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## TRIGONOMETRIC POLYNOMIALS AND LATTICE POINTS

J. CILLERUELO AND A. CORDOBA

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**ABSTRACT.** In this paper we study the distribution of lattice points on arcs of circles centered at the origin. We show that on such a circle of radius  $R$ , an arc whose length is smaller than  $\sqrt{2}R^{1/2-1(4[m/2]+2)}$  contains, at most,  $m$  lattice points. We use the same method to obtain sharp  $L^4$ -estimates for uncompleted, Gaussian sums

### I. INTRODUCTION AND STATEMENT OF RESULTS

Let us denote by  $r(n)$  the number of representations of the integer  $n$  as a sum of two squares, i.e.,  $r(n)$  is the number of lattice points on the circle  $x^2 + y^2 = n$ . This function plays an important role in the arithmetic of Gaussian integers and it satisfies the relation  $r(n) = 4 \sum_{d|n} \chi_1(d)$ , where  $\chi_1$  is the nonprincipal character (mod 4).

It is a well-known fact that  $r(n) = O(n^\varepsilon)$  for every  $\varepsilon > 0$ , and that  $r(n)$  is not  $O((\log n)^c)$  for any  $c$ . However the distribution of values of  $r(n)$  is rather irregular. The function  $R(n) = \sum_{k \leq n} r(k)$  has also been considered, and it was observed by Gauss that  $R(n) = \pi n + E(n)$ , with  $E(n) = O(n^{1/2})$ . To find the true order of magnitude of the error term  $E(n)$  is an outstanding problem in the number theory, i.e., the well-known lattice point problem.

There is a result attributed to Heron of Alexandria which says: in any triangle, the product of the lengths of its three sides is equal to four times the area of the triangle multiplied by the radius of the circumscribed circle:  $abc = 4\Delta R$ . This theorem has the following application: If  $\nu_1, \nu_2, \nu_3$  are three lattice points on the circle  $x^2 + y^2 = R^2$ , then  $\max\{\|\nu_1 - \nu_2\|, \|\nu_1 - \nu_3\|, \|\nu_2 - \nu_3\|\} \geq 2R^{1/3}$ . In particular, an arc of length  $2R^{1/3}$  contains, at most, two lattice points. This fact was first observed by Schinzel and used by Zygmund [1] to prove a Cantor-Lebesgue theorem in two variables.

Therefore it is a natural question to ask for which exponents  $\alpha$  there exists a finite constant  $N_\alpha$  satisfying the condition that any arc of length  $R^\alpha$ , in a circle centered at the origin and radius  $R$ , contains at most  $N_\alpha$  points uniformly in  $R$ .

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In this paper a new method is introduced to prove that the answer to this question is affirmative for every  $\alpha < \frac{1}{2}$ . More precisely, we have

**Theorem 1.** *On a circle of radius  $R$  centered at the origin, an arc  $\Gamma$  whose length is not greater than  $\sqrt{2}R^{1/2-1/(4\lfloor m/2\rfloor+2)}$ , contains, at most,  $m$  lattice points.*

Gaussian sums are an important object in number theory. It is known that

$$\left\| \sum_{N \leq k \leq 2N} e^{2\pi i k^2 x} \right\|_{L^p[0,1]} \simeq N^{1/2} \quad \text{for } 1 \leq p < 4$$

and that the weak- $L^4$  estimate also holds:

$$\mu \left\{ x : \left| \sum_{N \leq k \leq 2N} e^{2\pi i k^2 x} \right| \geq \alpha > 0 \right\} \leq C \frac{N^2}{\alpha^4}$$

where  $C < \infty$  is some universal constant and  $\mu$  denotes Lebesgue measure.

In the other direction, we have the equivalence

$$\left\| \sum_{N \leq k \leq 2N} e^{2\pi i k^2 x} \right\|_{L^4[0,1]} \simeq N^{1/2} (\log N)^{1/4}, \quad N \rightarrow \infty.$$

It is not difficult to observe that

$$\left\| \sum_{N \leq k \leq N+N^\alpha} e^{2\pi i k^2 x} \right\|_{L^4[0,1]} = (2N^{2\alpha})^{1/4} + O(1) \quad \text{for every } \alpha \leq \frac{1}{2}.$$

The method introduced in the proof of Theorem 1 allows us to provide estimates of this type for every  $\alpha$ ,  $\frac{1}{2} < \alpha < 1$ .

**Theorem 2.**

$$\int_0^1 \left| \sum_{N \leq k \leq N+N^\alpha} e^{2\pi i k^2 x} \right|^4 dx = 2N^{2\alpha} + O(N^{3\alpha-1+\varepsilon})$$

for every  $\varepsilon > 0$  and  $\alpha$ ,  $\frac{1}{2} < \alpha < 1$ .

## II. PROOF OF THEOREM 1

**A. Preliminary remarks and notation.** Let us recall the known fact that  $r(n) = 4 \sum_{d|n} \chi_1(d) = 4\{d_1(n) - d_3(n)\}$ , where  $d_j$ ,  $j = 1, 3$ , denotes the number of divisors of  $n$  which are conjugent with  $j$  module 4. The factor 4 takes into account the symmetry of the lattice.

If  $n = 2^\nu \prod_{p_j \equiv 1(4)} p_j^{\alpha_j} \prod_{q_k \equiv 3(4)} q_k^{\beta_k}$  is the prime factorization of the integer  $n$ , then  $r(n) = 0$  unless all the exponents  $\beta_k$  are even. In that case we have  $r(n) = 4 \prod (1 + \alpha_j)$ .

We also have the composition formula given by multiplication in the ring of Gaussian integers: If  $a^2 + b^2 = n$  and  $c^2 + d^2 = m$ , then  $(ac - bd)^2 + (ad + bc)^2 = mn$ . That is, in terms of norms we have  $N(a + bi) = n$  and  $N(c + di) = m$  imply that  $N((a + bi)(c + di)) = mn$ . Furthermore, if  $(m, n) = 1$  then each

representation of  $mn$  as a sum of two squares arises from decompositions of  $m$  and  $n$  given by the formula above. Therefore we shall associate lattice points with Gaussian integers:  $a^2 + b^2 = n$  determines a Gaussian integer  $a + bi = \sqrt{n}e^{2\pi i\Phi}$  for a suitable phase  $\Phi$ .

We have

(i)  $r(2^\nu) = 4$  and there are two possible situations, namely,  $\nu$  is even, in which case  $2^\nu = 0 + (\pm 2^{\nu/2})^2 = (\pm 2^{\nu/2})^2 + 0$  or  $\nu$  is odd, and then  $2^\nu = (\pm 2^{[\nu/2]})^2 + (\pm 2^{[\nu/2]})^2$ .

That is, the Gaussian integers corresponding to the four representations of  $2^\nu$  as a sum of two squares are the following:

$$2^{\nu/2}e^{2\pi i\{\Phi_0+t/4\}}, \quad t = 0, 1, 2, 3, \quad \text{where } \Phi_0 = \begin{cases} 0 & \text{if } \nu \text{ is even,} \\ \frac{1}{8} & \text{if } \nu \text{ is odd.} \end{cases}$$

(ii) If  $q \equiv 3(4)$  is a prime number, then  $r(q^{2\beta}) = 4$  and the representations are given by

$$q^{2\beta} = 0^2 + (\pm q^\beta)^2 = (\pm q^\beta)^2 + 0^2,$$

i.e., they correspond to the Gaussian integers  $q^\beta e^{2\pi it/4}$ ,  $t = 0, 1, 2, 3$ .

(iii) If  $p \equiv 1(4)$  is prime, then  $r(p^\alpha) = 4(1 + \alpha)$ .

Furthermore if  $p = a^2 + b^2$ ,  $a + bi = \sqrt{p}e^{2\pi i\Phi}$ ,  $a, b > 0$ , then the representations of  $p^\alpha$  as a sum of two squares are given by  $p^{\alpha/2}e^{2\pi i\{\gamma\Phi+t/4\}}$  where  $t = 0, 1, 2, 3$  and  $\gamma$  describes the set  $\Lambda_\alpha = \{\gamma \in \mathbb{Z}, |\gamma| \leq \alpha, \gamma \equiv \alpha \pmod{2}\}$ .

(iv) We can summarize the previous observations in the following manner. If  $n = 2^\nu \prod_{p_j \equiv 1(4)} p_j^{\alpha_j} \prod_{q_k \equiv 3(4)} q_k^{2\beta_k}$  then the Gaussian integers corresponding to the  $4 \prod (1 + \alpha_j)$  representations of  $n$  as a sum of two squares are given by the formula

$$(*) \quad \sqrt{ne}^{2\pi i\{\Phi_0 + \sum_j \gamma_j \Phi_j + t/4\}}, \quad t = 0, 1, 2, 3,$$

where  $\gamma_j$  describes the set  $\Lambda_{\alpha_j}$  and  $\Phi_0 = 0$  or  $\frac{1}{8}$  depending upon the parity of  $\nu$ .

*Remark.* The angles  $\Phi_j$  corresponding to different primes  $p_j \equiv 1(4)$  are linearly independent over the rationals, i.e., a relation  $\sum_{1 \leq j \leq N} a_j \Phi_j + a_0 = 0$ , with coefficients  $a_j \in \mathbb{Q}$ , implies necessarily that  $a_0 = a_1 = \dots = a_N = 0$ .

**B. End of the proof.** Let us suppose that for the integer

$$n_0 = 2^\nu \prod_{p_j \equiv 1(4)} p_j^{\alpha_j} \prod_{q_k \equiv 3(4)} q_k^{2\beta_k}$$

there is an arc, in the circle of radius  $R_0 = \sqrt{n_0}$  centered at the origin, which contains more than  $m$  points and whose length is  $\sqrt{2}R_0^\alpha$ . Then, formula (\*) implies that the same must be true for the circle of radius  $R = \sqrt{n}$ , where  $n = \prod_j p_j^{\alpha_j}$ . That is, we may assume the existence of  $m+1$  lattice points in such an arc of this circle. They will have representations  $\sqrt{ne}^{2\pi i\{\sum_s \gamma_s^s \Phi_s + t^s/4\}}$ ,  $s = 1, 2, \dots, m+1$ ,  $\gamma_j^s \in \{\gamma \in \mathbb{Z}, |\gamma| \leq \alpha_j, \gamma \equiv \alpha_j(2)\}$ , and  $t^s \in \{0, 1, 2, 3\}$ .

For each pair  $s \neq s'$  of such points, let us consider the quantity

$$\begin{aligned} \Psi^{s,s'} &= \sum_j \{ \gamma_j^s - \gamma_j^{s'} \} + \frac{t^s - t^{s'}}{4} \\ &= 2 \left\{ \sum_j \Phi_j \frac{\gamma_j^s - \gamma_j^{s'}}{2} + \frac{t^s - t^{s'}}{8} \right\} \end{aligned}$$

and observe that the  $\gamma_j^{s,s'} = (\gamma_j^s - \gamma_j^{s'})/2$  take always integer values.

We have two cases to consider, depending upon the parity of  $t^s$  and  $t^{s'}$  :

- (1) if  $t^s \equiv t^{s'}(2)$ , then  $(t^s - t^{s'})/8 = t^{s,s'}/4$  for some integer  $t^{s,s'}$  ;
- (2) if  $t^s \not\equiv t^{s'}(2)$ , then  $(t^s - t^{s'})/8 = \frac{1}{8} + t^{s,s'}/4$  for some integer  $t^{s,s'}$  .

In the first case, formula (\*) shows that  $\Psi^{s,s'}/2$  is the angle corresponding to a representation of  $\prod_j p_j^{|\gamma_j^{s,s'}|}$  as a sum of two squares.

The second case corresponds to a representation of  $2 \prod_j p_j^{|\gamma_j^{s,s'}|}$ .

If  $\Psi^{s,s'}/2$  is an integer, then the linear independence of  $\{1, \Phi_1, \Phi_2, \dots\}$  over the rationals, implies that  $t^s = t^{s'}$  and  $\gamma_j^{s,s'} = 0$  for every  $j$ , so that finally  $s = s'$ . Thus we have the following result. If  $s \neq s'$  then  $||| \Psi^{s,s'}/2 ||| > 0$ , where  $||| \cdot |||$  denotes distance to the integers.

Our previous considerations show that if  $s \neq s'$ , then  $\Psi^{s,s'}/2$  is the angle of a lattice point not on the  $x$  axis, but on the circle of radius

$$2^{\nu/2} \prod_j p_j^{|\gamma_j^{s,s'}|/2} \quad \text{where } \nu = 0 \text{ or } \nu = 1.$$

Therefore we have

$$||| \frac{\Psi^{s,s'}}{2} ||| > \frac{1}{2\pi\sqrt{2} \prod_j p_j^{|\gamma_j^{s,s'}|/2}}$$

by a simple argument.

On the other hand, our hypothesis on the location of the  $m + 1$  lattice points on an arc of length  $\sqrt{2}R^\alpha$ , implies the following inequality:

$$||| \frac{\Psi^{s,s'}}{2} ||| \leq \frac{1}{2\pi\sqrt{2}} R^{\alpha-1}.$$

We have  $m(m + 1)/2$  couples  $s, s'$ . Multiplying all these inequalities, we get

$$R^{(\alpha-1)m(m+1)/2} > \frac{1}{\prod_{s,s'} \prod_j p_j^{|\gamma_j^{s,s'}|}} = \frac{1}{\left( \prod_j p_j^{\sum_{s,s'} |\gamma_j^{s,s'}|} \right)^{1/2}}.$$

Next we want to estimate the maximum value of the sum

$$\sum_{s,s'} |\gamma_j^{s,s'}| = \frac{1}{2} \sum_{s,s'} |\gamma_j^s - \gamma_j^{s'}|$$

where, as was stated before,  $\gamma_j^s$  takes values in the set  $\{\gamma \in \mathbb{Z}, |\gamma| \leq \alpha_j, \gamma \equiv \alpha_j(2)\}$ .

If  $m + 1$  is even, the maximum value is obtained when  $(m + 1)/2$  of the numbers  $\gamma_j^s$  are equal to  $\alpha_j$  and the remaining  $(m + 1)/2$  are  $-\alpha_j$ .

If  $m + 1$  is odd, then the maximum takes place when  $(m + 2)/2$  of the  $\gamma_j^s$  are equal to  $\alpha_j$  and the remaining  $m/2$  are  $-\alpha_j$ . Therefore,

$$\sum_{s, s'} |\gamma_j^{s, s'}| \leq \alpha_j \frac{(m + 1)^2 - \delta(m)}{4},$$

where

$$\delta(m) = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ 1 & \text{if } m \text{ is even.} \end{cases}$$

We obtain

$$R^{(\alpha-1)m(m+1)/2} > \left( \prod_j P_j^{\alpha_j((m+1)^2 - \delta(m))/4} \right)^{-1/2} = R^{-((m+1)^2 - \delta(m))/4}$$

which yields

$$\alpha > 1 - \frac{(m + 1)^2 - \delta(m)}{2m(m + 1)} = \frac{1}{2} - \frac{1}{4[m/2] + 2}$$

and this completes the proof of Theorem 1.

Here, as in the rest of the paper, we have used the standard notation  $[x]$  to denote the integer part of the real number  $x$ .

### III. PROOF OF THEOREM 2

Let  $r_\alpha(n) = \text{Card}\{(h, k) \in Z^2: n = k^2 + h^2, N \leq k, h \leq N + N^\alpha\}$ . Let  $C_j = \{n: r_\alpha(n) = j\}$ . Then

$$\begin{aligned} \int_0^1 \left| \sum_{N \leq k \leq N + N^\alpha} e^{2\pi i k^2 x} \right|^4 dx &= \sum_n r_\alpha^2(n) = \sum_j j^2 \text{Card}(C_j) \\ &= \text{Card}(C_1) + 4 \text{Card}(C_2) + \sum_{j \geq 3} j^2 \text{Card}(C_j). \end{aligned}$$

Observe that  $\text{Card}(C_1) \leq [N^\alpha]$ , because the lattice points corresponding to points of  $C_1$  necessarily lie on the diagonal, i.e.,  $h = k$ .

Our next step is to compute  $\sum_{j \geq 3} j^2 \text{Card}(C_j)$ .

Let  $n \in \bigcup_{j \geq 3} C_j$ . Then  $n$  admits two different representations as a sum of two squares which are not placed symmetrically with respect to the diagonal of the first quadrant.

Let  $\sqrt{ne}^{2\pi i \Phi^1}$ ,  $\sqrt{ne}^{2\pi i \Phi^2}$  be the Gaussian integers corresponding to such points, where

$$\Phi^s = \sum_j \gamma_j^s \Phi_j + \frac{t^s}{4}, \quad s = 1, 2,$$

and we have used the notation of the previous theorem.

We have

$$\frac{(\Phi^1 \pm \Phi^2)}{2} = \sum_j \Phi_j \frac{(\gamma_j^1 \pm \gamma_j^2)}{2} + \frac{(t^1 \pm t^2)}{8},$$

$$\left| \frac{\Phi^1 - \Phi^2}{2} \right| < \frac{1}{4\pi} \left\{ \operatorname{arctg} \frac{N + N^\alpha}{N} - \operatorname{arctg} \frac{N}{N + N^\alpha} \right\} < \frac{1}{2\pi} N^{\alpha-1},$$

$$\left| \frac{(\Phi^1 + \Phi^2)}{2} - \frac{1}{8} \right| < \frac{1}{4\pi} \left\{ \operatorname{arctg} \frac{N + N^\alpha}{N} - \operatorname{arctg} \frac{N}{N + N^\alpha} \right\} < \frac{1}{2\pi} N^{\alpha-1}.$$

(i) If  $t^1 - t^2$  is even, then  $(\Phi^1 - \Phi^2)/2$  and  $(\Phi^1 + \Phi^2)/2$  are angles corresponding to representations of integers  $d_1, d_2$ , respectively, such that  $d_1 d_2 = n$ ,

$$d_1 = a_1^2 + b_1^2, \quad 0 < b_1 < N^{\alpha-1} a_1,$$

$$d_2 = a_2^2 + b_2^2, \quad 0 < |a_2 - b_2| < N^{\alpha-1} a_2.$$

Observe that  $a_2 = b_2$  implies  $(\Phi^1 + \Phi^2)/2 - \frac{1}{8} = 0$ , i.e., the two representations of  $n$  are symmetric with respect to the diagonal;  $b_1 = 0$  implies  $(\Phi^1 + \Phi^2)/2 = 0$ , i.e., the two representations of  $n$  are the same.

(ii) If  $t^1 - t^2$  is odd, then  $(\Phi^1 - \Phi^2)/2 + \frac{1}{8}$ ,  $(\Phi^1 + \Phi^2)/2 + \frac{1}{8}$  are angles corresponding to representations of  $d_1, d_2$  such that  $d_1 d_2 = n$ ,

$$d_1 = a_1^2 + b_1^2, \quad 0 < |a_1 - b_1| < N^{\alpha-1} a_1,$$

$$d_2 = a_2^2 + b_2^2, \quad 0 < b_2 < N^{\alpha-1} a_2.$$

In any case, given  $n \in \bigcup_{j \geq 3} C_j$ , we have a decomposition  $n = d_1 d_2$  such that

$$d_1 = a_1^2 + b_1^2, \quad 0 < b_1 < N^{\alpha-1} a_1,$$

$$d_2 = a_2^2 + b_2^2, \quad 0 < |a_2 - b_2| < N^{\alpha-1} a_2.$$

The number of pairs  $d_1, d_2$  satisfying the conditions above gives an upper bound for  $\operatorname{Card}(\bigcup_{j \geq 3} C_j)$ .

With  $d_1$  fixed we estimate the cardinality of the set of numbers  $d_2$  such that  $d_2 = a_2^2 + b_2^2$ ,  $0 < |a_2 - b_2| < N^{\alpha-1} a_2$ ,  $\sqrt{2}N < \sqrt{d_1 d_2} < \sqrt{2}(N + N^\alpha)$ , i.e.,

$$\frac{\sqrt{2}N}{\sqrt{d_1}} < \sqrt{a_2^2 + b_2^2} < \frac{\sqrt{2}N}{\sqrt{d_1}} (N + N^\alpha).$$

To do it we observe that if we fix  $a_2$ , then there are  $O(N^{\alpha-1} a_2)$  integers  $b_2$  such that  $|a_2 - b_2| < N^{\alpha-1} a_2$  and since  $\sqrt{a_2^2 + b_2^2} \simeq \sqrt{2} a_2 \simeq \sqrt{2} N / \sqrt{d_1}$ , we have that for  $d_1$  fixed, there are at most  $N^{\alpha-1} (N / \sqrt{d_1}) (N^\alpha / \sqrt{d_1}) = N^{2\alpha} / d_1$  integers  $d_2$  satisfying that set of inequalities. Therefore,

$$\operatorname{Card} \left\{ \bigcup_{j \geq 3} C_j \right\} < \sum_{a_1 < N} \sum_{0 < b_1 < N^{\alpha-1} a_1} \frac{N^{2\alpha}}{a_1^2 + b_1^2} < N^{3\alpha-1} \log N.$$

On the other hand, clearly

$$\sum_n r_\alpha(n) = \operatorname{Card}\{(k, h) : N \leq k, h \leq N + N^\alpha\} = N^{2\alpha} + O(N^\alpha)$$

and

$$\sum_n r_\alpha(n) = \sum_j \text{Card}(C_j) = \text{Card}(C_1) + 2 \text{Card}(C_2) + O\left(N^\varepsilon \text{Card}\left(\bigcup_{j \geq 3} C_j\right)\right)$$

for every  $\varepsilon > 0$ .

Thus

$$\text{Card}(C_2) = \frac{N^{2\alpha}}{2} + O(N^\alpha + N^{3\alpha-1+\varepsilon}) \quad \text{for every } \varepsilon > 0$$

which yields the theorem.  $\square$

*Remarks.* Theorem 1 was motivated by the search for a direct connection between the restriction properties of Fourier integrals and those of Fourier series coefficients to circles. We have some evidence that  $\alpha = \frac{1}{2}$  is sharp, but we have no proof.

It would be interesting to obtain a version of Theorem 2 with coefficients: is

$$\left\| \sum_{N \leq k \leq N+N^\alpha} a_k e^{2\pi i k^2 x} \right\|_4 \simeq \left( \sum |a_n|^2 \right)^{1/2}$$

uniformly in  $N$ ?

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