}}}{\xi_{j+1}^{k_2(2s_{j+1}-r_j)}\xi_{j+1}^{k_1 r_j}}\cdot\frac{(\omega-1)^{2s_{j+1}}}{(\omega^{k_2}-1)^{2s_{j+1}-r_j}(\omega^{k_1}-1)^{r_j}} \end{aligned}$$

which tends to

$$\frac{\xi_{j+1}^{2s_{j+1}(1-k_2)+r_j(k_2-k_1)}}{k_2^{2s_{j+1}-r_j}k_1^{r_j}}$$

as $\omega \to 1$. The relevant fact about this limit is that it is neither zero nor infinity. □

The next proof is quite similar to the previous one:

Proof of proposition 4.2. We will prove, for each diagonal $j_1 + j_2 = j$, that all r_{j_1,j_2}, $0 \leq j_2 \leq j$ are well defined. For each j we will do it by induction on j_2.

This is true for $r_{j,0}$ by Proposition 4.1. Suppose that r_{j_1,j_2} is well defined. Thus

$$F(z) = \Phi_{j_1,j_2}^{r_{j_1,j_2}}(z)F_{j_1,j_2}(z) \tag{4.4}$$

where $F_{j_1,j_2}(\xi) \notin \{0,\infty\}$ for all $\xi \in \phi_{k_1^{j_1}k_2^{j_2}}$.

We prove that r_{j_1-1,j_2+1} is also well defined and also that $r_{j_1-1,j_2+1} = s_{j_1,j_2+1} - r_{j_1,j_2}$.

In order to do this we write

$$F(z) = \Phi_{j_1-1,j_2+1}^{2s_{j_1,j_2+1}-r_{j_1,j_2}}(z)F_{j_1-1,j_2+1}(z).$$

What we have to prove is that $\lim_{z\to\xi} F_{j_1-1,j_2+1}(z) \notin \{0,\infty\}$ for any $\xi \in \phi_{j_1-1,j_2+1}$.

We use (4.4) in $F(z^{k_2})$ and (4) in $F(z^{k_1})$ to write the equation (2.2) in the form

$$F_{j_1-1,j_2+1}(z^{k_1}) = \frac{P_{j_1,j_2+1}^2(z)\Phi_{j_1,j_2+1}^{2s_{j_1,j_2+1}}(z)}{(1-z)^2\Phi_{j_1-1,j_2+1}^{2s_{j_1,j_2+1}-r_{j_1,j_2}}(z^{k_1})\Phi_{j_1,j_2}^{r_{j_1,j_2}}(z^{k_2})F_{j_1,j_2}(z^{k_2})}.$$

Now we make the substitution $z = \xi_{j_1,j_2+1}\omega$ for some arbitrary $\xi_{j_1,j_2+1} \in \phi_{k_1^{j_1}k_2^{j_2+1}}$. We obtain the expression

$$F_{j_1-1,j_2+1}(\xi_{j_1,j_2+1}^{k_1}\omega^{k_1}) = \frac{P_{j_1,j_2+1}^2(\xi_{j_1,j_2+1}\omega)}{(1-\xi_{j_1,j_2+1}\omega)^2} \cdot \frac{1}{F_{j_1,j_2}(\xi_{j_1,j_2+1}^{k_2}\omega^{k_2})}$$
$$\cdot \frac{\Phi_{j_1,j_2+1}^{2s_{j_1,j_2+1}}(\xi_{j_1,j_2+1}\omega)}{\Phi_{j_1-1,j_2+1}^{2s_{j_1,j_2+1}-r_{j_1,j_2}}(\xi_{j_1,j_2+1}^{k_1}\omega^{k_1})\Phi_{j_1,j_2}^{r_{j_1,j_2}}(\xi_{j_1,j_2+1}^{k_2}\omega^{k_2})}.$$

Now we let $\omega \to 1$. We observe that all the primitive roots of order $k_1^{j_1-1}k_2^{j_2-1}$ can be written in the form $\xi_{j_1,j_2+1}^{k_1}$ for a suitable $\xi_{j_1,j_2+1} \in \phi_{k_1^{j_1}k_2^{j_2+1}}$. As in the previous proposition, we show that the limit is neither 0 nor ∞, showing that this is the case for every factor in the previous equation.

For the first factor, it is clear that $P_{j_1,j_2+1}(\xi_{j_1,j_2+1}) \notin \{0,\infty\}$ by definition.

Since $\xi_{j_1,j_2+1}^{k_1} \in \phi_{k_1^{j_1-1}k_2^{j_2+1}}$, we use induction hypothesis to conclude that the limit in the second factor does not belong to $\{0,\infty\}$.

Finally, to study the third factor when $\omega \to 1$ we look at the cyclotomic polynomials which vanish at $\omega = 1$. It should be noticed that $\xi_{j_1,j_2+1}^{k_1} \in \phi_{k_1^{j_1-1}k_2^{j_2+1}}$ and $\xi_{j_1,j_2+1}^{k_2} \in \phi_{k_1^{j_1}k_2^{j_2}}$. The contribution of these factors is

$$\frac{(\xi_{j_1,j_2+1}\omega - \xi_{j_1,j_2+1})^{2s_{j_1,j_2+1}}}{(\xi_{j_1,j_2+1}^{k_1}\omega^{k_1} - \xi_{j_1,j_2+1}^{k_1})^{2s_{j_1,j_2+1}-r_{j_1,j_2}}(\xi_{j_1,j_2+1}^{k_2}\omega^{k_2} - \xi_{j_1,j_2+1}^{k_2})^{r_{j_1,j_2}}}.$$

which tends to a number which is neither zero nor infinity. This concludes the proof. □

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