GAPS IN SUMSETS OF s PSEUDO s-TH POWER SEQUENCES

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Abstract. We study the length of the gaps between consecutive members in the sumset sA when A is a pseudo s-th power sequence, with $s \geq 2$. We show that, almost surely, $\limsup(b_{n+1} - b_n)/\log(b_n) = s^s s!/\Gamma^s(1/s)$, where b_n are the elements of sA.

1. Introduction

Erdős and Rényi [3] proposed in 1960 a probabilistic model for sequences A growing like the s-th powers: they build a probability space $(\mathcal{U}, \mathcal{T}, P)$ and a sequence of independent random variables $(\xi_n)_{n\in\mathbb{N}}$ with values in $\{0,1\}$ and $P(\xi_n = 1) = \frac{1}{s}n^{-1+1/s}$; to any $u \in \mathcal{U}$, they associate the sequence of integers $A = A_u$ such that $n \in A_u$ if and only if $\xi_n(u) = 1$. In short, the events $\{n \in A\}$ are independent and $P(n \in A) = \frac{1}{s}n^{-1+1/s}$. The counting function of these random sequences A satisfies almost surely the asymptotic relation $|A \cap [1,x]| \sim x^{1/s}$, whence the terminology *pseudo s-th powers*. Erdős and Rényi studied the random variable $r_s(A,n)$ which counts the number of representations of n in the form $n = a_1 + \cdots + a_s,\ a_1 \leq \cdots \leq a_s,\ a_i \in A$. For the simplest case $s = 2$ they proved that $r_2(A,n)$ converges to a Poisson distribution with parameter $\pi/8$, when $n \to \infty$. They also claimed the analogous result for $s > 2$ but their analysis did not take into account the dependence of some events. J. H. Goguel [4] proved indeed that for each integer d, the sequence of the integers n such that $r_s(A,n) = d$ has almost surely the density $\lambda_s^d e^{-\lambda_s}/d!$, where $\lambda_s = \Gamma^s(1/s)/(s^s s!)$. B. Landreau [5] gave a proof of this result based on correlation inequalities and also showed that the sequence of random variables $(r_s(A,n))_n$ converges in law towards the Poisson distribution with parameter λ_s.

In particular, both the sets of the integers belonging, or not belonging, to $sA = \{a_1 + \cdots + a_s \ :\ a_i \in A\}$ have almost surely a positive density and it makes sense to study the length of the gaps in sA. The aim of the paper is to obtain a precise estimate for the maximal length of such gaps.

Theorem 1. *For any $s \geq 2$ the sequence $sA = (b_n)_n$, sum of s copies of a pseudo s-th power sequence A, satisfies almost surely*

$$\limsup_{n\to\infty} \frac{b_{n+1} - b_n}{\log b_n} = \frac{s^s s!}{\Gamma^s(1/s)}. \qquad (1)$$

We remark that this result is heuristically consistent with the easier fact that for a random sequence S with $P(n \in S) = 1 - e^{-\lambda}$, we have $\limsup(s_{m+1} - s_m)/\log s_m = 1/\lambda$ almost surely.

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2. Notation and general lemmas

2.1. **Notation.** We retain the notation of the introduction, for the probability space $(\mathcal{U}, \mathcal{T}, P)$ and the definition of the random sequences $A = A_u$, where the events $\{n \in A\}$ are independent and $P(n \in A) = \frac{1}{s} n^{-1+1/s}$. We further use the following notation.

i) We write ω to denote a set of distinct integers and we denote by E_ω and E_ω^c the events

$$E_\omega := \{\omega \subset A\} \qquad \text{and} \qquad E_\omega^c := \{\omega \not\subset A\}$$

respectively. We write $\omega \sim \omega'$ to mean that $\omega \cap \omega' \neq \emptyset$ but $\omega \neq \omega'$: e remark that $\omega \sim \omega'$ if and only if the events E_ω and $E_{\omega'}$ are distinct an dependent.

If $\omega = \{x_1, \dots, x_r\}$ we write

$$\sigma(\omega) = \{a_1 x_1 + \cdots + a_r x_r : \ a_1 + \cdots + a_r = s, \ a_i \geq 1\}$$

for the set of all integers that can be written as a sum of s integers using all the integers $x_1, \dots, x_r$. We denote by Ω_z the family of sets

$$\Omega_z = \{\omega : \ z \in \sigma(\omega)\}.$$

ii) Given $\alpha > 0$, we denote by I_i the interval $[i, i + \alpha \log i]$ and we denote by F_i the event

$$F_i := \{sA \cap I_i = \emptyset\}.$$

We denote by Ω_{I_i} the family of sets

$$\Omega_{I_i} = \{\omega : \ \sigma(\omega) \cap I_i \neq \emptyset\}.$$

iii) We finally let $\lambda_s = \frac{\Gamma^s(1/s)}{s! s^s}$.

2.2. **Probabilistic lemmas.** We use the following generalization of the Borel-Cantelli Lemma, proved indeed by P. Erdős and A. Rényi in 1959 [2].

Theorem 2 (Borel-Cantelli Lemma). *Let $(F_i)_{i \in \mathbb{N}}$ be a sequence of events and let $Z_n = \sum_{i \leq n} P(F_i)$.*

If the sequence $(Z_n)_n$ is bounded, then, with probability 1, only finitely many of the events F_i occur.

If the sequence $(Z_n)_n$ tends to infinity and

$$\lim_{n \to \infty} \frac{\sum_{1 \leq i < j \leq n} P(F_i \cap F_j) - P(F_i)P(F_j)}{Z_n^2} = 0,$$

then, with probability 1, infinitely many of the events F_i occur.

Theorem 3 (Janson's Correlation Inequality [1]). *Let $(E_\omega)_{\omega \in \Omega}$ be a finite collection of events which are intersections of elementary independent events and assume that $P(E_\omega) \leq 1/2$ for any $\omega \in \Omega$. Then*

$$\prod_{\omega \in \Omega} P(E_\omega^c) \leq P\big(\bigcap_{\omega \in \Omega} E_\omega^c\big) \leq \prod_{\omega \in \Omega} P(E_\omega^c) \times \exp\big(2 \sum_{\omega \sim \omega'} P(E_\omega \cap E_{\omega'})\big),$$

where $\omega \sim \omega'$ means that the events E_ω and $E_{\omega'}$ are distinct and dependent.

2.3. A technical lemma.

Lemma 1. *Given $1 \le t \le s-1$ and positive integers $a_1, \dots, a_t$ we have, as z tends to infinity:*

i)
$$\sum_{\substack{x_1,\dots,x_t \\ a_1x_1+\cdots+a_tx_t=z}} (x_1\cdots x_t)^{-1+1/s} \ll z^{-1+t/s}.$$

ii)
$$\sum_{\substack{x_1,\dots,x_t \\ a_1x_1+\cdots+a_tx_t<z}} (x_1\cdots x_t)^{-1+1/s}\big(z-(a_1x_1+\cdots+a_tx_t)\big)^{-2t/s} \ll z^{-1/s}\log z.$$

iii)
$$\sum_{\substack{1\le x_1<\cdots<x_s \\ x_1+\cdots+x_s=z}} (x_1\cdots x_s)^{-1+1/s} \sim s^s\lambda_s.$$

Proof. i) We have

$$\begin{aligned}\sum_{\substack{x_1,\dots,x_t \\ a_1x_1+\cdots+a_tx_t=z}} (x_1\cdots x_t)^{-1+1/s} &= (a_1\cdots a_t)^{1-1/s} \sum_{\substack{x_1,\dots,x_t \\ a_1x_1+\cdots+a_tx_t=z}} (a_1x_1\cdots a_tx_t)^{-1+1/s} \\ &\le (a_1\cdots a_t)^{1-1/s} \sum_{\substack{y_1,\dots,y_t \\ y_1+\cdots+y_t=z}} (y_1\cdots y_t)^{-1+1/s}.\end{aligned}$$

If $y_1+\cdots+y_t=z$ then at least one of them, say y_t, is greater than z/t and is determined by $y_1,\dots,y_{t-1}$. Thus,

$$\begin{aligned}\sum_{\substack{x_1,\dots,x_t \\ a_1x_1+\cdots+a_tx_t=z}} (x_1\cdots x_t)^{-1+1/s} &\ll z^{-1+1/s} \sum_{y_1,\dots,y_{t-1}<z} (y_1\cdots y_{t-1})^{-1+1/s} \\ &\ll z^{-1+1/s}\left(\sum_{y<z} y^{-1+1/s}\right)^{t-1} \\ &\ll z^{-1+1/s}(z^{1/s})^{t-1} \ll z^{-1+t/s}.\end{aligned}$$

ii) We have

$$\begin{aligned} &\sum_{\substack{x_1,\dots,x_t \\ a_1x_1+\cdots+a_tx_t<z}} (x_1\cdots x_t)^{-1+1/s}\big(z-(a_1x_1+\cdots+a_tx_t)\big)^{-2t/s} \\ &= \sum_{m<z}(z-m)^{-2t/s} \sum_{\substack{x_1,\dots,x_t \\ a_1x_1+\cdots+a_tx_t=m}} (x_1\cdots x_t)^{-1+1/s} \\ (\text{ by i) }) &\ll \sum_{m<z}(z-m)^{-2t/s}m^{-1+t/s} \\ &\ll \sum_{m\le z/2}(z-m)^{-2t/s}m^{-1+t/s} + \sum_{z/2<m<z}(z-m)^{-2t/s}m^{-1+t/s} \\ &\ll z^{-2t/s}z^{t/s} + z^{-1+t/s}\sum_{z/2<m<z}(z-m)^{-2t/s} \\ &\ll z^{-t/s} + z^{-1+t/s}\left(1+\log z + z^{1-2t/s}\right) \\ &\ll z^{-t/s} + z^{-1+t/s}\log z \\ &\ll z^{-1/s}\log z.\end{aligned}$$

Remark 1. *Except in the case when $s=2$ and $t=1$, the upper bound in ii) may be replaced by $z^{-1/s}$.*

iii) It follows from Lemma 3 of [5].

□

3. Proof of Theorem 1

3.1. Combinatorial lemmas.

Lemma 2. *We have*

$$\sum_{\omega\in\Omega_z} P(E_\omega) \sim \lambda_s$$

as $z\to\infty$.

Proof.

$$\sum_{\omega\in\Omega_z} P(E_\omega) = \sum_{\substack{\omega\in\Omega_z\\|\omega|=s}} P(E_\omega) + \sum_{\substack{\omega\in\Omega_z\\|\omega|\le s-1}} P(E_\omega). \tag{2}$$

The main contribution comes from the first sum.

$$\sum_{\substack{\omega\in\Omega_z\\|\omega|=s}} P(E_\omega) = \frac{1}{s^s}\sum_{\substack{1\le x_1<\cdots<x_s\\x_1+\cdots+x_s=z}} (x_1\cdots x_s)^{-1+1/s} \sim \lambda_s$$

as $z\to\infty$, by Lemma 1 iii). For the second sum we have

$$\begin{aligned}\sum_{\substack{\omega\in\Omega_z\\|\omega|\le s-1}} P(E_\omega) \quad &\le \quad \sum_{r\le s-1}\sum_{\substack{a_1,\dots,a_r\\a_1+\cdots+a_r=s}}\sum_{a_1x_1+\cdots+a_rx_r=z}(x_1\cdots x_r)^{-1+1/s}\\ (\text{Lemma } 1, i)) \quad &\ll \quad \sum_{r\le s-1} z^{\frac{r}{s}-1} \ll z^{-1/s}.\end{aligned}$$

.

□

Lemma 3. *For any $z\le z'$ we have*

$$\sum_{\substack{\omega\sim\omega'\\ \omega\in\Omega_z,\ \omega'\in\Omega_{z'}}} P(E_\omega\cap E_{\omega'}) \ll z^{-1/s}\log z.$$

Proof. If $\omega\in\Omega_z$ then there exist some $r\le s$ and some positive integers $a_1,\dots,a_r$ with $a_1+\cdots+a_r=s$ such that $a_1x_1+\cdots+a_rx_r=z$. Thus, any pair of sets $\omega\sim\omega'$ with $\omega\in\Omega_z,\ \omega'\in\Omega_{z'},\ z\le z'$ is of the form

$$\begin{aligned}\omega &= \{x_1,\dots,x_t,u_{t+1},\dots,u_r\}\\ \omega' &= \{x_1,\dots,x_t,v_{t+1},\dots,v_{r'}\}\end{aligned}$$

with $1\le t\le r,r'\le s$ and positive integers $a_1,\dots,a_r$ and $b_1,\dots,b_{r'}$ with

$$\begin{aligned}a_1x_1+\cdots+a_tx_t+a_{t+1}u_{t+1}+\cdots+a_ru_r &= z\\ b_1x_1+\cdots+b_tx_t+b_{t+1}v_{t+1}+\cdots+b_{r'}v_{r'} &= z'.\end{aligned}$$

Of course if $r=t$ then $\omega=\{x_1,\dots,x_r\}$ and $r'\ge t+1$. Otherwise $\omega=\omega'$. And similarly, when $r'=t$, we have $r\ge t+1$.

Given $z,z',t,r,r',a_1,\dots,a_r,b_1,\dots,b_{r'}$ we estimate the sum

$$\sum_{\omega\sim\omega'}^{*} P(E_\omega\cap E_{\omega'})$$

where the sum is extended to the pairs $\omega \sim \omega'$ satisfying the above conditions. We distinguish several cases according to the values of r and r'.

- If $r \geq t+1$ and $r' \geq t+1$, we have

$$\sum_{\omega\sim\omega'}^{*} P(E_\omega \cap E_{\omega'})$$

$$\leq \sum_{\substack{x_1,\dots,x_t \\ a_1x_1+\cdots+a_tx_t<z \\ b_1x_1+\cdots+b_tx_t<z'}} (x_1\cdots x_t)^{-1+1/s} \times \left(\sum_{\substack{u_{t+1},\dots,u_r, \\ a_{t+1}u_{t+1}+\cdots+a_ru_r \\ =z-(a_1x_1+\cdots+a_tx_t)}} (u_{t+1}\cdots u_r)^{-1+1/s} \right)$$

$$\times \left(\sum_{\substack{v_{t+1},\dots,v_{r'}, \\ b_{t+1}v_{t+1}+\cdots+b_{r'}v_{r'} \\ =z'-(b_1x_1+\cdots+b_tx_t)}} (v_{t+1}\cdots v_{r'})^{-1+1/s} \right)$$

By Lemma 1 i) we have

$$\sum_{\omega\sim\omega'}^{*} P(E_\omega \cap E_{\omega'})$$

$$\ll \sum_{\substack{x_1,\dots,x_t \\ a_1x_1+\cdots+a_tx_t<z \\ b_1x_1+\cdots+b_tx_t<z'}} (x_1\cdots x_t)^{-1+\frac{1}{s}} \left(z-(a_1x_1+\cdots+a_tx_t)\right)^{\frac{r-t}{s}-1} \left(z'-(b_1x_1+\cdots+b_tx_t)\right)^{\frac{r'-t}{s}-1}$$

$$\ll \sum_{\substack{x_1,\dots,x_t \\ a_1x_1+\cdots+a_tx_t<z \\ b_1x_1+\cdots+b_tx_t<z'}} (x_1\cdots x_t)^{-1+1/s} \left(z-(a_1x_1+\cdots+a_tx_t)\right)^{-t/s} \left(z'-(b_1x_1+\cdots+b_tx_t)\right)^{-t/s}$$

Using the inequality $AB \leq A^2+B^2$, we get

$$\begin{aligned} \sum_{\omega\sim\omega'}^{*} P(E_\omega \cap E_{\omega'}) &\leq \sum_{\substack{x_1,\dots,x_t \\ a_1x_1+\cdots+a_tx_t<z}} (x_1\cdots x_t)^{-1+1/s} \left(z-(a_1x_1+\cdots+a_tx_t)\right)^{-2t/s} \\ &+ \sum_{\substack{x_1,\dots,x_t \\ b_1x_1+\cdots+b_tx_t<z'}} (x_1\cdots x_t)^{-1+1/s} \left(z'-(b_1x_1+\cdots+b_tx_t)\right)^{-2t/s} \\ (\text{Lemma } 1, ii)) &\ll z^{-1/s}\log z. \end{aligned}$$

- $r = t$ and $r' \geq t+1$. In this case we have

$$\begin{aligned}
\sum_{\omega\sim\omega'}^{*} P(E_\omega \cap E_{\omega'}) &\leq \sum_{\substack{x_1,\dots,x_t\\ a_1x_1+\cdots+a_tx_t=z\\ b_1x_1+\cdots+b_tx_t<z'}} (x_1\cdots x_t)^{-1+1/s}\\
&\times \sum_{\substack{v_{t+1},\dots,v_{r'}\\ b_{t+1}v_{t+1}+\cdots+b_{r'}v_{r'}\\ =z'-(b_1x_1+\cdots+b_tx_t)}} (v_{t+1}\cdots v_{r'})^{-1+1/s}\\
\text{(Lemma 1 i))}\quad &\leq \sum_{\substack{x_1,\dots,x_t\\ a_1x_1+\cdots+a_tx_t=z\\ b_1x_1+\cdots+b_tx_t<z'}} (x_1\cdots x_t)^{-1+1/s} \times (z'-(b_1x_1+\cdots+b_tx_t))^{\frac{r'-t}{s}-1}\\
&\leq \sum_{\substack{x_1,\dots,x_t\\ a_1x_1+\cdots+a_tx_t=z}} (x_1\cdots x_t)^{-1+1/s} \ll z^{\frac{t}{s}-1} \ll z^{-1/s}.
\end{aligned}$$

- $r' = t$ and $r \geq t+1$ is similar to the previous one.

□

Lemma 4. *Let $\alpha > 0$ and the interval $I_i = [i, i+\alpha\log i]$. For any $i \leq j$ we have*

$$\sum_{\substack{\omega\sim\omega'\\ \omega\in\Omega_{I_i},\ \omega'\in\Omega_{I_j}}} P(E_\omega \cap E_{\omega'}) \ll i^{-1/s}(\log i)^2(\log j).$$

Proof.

$$\begin{aligned}
\sum_{\substack{\omega\sim\omega'\\ \omega\in\Omega_{I_i},\ \omega'\in\Omega_{I_j}}} P(E_\omega \cap E_{\omega'}) &\leq \sum_{z\in I_i,\ z'\in I_j} \sum_{\substack{\omega\sim\omega'\\ \omega\in\Omega_z,\ \omega'\in\Omega_{z'}}} P(E_\omega \cap E_{\omega'})\\
&\ll \sum_{z\in I_i,\ z'\in I_j} z^{-1/s}\log z \ll (\log i)^2(\log j) i^{-1/s}.
\end{aligned}$$

□

Lemma 5. *We have*

$$\prod_{\omega\in\Omega_{I_i}} P(E_\omega^c) = i^{-\alpha\lambda_s+o(1)}.$$

Proof. We observe that

$$\prod_{z\in I_i}\prod_{\omega\in\Omega_z} P(E_\omega^c) \leq \prod_{\omega\in\Omega_{I_i}} P(E_\omega^c) \leq \prod_{\substack{\omega\in\Omega_{I_i}\\ |\omega|=s}} P(E_\omega^c) = \prod_{z\in I_i}\prod_{\substack{\omega\in\Omega_z\\ |\omega|=s}} P(E_\omega^c).$$

Writing $P(E_\omega^c) = 1 - P(E_\omega)$ and taking logarithms we have

$$\begin{aligned}
\log\left(\prod_{z\in I_i}\prod_{\omega\in\Omega_z} P(E_\omega^c)\right) &= \sum_{z\in I_i}\sum_{\omega\in\Omega_z} \log(1-P(E_\omega))\\
&\sim -\sum_{z\in I_i}\sum_{\omega\in\Omega_z} P(E_\omega)\\
\text{(Lemma 2)}\quad &\sim -\sum_{z\in I_i}\lambda_s\\
&\sim -\alpha\lambda_s\log i.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\log\left(\prod_{z\in I_i}\prod_{\substack{\omega\in\Omega_z\\|\omega|=s}} P(E_\omega^c)\right) &= \sum_{z\in I_i}\sum_{\substack{\omega\in\Omega_z\\|\omega|=s}} \log(1-P(E_\omega))\\
&\sim -\sum_{z\in I_i}\sum_{\substack{\omega\in\Omega_z\\|\omega|=s}} P(E_\omega)\\
&= -\sum_{z\in I_i}\sum_{\substack{x_1<\cdots<x_s\\x_1+\cdots+x_s=z}} \frac{1}{s^s}(x_1\cdots x_s)^{-1+1/s}\\
\text{(Lemma 1 iii))} &\sim -\lambda_s\alpha\log i.
\end{aligned}$$

□

Lemma 6. *We have*

$$P(F_i)=i^{-\alpha\lambda_s+o(1)}.$$

Proof. We observe that

$$F_i=\bigcap_{\omega\in\Omega_{I_i}} E_\omega^c.$$

Since $P(E_\omega)\le 1/2$ for any ω, Theorem 3 applies and we have

$$\prod_{\omega\in\Omega_{I_i}} P(E_\omega^c)\le P(F_i)\le \prod_{\omega\in\Omega_{I_i}} P(E_\omega^c)\times\exp\left(2\sum_{\substack{\omega\sim\omega'\\\omega,\omega'\in\Omega_{I_i}}} P(E_\omega\cap E_{\omega'})\right).$$

After Lemma 5 we only need to prove

$$\sum_{\substack{\omega\sim\omega'\\\omega,\omega'\in\Omega_{I_i}}} P(E_\omega\cap E_{\omega'})=o(1).$$

But it is a consequence of Lemma 4 with $j=i$.

$$\sum_{\substack{\omega\sim\omega'\\\omega,\ \omega'\in\Omega_{I_i}}} P(E_\omega\cap E_{\omega'})\ll i^{-1/s+o(1)}.$$

□

Lemma 7. *If $i<j$ and $I_i\cap I_j=\emptyset$ then*

$$\prod_{\omega\in\Omega_{I_i}\cup\Omega_{I_j}} P(E_\omega^c)\le P(F_i)P(F_j)(1+O(j^{-1/s}\log j)).$$

Proof. It is clear that

$$\prod_{\omega\in\Omega_{I_i}\cup\Omega_{I_j}} P(E_\omega^c)=\left(\prod_{\omega\in\Omega_{I_i}} P(E_\omega^c)\right)\left(\prod_{\omega\in\Omega_{I_j}} P(E_\omega^c)\right)\left(\prod_{\omega\in\Omega_{I_i}\cap\Omega_{I_j}} P(E_\omega^c)\right)^{-1}.$$

The lower bound of the Janson's inequality, applied to the first two products, gives

$$\prod_{\omega\in\Omega_{I_i}\cup\Omega_{I_j}} P(E_\omega^c)\le P(F_i)P(F_j)\left(\prod_{\omega\in\Omega_{I_i}\cap\Omega_{I_j}} P(E_\omega^c)\right)^{-1}.$$

The logarithm of the last factor is

$$-\sum_{\omega\in\Omega_{I_i}\cap\Omega_{I_j}}\log(1-P(E_\omega)) \quad\sim\quad \sum_{\omega\in\Omega_{I_i}\cap\Omega_{I_j}}P(E_\omega)$$

Since $I_i\cap I_j=\emptyset$, if $\omega\in\Omega_{I_i}\cap\Omega_{I_j}$ then $|\omega|\le s-1$. Thus

$$\begin{aligned}\sum_{\omega\in\Omega_{I_i}\cap\Omega_{I_j}}P(E_\omega) &\le \sum_{\substack{\omega\in\Omega_{I_j}\\ |\omega|\le s-1}}P(E_\omega)\\ &\le \sum_{z\in I_j}\sum_{r\le s-1}\sum_{a_1+\cdots+a_r=s}\sum_{\substack{x_1,\dots,x_r\\ a_1x_1+\cdots+a_rx_r=z}}(x_1\cdots x_r)^{-1+1/s}\\ \text{(Lemma 1 } i)) &\ll j^{-1/s}(\log j).\end{aligned}$$

Thus

$$\left(\prod_{\omega\in\Omega_{I_i}\cap\Omega_{I_j}}P(E_\omega^c)\right)^{-1}\le 1+O(j^{-1/s}(\log j))$$

which ends the proof of the Lemma. □

3.2. **End of the proof.** After these Lemmas we are ready to finish the proof of Theorem 1.

If $\alpha>1/\lambda_s$ then

$$\sum_i P(F_i)=\sum_i i^{-\alpha\lambda_s+o(1)}<\infty$$

and Theorem 2 implies that with probability 1 only finite many events F_i occur. This proves that

$$\limsup_{k\to\infty}\frac{b_{k+1}-b_k}{\log b_k}\le 1/\lambda_s.$$

If $\alpha<1/\lambda_s$ then

$$Z_n=\sum_{i\le n}P(F_i)=\sum_{i\le n}i^{-\alpha\lambda_s+o(1)}=n^{1-\alpha\lambda_s+o(1)}\to\infty.$$

If in addition

$$\lim_{n\to\infty}\frac{\sum_{1\le i<j\le n}P(F_i\cap F_j)-P(F_i)P(F_j)}{Z_n^2}=0, \tag{3}$$

Theorem 2 implies that with probability 1 infinitely many events F_i occur and

$$\limsup_{k\to\infty}\frac{b_{k+1}-b_k}{\log b_k}\ge 1/\lambda_s.$$

We next prove (3). We observe that

$$F_i\cap F_j=\bigcap_{\omega\in\Omega_{I_i}\cup\Omega_{I_j}}E_\omega^c,$$

so we can use Janson inequality to get

$$P(F_i\cap F_j)\quad\le\quad\prod_{\omega\in\Omega_{I_i}\cup\Omega_{I_j}}P\big(E_\omega^c\big)\times\exp\left(2\sum_{\substack{\omega\sim\omega'\\ \omega,\omega'\in\Omega_{I_i}\cup\Omega_{I_j}}}P(E_\omega\cap E_{\omega'})\right).$$

Observe that

$$\begin{aligned}\sum_{\substack{\omega\sim\omega'\\ \omega,\omega'\in\Omega_{I_i}\cup\Omega_{I_j}}} P(E_\omega\cap E_{\omega'}) &\le \sum_{\substack{\omega\sim\omega'\\ \omega,\omega'\in\Omega_{I_i}}} P(E_\omega\cap E_{\omega'})\\ &+ \sum_{\substack{\omega\sim\omega'\\ \omega,\omega'\in\Omega_{I_j}}} P(E_\omega\cap E_{\omega'})\\ &+ \sum_{\substack{\omega\sim\omega'\\ \omega\in\Omega_{I_i},\ \omega'\in\Omega_{I_j}}} P(E_\omega\cap E_{\omega'}).\end{aligned}$$

Applying Lemma 4 to the three sums we have

$$\sum_{\substack{\omega\sim\omega'\\ \omega,\omega'\in\Omega_{I_i}\cup\Omega_{I_j}}} P(E_\omega\cap E_{\omega'}) \ll i^{-1/s}(\log i)^3 + j^{-1/s}(\log j)^3 + i^{-1/s}(\log i)^2(\log j),$$

and so

$$\exp\left(2\sum_{\substack{\omega\sim\omega'\\ \omega,\omega'\in\Omega_{I_i}\cup\Omega_{I_j}}} P(E_\omega\cap E_{\omega'})\right) \le 1 + O\left(i^{-1/s}(\log i)^2(\log j)\right). \tag{4}$$

Thus,

$$P(F_i\cap F_j) \le \prod_{\omega\in\Omega_{I_i}\cup\Omega_{I_j}} P\big(E_\omega^c\big)\times\big(1+O(i^{-1/s}(\log i)^2(\log j))\big). \tag{5}$$

Since $\alpha<\lambda_s$, the number $\beta=(1-\alpha\lambda_s)/2$ is positive. Now we split the sum in (3) into three sums:

$$\begin{aligned}\Delta_{1n} &= \sum_{\substack{1\le i<j\le n\\ n^\beta<i<j-\alpha\log j}} P(F_i\cap F_j)-P(F_i)P(F_j)\\ \Delta_{2n} &= \sum_{\substack{1\le i<j\le n\\ i\le n^\beta}} P(F_i\cap F_j)-P(F_i)P(F_j)\\ \Delta_{3n} &= \sum_{\substack{1\le i<j\le n\\ j-\log j\le i\le j}} P(F_i\cap F_j)-P(F_i)P(F_j)\end{aligned}$$

i) Estimate of Δ_{1n}. Since in this case we have $I_i\cap I_j=\emptyset$, we can apply Lemma 7 to (5) to get

$$\prod_{\omega\in\Omega_{I_i}\cup\Omega_{I_j}} P\big(E_\omega^c\big) \le P(F_i)P(F_j)(1+O(j^{-1/s}\log j)).$$

This inequality and (5) gives

$$P(F_i\cap F_j) \le P(F_i)P(F_j)\times\big(1+O(i^{-1/s}(\log i)^2(\log j))\big),$$

so

$$\begin{aligned}P(F_i\cap F_j)-P(F_i)P(F_j) &\ll P(F_i)P(F_j)i^{-1/s}(\log i)^2(\log j)\\ &\ll n^{-\beta/s+o(1)}P(F_i)P(F_j).\end{aligned}$$

Thus

$$\Delta_{1n} \ll n^{-\beta/s+o(1)} \sum_{i,j\le n} P(F_i)P(F_j) \ll n^{-\beta/s+o(1)} Z_n^2. \tag{6}$$

ii) Estimate of Δ_{2n}. In this case we use the crude estimate

$$P(F_i \cap F_j) - P(F_i)P(F_j) \le P(F_i \cap F_j) \le P(F_j). \tag{7}$$

We have

$$\Delta_{2n} \le \sum_{j\le n}\sum_{i\le j^\beta} P(F_j) \le \sum_{j\le n} j^\beta P(F_j) \le n^\beta Z_n \le n^{-\beta+o(1)} Z_n^2, \tag{8}$$

since $Z_n = n^{1-\alpha\lambda_s+o(1)} = n^{2\beta+o(1)}$.

iii) Estimate of Δ_{3n}. Again we use (7) and we have

$$\Delta_{3n} \le \sum_{j\le n}\sum_{j-\alpha\log j\le i\le j} P(F_j) \le \alpha\log n \sum_{j\le n} P(F_j) \le n^{-2\beta+o(1)} Z_n^2. \tag{9}$$

Finally, using the estimates in (6),(8) and(9) we have

$$\frac{\sum_{1\le i<j\le n} P(F_i\cap F_j) - P(F_i)P(F_j)}{Z_n^2} \ll n^{-\beta/s+o(1)} + n^{-\beta+o(1)} + n^{-2\beta+o(1)} \to 0.$$

This ends the proof of (3) and hence that of Theorem 1.

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References

[1] R. B. Boppona and J. H. Spencer, *A useful elementary correlation inequality,* J. Comb. Th. Ser. A, 50 (1989), 305-307.

[2] P. Erdős and A. Rényi, *On Cantor's series with convergent* $\sum 1/q_n$, Ann. Univ. Sci. Budapest. Eötvös. sect. math. 2 (1959), 93-109.

[3] P. Erdős and A. Rényi, *Additive properties of random sequences of positive integers,* Acta Arith. 6 (1960), 83–110.

[4] J. H. Goguel, *Über Summen von zufälligen Folgen natürlischen Zahlen,* J. Reine Angew. Math. 278/279 (1975), 63 - 77.

[5] B. Landreau, *Étude probabiliste des sommes des puissances s-ièmes.* Compositio Mathematica 99 (1995), 1 - 31.

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