Power Values of Palindromes

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Abstract

We show that for a fixed integer base $g \geq 2$ the palindromes to base g which are k-powers form a very thin set in the set of all base g palindromes.

1 Introduction

For a fixed integer base $g \geq 2$ consider the *base g representation* of an arbitrary natural number $n \in \mathbb{N}$:

$$n = \sum_{k=0}^{L-1} a_k(n) g^k, \qquad (1)$$

where $a_k(n) \in \{0, 1, \dots, g-1\}$ for each $k = 0, 1, \dots, L-1$, and the leading digit $a_{L-1}(n)$ is *nonzero*. The integer n is said to be a *base g palindrome* if its digits satisfy the symmetry condition:

$$a_k(n) = a_{L-1-k}(n) \qquad \text{for all } k = 0, 1, \dots, L-1. \qquad (2)$$

When the base g is understood, we will refer to these numbers simply as *palindromes.*

It has recently been shown in [1] that almost all palindromes are composite. In [6], it has been shown that almost all Fibonacci numbers are not palindromes, and the argument there applies to some other similar sequences. For an integer $a \geq 2$, the smallest positive integer k such that a^k is not a base g palindrome has been estimated in [4] as $\exp\left(O\left((\log H)^3 \log\log H\right)\right)$, where $H = \max\{a, g\}$. Several more results about the prime divisors and other arithmetic properties of palindromes can be found in [2, 3].

Square values of palindromes have been investigated in [5], where some constructions of infinite families of palindromes which are perfect squares are given.

Here, we continue the study of k-power values of palindromes and show that they form a very thin set in the set of all palindromes. We also show that this set is larger than standard heuristic arguments suggest.

Throughout the paper, implied constants in the symbols O and $\ll$ may depend on the base g (we recall that the notations $U = O(V)$ and $U \ll V$ are equivalent to the assertion that the inequality $|U| \leq cV$ holds with some positive constant c).

2 Upper bound

Let $\mathcal{P}_{g,L}$ denote the set of all palindromes (2) of length L; that is, the set of positive integers satisfying both (1) and (2).

We also denote by $\mathcal{Q}_{g,L}^k$ the set of $n \in \mathcal{P}_{g,L}$ which are k-powers.

Theorem. *The inequality*

$$\#\mathcal{Q}_{g,L}^k \ll (\#\mathcal{P}_{g,L})^{1/k}$$

holds for all $L \geq 1$.

Proof. We may assume that L is large. Let $M = \lfloor (L-1)/(2k) \rfloor$. We write $\mathcal{Q}_{g,L}^k = \sum_{0 \leq a < g^M} \mathcal{Q}_{g,L,a}^k$ where $\mathcal{Q}_{g,L,a}^k = \{x^k \in \mathcal{P}_{g,L},\ x \equiv a \pmod{g^M}\}$.

We observe that $\#\mathcal{Q}_{g,L,a}^k = 0$ for those a such that the last digit of $a_{g^M}^k$ in base g is 0. Thus, we assume that the last digit of $a_{g^M}^k$ is different of zero. Then, if x^k is a palindrome for some positive integer x, its first M digits are the mirror reflection of the base g representation of $a_{g^M}^k$. We write b for this number of M digits. For $x^k \in \mathcal{Q}_{g,L,a}^k$, we have $bg^{L-M} \leq x^k < (b+1)g^{L-M}$. Thus, $\left(bg^{L-M}\right)^{1/k} \leq x < \left((b+1)g^{L-M}\right)^{1/k}$. The number of integers in the arithmetic progression $x \equiv a \pmod{g^M}$ lying in the above interval is bounded above by $\frac{1}{g^M}\left(\left((b+1)g^{L-M}\right)^{1/k} - \left(bg^{L-M}\right)^{1/k}\right) + 1$.

So,

$$\begin{aligned} \#\mathcal{Q}_{g,L}^k &\leq g^M \max_a \#\mathcal{Q}_{g,L,a}^k \\ &\leq \left((b+1)g^{L-M}\right)^{1/k} - \left(bg^{L-M}\right)^{1/k} + g^M \\ &\leq g^{\frac{L-M}{k}} \frac{1}{k}(g^{M-1})^{\frac{1}{k}-1} + g^M \leq \frac{1}{k} g^{\frac{L}{k}-M+1-\frac{1}{k}} + g^M, \end{aligned}$$

which gives the desired result. □

3 Lower bound

Most certainly, our result is not tight and there should be very few palindromes which are k-powers. We note that the standard naïve heuristic predictions suggests that

$$\#\mathcal{Q}_{g,L}^2 \approx \sum_{n \in \mathcal{P}_{g,L}} \frac{1}{n^{1/2}} \sim L \log g$$

and

$$\#\mathcal{Q}_{g,L}^k \approx \sum_{n \in \mathcal{P}_{g,L}} \frac{1}{n^{1-1/k}} < \infty$$

for $k \geq 3$.

However, the above heuristic is wrong and in fact it is easy to show that if $g > k!$, then there are infinitely many palindromic k-powers. To see this, observe that the polynomial $(x^H + 1)^k$ is symmetric and all its coefficients are at most $k!$. Thus, for $x = g^\ell$ and $g > k!$, we obtain palindromic kth-powers. But the following theorem is stronger and unexpected.

Theorem. *Given $k \geq 2$, there exists a positive constant $c = c(k)$ depending on k such that if $g \geq g(k)$, then*

$$\#\mathcal{Q}_{g,L}^k \gg L^{cg^{1/\lfloor k/2 \rfloor}}.$$

Proof. It is clear that the k-power of a symmetric polynomial is also symmetric. So, we consider $f(x) = \sum_{a \in A} x^a$ for a symmetric set A with $\max A = L$ and $\min A = 0$. We have that

$$f^k(x) = \sum_n r_k(n, A) x^n$$

where

$$r_k(n, A) = \#\{(a_1, \dots, a_k) : n = a_1 + \cdots + a_k,\ a_i \in A\}.$$

Of course, if $\max r_k(n, A) \leq g - 1$, then $\sum_n r_k(n, A) g^n$ is a palindromic k-power since $r_k(kL - n, A) = r(n, A)$.

Next, we give a lower bound for the number of symmetric sets A with $\max A = L,\ \min A = 0$, and $\max r_k(n, A) \leq g - 1$.

Let $H = \lfloor (L-1)/2 \rfloor\}$, and let $B \subset \{1, \dots, H\}$ be a subset with the property that all the quantities $\sum_{b \in U} b - \sum_{b \in U'} b$, with disjoint multisets U and U' of B, are distinct $\pmod L$. We will refer to this property as *property* P.

Claim 1. *If B satisfies property P and $|B| \geq 2$, then set $A = \{0, L\} \cup B \cup (L - B)$ is symmetric and satisfies*

$$\max r_k(n, A) \leq 2k!(\#B)^{\lfloor k/2 \rfloor}.$$

Proof. The summands of any representation of n as a sum of k elements of A can be ordered as

$$n = \sum_{b \in U_1} b + \sum_{b \in U_2} (L - b) + \sum_{b \in U_3} (b + (L - b)) + \sum_{x \in U_4} x,$$

where U_1, U_2, U_3 are non decreasing sequences of elements of B with $U_1 \cap U_2 = \emptyset$, U_4 is a non decreasing sequence of elements of $\{0, L\}$, and $\#U_1 + \#U_2 + 2\#U_3 + \#U_4 = k$.

Since $n \equiv \sum_{b\in U_1} b - \sum_{b\in U_2} b \pmod{L}$, and B has property P, the sequences U_1 and U_2 are determined by n. We observe also that, given n, the sequence U_4 is determined by $\#U_3$. Thus, the different representations of n in this form all come from the $\#B$ possible elections for each $b_i,\ 1 \le i \le \#U_3$, and the $k!$ different order in the presentations of the k elements. Since $\#U_3 \le k/2$, we have that

$$r_k(n, A) \le k! \sum_{r=0}^{\lfloor k/2 \rfloor} (\#B)^r \le 2k!(\#B)^{\lfloor k/2 \rfloor}.$$

□

So, each set $B \subset \{1, \ldots, H\}$ with $2 \le \#B \le \left(\frac{g-1}{2k!}\right)^{1/\lfloor k/2 \rfloor}$ satisfying property P provides the k-power palindrome $\left(g^L + \sum_{b\in B}(g^b + g^{L-b}) + 1\right)^k$.

Next, we estimate from below the number of subsets $B \subset \{1, \ldots, H\}$, with cardinality $t = \lfloor \left(\frac{g-1}{2k!}\right)^{1/\lfloor k/2 \rfloor} \rfloor$ satisfying property P.

We observe that B doesn't satisfies property P if there exist disjoint multisets U_1 and U_2 with elements in B such that $\sum_{b\in U_1} b - \sum_{b\in U_2} b = jL$, for some $j \in [-k, k]$.

In the first step, we choose any element $b_1 \in B$ from $\{1, \ldots, H\}$, except those elements of B which are in the form $L, L/2, \ldots, L/k$. If such an element cannot be choen, then B cannot satisfy property P.

Assume that $r \in \{1, \ldots, t-1\}$, and $b_1, \ldots, b_r$ have been chosen. We take b_{r+1} to be any of the elements of $\{1, \ldots, H\}$ except for the previous ones, and those elements x such that there exists disjoint multisets U_1 and U_2 destroying property P, one of them containing x. Since the number of exceptions depends on t and k, but not on H, we have that once $b_1, \ldots, b_r$ are chosen, we have $H + O_{k,t}(1)$ possibilities for b_{r+1}. Thus, the number of such sets of cardinality t chosen in this way is $(H + O_{k,t}(1))^t$. But, since the same set can be ordered in $t!$ different ways, we have that the number of sets B satisfying property P is $\ge (H + O_{t,k}(1))^t/t! \gg L^t$, as $L \to \infty$.

Finally, it easy to check that for $g > 2^{\lfloor k/2 \rfloor + 1}k! + 1$, we have that $t \ge cg^{1/\lfloor k/2 \rfloor}$, where $c = \frac{1}{2}(4k!)^{-1/\lfloor k/2 \rfloor}$. □

Certainly, obtaining tighter lower and upper bounds on $\#\mathcal{Q}_{g,L}$ is an interesting open question.

On the other hand, we have not been able to produce a similar explicit construction of k-powers palindromes for $g \leq k!$. In particular we don't know if there are infinitely many squares among binary palindromes (see also [5], where this questions has also been mentioned).

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