UPPER AND LOWER BOUNDS FOR FINITE $B_h[g]$ SEQUENCES

JAVIER CILLERUELO, IMRE Z. RUZSA AND CARLOS TRUJILLO

Abstract. We give a non trivial upper bound, $F_h(g,N)$, for the size of a $B_h[g]$ subset of $\{1,....,N\}$ when $g>1$. In particular, we prove

$$F_2(g,N) \le 1.864(gN)^{1/2}+1$$

$$F_h(g,N) \le \frac{1}{(1+\cos^h(\pi/h))^{1/h}}(hh!gN)^{1/h}, \quad h>2.$$

On the other hand we exhibit $B_2[g]$ subsets of $\{1,....,N\}$ with

$$\frac{g+[g/2]}{\sqrt{g+2[g/2]}}N^{1/2}+o(N^{1/2}) \qquad \text{elements.}$$

1. Upper bounds

Let $h\ge 2, g\ge 1$ be integers. A subset A of integers is called a $B_h[g]$-sequence if for every positive integer m, the equation

$$m = x_1+\cdots+x_h, \qquad x_1\le\cdots\le x_h, \qquad x_i\in A$$

has, at most, g distinct solutions.

Let $F_h(g,N)$ denote the maximum size of a $B_h[g]$ sequence contained in $[1,N]$. If A is a $B_h[g]$ subset of $\{1,...,N\}$, then $\binom{|A|+h-1}{h}\le ghN$, which implies the trivial upper bound

$$F_h(g,N)\le (ghh!N)^{1/h} \tag{1.1}$$

For $g=1$, $h=2$, it is possible to take advantage of counting the differences x_i-x_j instead of the sums x_i+x_j, because the differences are all distinct. In this way, P. Erdős and P. Turán [2] proved that $F_2(1,N)\le N^{1/2}+O(N^{1/4})$, which is the best possible except for the estimate of error term.

For $h=2m$, Jia [4] proved $F_{2m}(1;N)\le (m(m!)^2)^{1/2m}N^{1/2m}+O(N^{1/4m})$. A similar upper bound for $F_{2m-1}(1,N)$ has been proved independently by S.Chen [1] and S.W.Graham [3]: $F_{2m-1}(1,N)\le ((m!)^2)^{1/2m-1}N^{1/2m-1}+O(N^{1/4m-2})$.

However, for $g>1$, the situation is completely different because the same difference can appear many times, and, for $g>1$ nothing better than (1.1) is known. In this paper we improve this trivial upper bound.

Theorem 1.1.

$$F_2(g,N)\le 1.864(gN)^{1/2}+1$$

$$F_h(g,N)\le \frac{1}{(1+\cos^h(\pi/h))^{1/h}}(hh!gN)^{1/h}, \quad h>2$$

Proof. Let $A \subset [1, N]$ a $B_h[g]$ sequence. $|A| = k$. Put $f(t) = \sum_{a\in A} e^{iat}$. We have $f(t)^h = \sum_{n=h}^{hN} r_h(n)e^{int}$ where $r_h(n) = \#\{n = a_1 + \cdots + a_h; a_i \in A\}$

$$f(t)^h = h!g \sum_{n=h}^{hN} e^{int} - \sum_{n=h}^{hN}(h!g - r_h(n))e^{int} = h!gp(t) - q(t)$$

Since $r_h(n) \le h!g$, we have

$$\sum_{n=h}^{hN} |h!g - r_h(n)| = \sum_{n=h}^{hN}(h!g - r_h(n)) =$$

$$= (h(N-1)+1)h!g - \sum r_h(n) = (h(N-1)+1)h!g - k^h$$

thus

$$|q(t)| \le hh!gN - k^h$$

for every value of t.

$p(t)$ is just a geometrical series and we can express it as

$$p(t) = e^{hit}\frac{1 - e^{i(h(N-1)+1)t}}{1 - e^{it}}$$

if $0 < t < 2\pi$. We shall use only the property that at values of the form $t = jt_h$, $t_h = \frac{2\pi}{h(N-1)+1}$ with integer j, $1 \le j \le h(N-1)$, we have $p(t) = 0$, thus $f(t)^h = q(t)$. Consequently

$$|f(jt_h)| \le (hh!gN - k^h)^{1/h} \text{ for any integer } j, \quad 1 \le j \le h(N-1).$$

Since the midpoint of the interval $[1, N]$ is $(N+1)/2$, it will be useful to express f as

$$f(t) = \exp\left(\frac{N+1}{2}it\right) f^*(t),$$

where

$$f^*(t) = \sum_{a\in A} \exp\left(\left(a - \frac{N+1}{2}\right)it\right).$$

Now we consider a function $F(x) = \sum_{j=1}^{h(N-1)} b_j \cos(jx)$ satisfying $F(x) \ge 1$ for $|x| \le \pi/h$. We define $C_F = \sum |b_j|$.

We are looking for a lower and an upper bound for $Re\left(\sum_{j=1}^{h(N-1)} b_j f^*(jt_h)\right)$.

$$Re\left(\sum_{j=1}^{h(N-1)} b_j f^*(jt_h)\right) \le \sum_{j=1}^{h(N-1)} |b_j||f^*(jt_h)| = \sum_{j=1}^{h(N-1)} |b_j||f(jt_h)| \le \tag{1.2}$$

$$\le \left(\sum_{j=1}^{h((N-1)} |b_j|\right) (hh!gN - k^h)^{1/h} = C_F(hh!gN - k^h)^{1/h}.$$

On the other hand

$$Re\left(\sum_{j=1}^{h(N-1)} b_j f^*(jt_h)\right) = Re\left(\sum_{a\in A}\sum_{j=1}^{h(N-1)} b_j e^{i(a-\frac{N-1}{2})t_h j}\right) = \tag{1.3}$$

$$= \sum_{a\in A} \sum_{j=1}^{h(N-1)} b_j \cos\left(\left(a - \frac{N-1}{2}\right) t_h j\right) = \sum_{a\in A} F\left(\left(a - \frac{N-1}{2}\right) t_h\right) \geq k,$$

because $|(a - \frac{N-1}{2})t_h| \leq \pi/h$ for any integer $a \in A$.
From (1.2.) and (1.3.) we have

$$|A| = k \leq \frac{1}{(1 + \frac{1}{C_F^h})^{1/h}} (hh!gN)^{1/h}.$$

For $h > 2$, we take $F(x) = \frac{1}{\cos(\pi/h)} \cos(x)$, with $C_F = \frac{1}{\cos(\pi/h)}$ and this proves the theorem for $h > 2$.

For $h = 2$, we can take $F(x) = 2\cos(x) - \cos(2x)$, $C_F = 3$, which gives $|A| \leq \frac{6}{\sqrt{10}}\sqrt{gN}$, a nontrivial upper bound.

However, an infinite series gives a better result. Take the function

$$F(x) = \begin{cases} 1, & |x| \leq \pi/2 \\ 1 + \pi\cos(x), & \pi/2 < |x| \leq \pi. \end{cases}$$

It is easy to see that

$$F(x) = \frac{\pi}{2}\cos(x) + 2\sum_{n=2}^{\infty} \frac{\cos(\pi n/2)}{n^2 - 1} \cos(nx).$$

This series satisfies the following: $F(x) = 1$ for $|x| \leq \pi/2$ with

$$C_F = \pi/2 + 2\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \pi/2 + 2\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \pi/2 + 1.$$

However we must truncate the series to the integers $n \leq 2(N-1)$. Let

$$F_T(x) = \frac{\pi}{2}\cos(x) + 2\sum_{n=2}^{2(N-1)} \frac{\cos(\pi n/2)}{n^2 - 1} \cos(nx).$$

Observe that

$$|F_T(x) - F(x)| \leq 2\sum_{2N-1}^{\infty} \frac{1}{n^2 - 1} = \frac{1}{2N - 2}.$$

Now we consider the polynomial $F^*(x) = \frac{2N-2}{2N-3} F_T(x)$. If $|x| < \pi/2$ we have

$$|F^*(x)| = \frac{2N-2}{2N-3}|F(x) + F_T(x) - F(x)| \geq \frac{2N-2}{2N-3}\left(|F(x)| - |F_T(x) - F(x)|\right) \geq$$

$$\geq \frac{2N-2}{2N-3}\left(1 - \frac{1}{2N-2}\right) = 1$$

and $C_{F^*} \leq \frac{2N-2}{2N-3}(\pi/2 + 1)$.
Thus

$$|A| = k \leq \frac{2}{\left(1 + \frac{1}{C_{F^*}^2}\right)^{1/2}} (gN)^{1/2}.$$

A simple calculation gives

$$\frac{2}{\left(1+\frac{1}{C_{F^*}^2}\right)^{1/2}} - \frac{2}{\left(1+\frac{1}{C_F^2}\right)^{1/2}} \leq \frac{1}{N}.$$

Then

$$|A| = k \leq \frac{2}{\left(1+\frac{1}{C_F^2}\right)^{1/2}}(gN)^{1/2} + \sqrt{\frac{g}{N}} =$$

$$= \frac{2\pi+4}{\sqrt{\pi^2+4\pi+8}}\sqrt{gN} + \sqrt{\frac{g}{N}} \leq 1.864\sqrt{gN}+1$$

because, obviously, $g \leq N$.

□

2. Lower bounds

Now we are interested in finite $B_2[g]$ sequences as dense as possible. Kolountzakis [6] exhibits a $B_2[2]$ subset of $\{1,...,N\}$ with $\sqrt{2}N^{1/2}+o(N^{1/2})$ elements taking $A=(2A_0)\cup(2A_0+1)$ with A_0 a $B_2[1]$ sequence contained in $\{1,...[N/2]\}$.

In general it is easy to construct a $B_2[g]$ subset of $\{1,...,N\}$ with $(gN)^{1/2}+o(N^{1/2})$ elements. In the sequel we improve these results

Theorem 2.1.

$$F_2(g,n) \geq \frac{g+[g/2]}{\sqrt{g+2[g/2]}}N^{1/2}+o(N^{1/2}). \tag{2.1}$$

For $g=2$ theorem 2 gives

$$F_2(2,N) \geq \frac{3}{2}N^{1/2}+o(N^{1/2}).$$

In general, for g even we get

$$F_2(g,N) \geq \frac{3}{2\sqrt{2}}(gN)^{1/2}+o(N^{1/2}).$$

And for g odd,

$$F_2(g,N) \geq \frac{3-(1/g)}{2\sqrt{2-(1/g)}}(gN)^{1/2}+o(N^{1/2})$$

Remark. Jia's constructions of $B_h(g)$ sequences in [5] does not work (Jia, personal communication). In the last step of the proof of theorem 3.1. of [5] we cannot deduce from the hypothesis that $\{b_{s1},\dots,b_{sh}\}=\{b_{t1},\dots,b_{th}\}$. Jia's argument can be modified if we define $g_a(h,m)$ as the number of solutions of the equation $a\equiv x_1+\cdots+x_h \pmod m$, $\quad 0\leq x_i\leq m-1$. It would imply the result $|B|=\sqrt{gN}+o(\sqrt{N})$. But for $g=2$ it is the Kolountzakis's construction [6].

We need some definitions and lemmas in order to construct $B_2[g]$ sequences satisfying Theorem 2.1.

Definition 2.1. We say that $a_0,a_1,...,a_k$ satisfies the $B^*[g]$ condition if the equation $a_i+a_j=r$ has at most g solutions. (Here, $a_i+a_j=a_j+a_i$ counts as two solutions if $i\neq j$).

Definition 2.2. We say that a sequence of integers C is a $B_2 \pmod m$ sequence if $c_i + c_j \equiv c_k + c_l \pmod m$ implies $\{c_i, c_j\} = \{c_k, c_l\}$.

Lemma 2.2. *If $a_0, a_1, .., a_k$ satisfies the $B^*[g]$ condition, and C is a $B_2 \pmod m$ sequence, then the sequence $B = \cup_{i=0}^{k}(C + ma_i)$ is a $B_2[g]$ sequence.*

Proof. If $b_1 + b_1' = b_2 + b_2' = ... = b_{g+1} + b_{g+1}'$, $\quad b_j, b_j' \in B$ we can write
$b_j = c_j + a_{i_j} m$
$b_j' = c_j' + a_{i_j}' m$, $\quad c_j, c_j' \in C$, $a_{i_j}, a_{i_j}' \in \{a_0, .., a_k\}$ where we have ordered the pairs b_j, b_j' such that $c_j \le c_j'$.

Then we have $c_j + c_j' \equiv c_k + c_k' \pmod m$ for all j, k, which implies $c_j = c_k$, $c_j' = c_k'$.

On the other hand, all the $g+1$ sums $a_{i_j} + a_{i_j'}$ are equal. Thus there exists j, k such that $a_{i_j} = a_{i_k}$, $a_{i_j'} = a_{i_k'}$

Then, for these j, k, we have $b_j = b_k$ and $b_j' = b_k'$. ☐

Lemma 2.3. *The subset*

$$A^g = A_1^g \cup A_2^g = \{k; \quad 0 \le k \le g-1\} \cup \{g - 1 + 2k; \quad 1 \le k \le [g/2]\}$$

satisfies the condition $B^*[g]$.

Proof. Let

$$r(m) = \#\{a; \quad a, m - a \in A^g\}$$

$$r_{ij}(m) = \#\{a; \quad a \in A_i^g, m - a \in A_j^g\}, \qquad 1 \le i, j \le 2$$

We have $r(m) = r_{11}(m) + 2r_{12}(m) + r_{22}(m)$, because $r_{12} = r_{21}$

With this notation we will prove that $r(m) \le g$ for any integer m. First we study the functions r_{ij}.

• $r_{11}(m)$

If $a, m - a \in A_1^g$, then $0 \le a \le g - 1$ and $0 \le m - a \le g - 1$, which implies

$$\max\{0, m - g + 1\} \le a \le \min\{g - 1, m\}.$$

Then

$$r_{11}(m) = \max\{0, \min\{g - 1, m\} - \max\{0, m - g + 1\} + 1\},$$

and

$$r_{11}(m) = \begin{cases} m + 1, & 0 \le m \le g - 1 \\ 2g - m - 1, & g \le m \le 2g - 1 \\ 0, & 2g - 1 \le m \end{cases}$$

• $r_{12}(m)$

If $a \in A_2^g$, $m - a \in A_1^g$, then $a = g - 1 + 2k$, $1 \le k \le [g/2]$ and

$$0 \le m - (g - 1 + 2k) \le g - 1, \qquad \text{which implies}$$

$$\max\{1, \frac{m - 2g + 2}{2}\} \le k \le \min\{[g/2], \frac{m - g + 1}{2}.\}$$

Since the k's are integers, we can write

$$\max\{1, [\frac{m - 2g + 3}{2}]\} \le k \le \min\{[g/2], [\frac{m - g + 1}{2}]\}.$$

Then

$$r_{12}(m) = \begin{cases} 0, & m \le g \\ [\frac{m-g+1}{2}], & g \le m \le 2g-1 \\ [\frac{g}{2}] - [\frac{m-2g+1}{2}], & 2g \le m \le 3g-1 \\ 0, & 3g-1 \le m \end{cases}$$

• $r_{22}(m)$

Obviously, if m is odd then $r_{22}(m) = 0$.

If $a, m-a \in A_2^g$, then $a = g-1+2k$, $m-a = g-1+2j$, $1 \le j, k \le [g/2]$ we have

$$1 \le j = m/2 - (g-1) - k \le [g/2],$$

which implies, if m is even, that

$$\max\{1, m/2 - g - [g/2] + 1\} \le k \le \min\{m/2 - g, [g/2]\}.$$

Then

$$r_{22}(m) = \max\{0, \min\{m/2 - g, [g/2]\} - \max\{1, m/2 - g - [g/2] + 1\} + 1\}$$

Therefore, if m is even

$$r_{22}(m) = \begin{cases} 0, & m < 2g \\ m/2 - g, & 2g \le m \le 3g-1 \\ g + 2[g/2] - m/2, & 3g \le m \le 4g-2 \\ 0, & 4g-2 < m \end{cases}$$

Now, we are ready to calculate $r(m)$.

• $m \le g-1$.

$r(m) = r_{11}(m) = m+1 \le g$

• $g \le m \le 2g-1$.

$r(m) = r_{11}(m) + 2r_{12}(m) = 2g-m-1+2[\frac{m-g+1}{2}] \le 2g-m-1+m-g+1 = g$.

•$2g \le m \le 3g-1$.

If m is odd, $r(m) = 2r_{12}(m) = 2([g/2] - [\frac{m-2g+1}{2}]) \le g$. If m is even, $r(m) = 2r_{21}(m) + r_{22}(m) = 2([g/2] - [\frac{m-2g+1}{2}]) + m/2 - g = 2[g/2] - (m-2g) + m/2 - g = 2[g/2] + g - m/2 \le 2[g/2] + g - (2g)/2 \le g$.

• $3g \le m \le 4g-2$

If m is odd, $r(m) = 0$ If m is even, $r(m) = r_{22}(m) = g + 2[g/2] - m/2 \le g + 2[g/2] - (3g)/2 \le g/2 < g$.

□

Proof. (Theorem 2.1)

It is known [2], that for $m = p^2 + p + 1$, p prime, there exists a B_2 (mod m) sequence C_m such that $|C_m| = p+1$ and $C_m \subset [1, m]$

Let us take

$$B = \cup_{i=0}^{k} (C_m + ma_i),$$

where $A^g = \{a_0, a_1, .., a_k\}$ is defined in lemma 2.2.

Observe that $B \subset [1, m(1+a_k)]$, where $a_k = g-1+2[g/2]$. Observe, also, that $|B| = |A^g||C_m| = (g+[g/2])(p+1)$. Then $F_2[g, m(g+2[g/2])] \ge (g+[g/2])(p+1)$.

For any integer n we can choose a prime p such that

$$n - o(n) \le (p^2+p+1)(g+2[g/2]) \le n$$

Then

$$F_2[g,n] \geq F_2[g, m(g+2[g/2])] \geq (g+[g/2])(p+1) \geq$$
$$\geq \frac{g+[g/2]}{\sqrt{g+2[g/2]}} n^{1/2} + o(n^{1/2})$$

□

REFERENCES

[1]. S.Chen, "On Sidon sequences of even orders". *Acta Arithm.* **64** (1993), 325-330

[2]. P.Erdős and P.Turán, "On a problem of Sidon in additive number theory and some related problems". *J.London Math. Soc.* **16** (1941) 212-215; addendum by Erdős. *J.London Math. Soc.* **19** (1994), 208.

[3]. S.W.Graham, "B_h sequences". *Analytic number theory, Vol. 1* (Allerton Park, IL, 1995), 431-449

[4]. X.-D.Jia, "On finite Sidon sequences". *J.Number Theory.* **44** (1993), 84-92.

[5]. X.-D.Jia, "$B_h[g]$ sequences with large Upper Density". *J.Number Theory* **56** (1996),298-308.

[6]. M.N.Kolountzakis, "The density of $B_h[g]$ sequences and the minimun of dense cosine sums".*Journal of Number Theory* **56,** 4-11 1996).

[7]. J.Singer, "A theorem in finite projective geometry and some applications to number theory", *Trans.Am.math.Soc.* **43** (1938) 377-85.

JAVIER CILLERUELO
Departamento de Matemáticas.
Universidad Autónoma de Madrid.
28049 Madrid. España.
e-mail: franciscojavier.cilleruelo uam.es

IMRE RUZSA
Mathematical Institute of The Hungarian Academy of Sciences,
Budapest, Pf. 127, H-1364, Hungary
e-mail: ruzsa math-inst.hu

CARLOS TRUJILLO
Departamento de Matemáticas.
Universidad del Cauca.
Calle 5, No. 4-70
Popayan, Colombia
e-mail: trujillo atenea.ucauca.edu.co