# $k$-fold Sidon sets

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**Abstract**

Let $k \geq 1$ be an integer. A set $A \subset \mathbb{Z}$ is a *$k$-fold Sidon set* if $A$ has only trivial solutions to each equation of the form $c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0$ where $0 \leq |c_i| \leq k$, and $c_1 + c_2 + c_3 + c_4 = 0$. We prove that for any integer $k \geq 1$, a $k$-fold Sidon set $A \subset [N]$ has at most $(N/k)^{1/2} + O((Nk)^{1/4})$ elements. Indeed we prove that given any $k$ positive integers $c_1 < \cdots < c_k$, any set $A \subset [N]$ that contains only trivial solutions to $c_i(x_1 - x_2) = c_j(x_3 - x_4)$ for each $1 \leq i \leq j \leq k$, has at most $(N/k)^{1/2} + O((c_k^2 N/k)^{1/4})$ elements. On the other hand, for any $k \geq 2$ we can exhibit $k$ positive integers $c_1, \dots, c_k$ and a set $A \subset [N]$ with $|A| \geq (\frac{1}{k} + o(1))N^{1/2}$, such that $A$ has only trivial solutions to $c_i(x_1 - x_2) = c_j(x_3 - x_4)$ for each $1 \leq i \leq j \leq k$.

## 1 Introduction

Let $\Gamma$ be an abelian group. A set $A \subset \Gamma$ is a *Sidon* set if $a + b = c + d$ and $a, b, c, d \in A$ implies $\{a, b\} = \{c, d\}$. Sidon sets in $\mathbb{Z}$ and in the group $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ have been studied extensively. Erdős and Turán [5] proved that a Sidon set $A \subset [N]$ has at most $N^{1/2} + O(N^{1/4})$ elements. Constructions of Singer [10], Bose and Chowla [2], and Ruzsa [9] show that this upper bound is asymptotically best possible. It is a prize problem of Erdős [4] to determine whether or not the error term is bounded. For more on Sidon sets we recommend O'Bryant's survey [8].

Let

$$c_1x_1 + \cdots + c_rx_r = 0 \tag{1}$$

be an integer equation where $c_i \in \mathbb{Z}\backslash\{0\}$, and $c_1 + \cdots + c_r = 0$. Call such an equation an *invariant equation*. A solution $(x_1, \dots, x_r) \in \mathbb{Z}^r$ to (1) is *trivial* if there is a partition of $\{1, \dots, r\}$ into nonempty sets $T_1, \dots, T_m$ such that for every $1 \leq i \leq m$, we have $\sum_{j \in T_i} c_j = 0$, and $x_{j_1} = x_{j_2}$ whenever $j_1, j_2 \in T_i$. A natural extremal problem is to determine the maximum size of a set $A \subset [N]$ with only trivial solutions to (1). This

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problem was investigated in detail by Ruzsa [9]. One of the important open problems from [9] is the genus problem. Given an invariant equation $E : c_1x_1 + \cdots + c_rx_r = 0$, the *genus* $g(E)$ is the largest integer $m$ such that there is a partition of $\{1, \ldots, r\}$ into nonempty sets $T_1, \ldots, T_m$, such that $\sum_{j \in T_i} c_j = 0$ for $1 \leq i \leq m$. Ruzsa proved that if $E$ is an invariant equation and $A \subset [N]$ has only trivial solutions to $E$, then $|A| \leq c_E N^{1/g(E)}$. Here $c_E$ is a positive constant depending only on the equation $E$. Determining if there are sets $A \subset [N]$ with $|A| = N^{1/g(E)-o(1)}$ and having only trivial solutions to $E$ is open for most equations. In particular, the genus problem is open for the equation $2x_1 + 2x_2 = 3x_3 + x_4$. This equation has genus 1 but the best known construction [9] gives a set $A \subset [N]$ with $|A| \geq cN^{1/2}$ where $c > 0$ is a positive constant. More generally, Ruzsa showed that for any four variable equation $E : c_1x_1 + c_2x_2 = c_3x_3 + c_4x_4$ with $c_1 + c_2 = c_3 + c_4$ and $c_i \in \mathbb{N}$, there is a set $A \subset [N]$ with only trivial solutions to $E$ and $|A| \geq c_E N^{1/2-o(1)}$. In this paper we consider special types of four variable invariant equations.

Let $k \geq 1$ be an integer. A set $A \subset \mathbb{Z}$ is a *k-fold Sidon* set if $A$ has only trivial solutions to each equation of the form

$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0$$

where $0 \leq |c_i| \leq k$, and $c_1 + c_2 + c_3 + c_4 = 0$. A 1-fold Sidon set is a Sidon set. A 2-fold Sidon set has only trivial solutions to each of the equations

$$x_1 + x_2 - x_3 - x_4 = 0, \qquad 2x_1 + x_2 - 2x_3 - x_4 = 0, \qquad 2x_1 - x_2 - x_3 = 0.$$

One can also define $k$-fold Sidon sets in $\mathbb{Z}_N$. We must add the condition that $N$ is relatively prime to all integers in the set $\{1, 2, \ldots, k\}$. The reason for this is that if a coefficient $c_i \in \{1, 2, \ldots, k\}$ has a common factor with $N$, then in $\mathbb{Z}_N$ one could have $c_i(a_1 - a_2) = 0$ with $a_1 \neq a_2$. In this case, if $|A| \geq 3$, we can choose $a_3 \in A \backslash \{a_1, a_2\}$, and obtain the nontrivial solution $(x_1, x_2, x_3, x_4) = (a_1, a_2, a_3, a_3)$ to the equation $c_i(x_1 - x_2) + x_3 - x_4 = 0$.

Lazebnik and Verstraëte [6] were the first to define $k$-fold Sidon sets. They conjectured the following.

**Conjecture 1.1 (Lazebnik, Verstraëte [6])** *For any integer $k \geq 3$, there is a positive constant $c_k > 0$ such that for all integers $N \geq 1$, there is a $k$-fold Sidon set $A \subset [N]$ with $|A| \geq c_k N^{1/2}$.*

This conjecture is still open. Lazebnik and Verstraëte proved that for infinitely many $N$, there is a 2-fold Sidon set $A \subset \mathbb{Z}_N$ with $|A| \geq \frac{1}{2}N^{1/2} - 3$. Axenovich [1] and Verstraëte (unpublished) observed that one can adapt Ruzsa's construction for four variable equations (Theorem 7.3, [9]) to construct $k$-fold Sidon sets $A \subset [N]$ or $A \subset \mathbb{Z}_N$ with $|A| \geq c_k N^{1/2} e^{-c_k \sqrt{\log N}}$ for any $k \geq 3$. An affirmative answer to Conjecture 1.1, even in the case when $k = 3$, would have applications to hypergraph Turán problems [6] and extremal graph theory [11].

Since any $k$-fold Sidon set is a Sidon set, the trivial upper bound $|A| \leq \sqrt{N - 3/4} + 1/2$ for a Sidon set $A \subset \mathbb{Z}_N$, and the Erdős-Turán bound $|A| \leq N^{1/2} + O(N^{1/4})$ for any

Sidon set $A \subset [N]$, also hold for $k$-fold Sidon sets. We will obtain better upper bounds for $k$-fold Sidon sets. Instead of considering all the possible equations $c_1x_1+c_2x_2+c_3x_3+c_4x_4 = 0$ with $c_1 + c_2 + c_3 + c_4 = 0$, we will take advantage only of the equations of the form

$$c_1(x_1 - x_2) = c_2(x_3 - x_4).$$

For any $c_1, \ldots, c_k$ with $(c_i, N) = 1$, if $A \subset \mathbb{Z}_N$ contains only trivial solutions to $c_i(x_1 - x_2) = c_j(x_3 - x_4)$ for each $1 \le i \le j \le k$, then

$$|A| \le \sqrt{\frac{N-1}{k} + \frac{1}{4}} + \frac{1}{2}. \tag{2}$$

To see this, consider all elements of the form $c_i(x - y)$ where $1 \le i \le k$, and $x \ne y$ are elements of $A$. All of these elements are distinct and nonzero. Therefore, $k|A|(|A|-1) \le N - 1$ which is equivalent to (2).

The short counting argument used to obtain (2) does not work in $\mathbb{Z}$. Using a more sophisticated argument, we can show that a bound similar to (2) does hold in $\mathbb{Z}$.

**Theorem 1.2** *Let $k \ge 1$ be an integer and $1 \le c_1 < c_2 < \cdots < c_k$ be a set of $k$ distinct integers. If $A \subset \mathbb{Z}_N$ is a set with only trivial solutions to $c_i(x_1 - x_2) = c_j(x_3 - x_4)$ for each $1 \le i \le j \le k$, then*

$$|A| \le \left(\frac{N}{k}\right)^{1/2} + O\left(\left(\frac{c_k^2 N}{k}\right)^{1/4}\right).$$

Taking $c_j = j$ for $1 \le j \le k$, we have the following corollary.

**Corollary 1.3** *If $k \ge 1$ is an integer and $A \subset [N]$ is a $k$-fold Sidon set, then*

$$|A| \le \left(\frac{N}{k}\right)^{1/2} + O((kN)^{1/4}).$$

It is natural to ask if we can improve Corollary 1.3 if we make full use of the assumption that $A$ is a $k$-fold Sidon set. For example, the bound $|A| \le (N/3)^{1/2} + O(N^{1/4})$ holds under the assumption that $A \subset [N]$ has only trivial solutions to $c_1(x_1 - x_2) = c_2(x_3 - x_4)$ for each $1 \le c_1 \le c_2 \le 3$. A 3-fold Sidon set additionally has only trivial solutions to $2x_1 + 2x_2 = 3x_3 + x_4$. Our argument does not capture this property. It is not known if this additional assumption would improve the upper bound $|A| \le (N/3)^{1/2} + O(N^{1/4})$.

The method used by Lazebnik and Verstraëte to construct 2-fold Sidon sets is rather robust. Using this method, we prove the following theorem.

**Theorem 1.4** *There exist $k$ distinct integers $c_1, \ldots, c_k$ and infinitely many $N$, such that there is a set $A \subset \mathbb{Z}_N$ with*

$$|A| \ge \frac{N^{1/2}}{k}(1 - o(1))$$

*and having only trivial solutions to $c_i(x_1 - x_2) = c_j(x_3 - x_4)$ for each $1 \le i \le j \le k$.*

The next section contains the proof of Theorem 1.2. Section 3 contains the proof of Theorem 1.4.

## 2 Proof of Theorem 1.2

For finite sets $B, C \subset \mathbb{Z}$, define

$$r_{B-C}(x) = |\{(b,c) : b - c = x, b \in B, c \in C\}|.$$

The following useful lemma has appeared in the literature (see [3] or [9]).

**Lemma 2.1** *For any finite sets $B, C \subset \mathbb{Z}$,*

$$\frac{(|B||C|)^2}{|B+C|} \leq |B||C| + \sum_{x \neq 0} r_{B-B}(x) r_{C-C}(x). \tag{3}$$

**Proof.** By Cauchy-Schwarz,

$$\begin{aligned} \frac{(|B||C|)^2}{|B+C|} &= \frac{\left(\sum_{x \in B+C} r_{B+C}(x)\right)^2}{|B+C|} \leq \sum_x r_{B+C}^2(x) \\ &= \sum_x r_{B-B}(x) r_{C-C}(x) = |B||C| + \sum_{x \neq 0} r_{B-B}(x) r_{C-C}(x). \end{aligned}$$

■

**Proof of Theorem 1.2.** Let $1 \leq c_1 < c_2 < \cdots < c_k$ be $k$ distinct integers. Let $A \subset [N]$ be a set with only trivial solutions to $c_i(x_1 - x_2) = c_j(x_3 - x_4)$ for each $1 \leq i \leq j \leq k$. Let

$$B_{r,i} = \{x : c_r x + i \in A\}$$

for $1 \leq r \leq k$ and $0 \leq i \leq c_r - 1$. Therefore,

$$|A| = \sum_{i=0}^{c_r - 1} |\{a \in A : \ a \equiv i \pmod{c_r}\}| = \sum_{i=0}^{c_r-1} |B_{r,i}|$$

so by Cauchy-Schwarz,

$$|A|^2 = \left( \sum_{i=0}^{c_r-1} |B_{r,i}| \right)^2 \leq c_r \sum_{i=0}^{c_r-1} |B_{r,i}|^2. \tag{4}$$

For any $y \neq 0$,

$$\sum_{r=1}^{k} \sum_{i=0}^{c_r-1} r_{B_{r,i} - B_{r,i}}(y) \leq 1. \tag{5}$$

To see this, suppose

$$y = x_1 - x_2 = x_3 - x_4 \tag{6}$$

where $x_1, x_2 \in B_{r,i}$ and $x_3, x_4 \in B_{r',i'}$ for some $1 \leq r, r' \leq k$, $1 \leq i \leq c_r - 1$, and $1 \leq i' \leq c_{r'} - 1$. There are elements $a_1, a_2, a_3, a_4 \in A$ such that

$$c_r x_1 + i = a_1, \quad c_r x_2 + i = a_2, \quad c_{r'} x_3 + i' = a_3, \quad \text{and} \quad c_{r'} x_4 + i' = a_4.$$

Then (6) implies

$$\frac{1}{c_r}(a_1 - i) - \frac{1}{c_r}(a_2 - i) = \frac{1}{c_{r'}}(a_3 - i') - \frac{1}{c_{r'}}(a_4 - i'),$$

thus $c_{r'}(a_1 - a_2) = c_r(a_3 - a_4)$. Since $y \neq 0$, we have $a_1 \neq a_2$ and $a_3 \neq a_4$ and the we would have a non trivial solution of the equation.

Let $C = \{0, 1, \ldots, m-1\}$. For any $1 \leq r \leq k$ and $0 \leq i \leq c_r - 1$, the set $B_{r,i} + C$ is contained in the interval $\{0, 1, \ldots, N/c_r + m - 1\}$. This gives the trivial estimate $|B_{r,i} + C| \leq N/c_r + m$. By Lemma 2.1,

$$\frac{|B_{r,i}|^2 m^2}{N/c_r + m} \leq |B_{r,i}|m + \sum_{y \neq 0} r_{B_{r,i}-B_{r,i}}(y) r_{C-C}(y).$$

We sum this inequality over all $1 \leq r \leq k$ and $0 \leq i \leq c_r - 1$ to get

$$\begin{aligned} m^2 \sum_{r=1}^{k} \frac{1}{N/c_r + m} \sum_{i=0}^{c_r - 1} |B_{r,i}|^2 &\leq \sum_{r=1}^{k} \sum_{i=0}^{c_r-1} |B_{r,i}|m \\ &+ \sum_{y \neq 0} \sum_{r=1}^{k} \sum_{i=0}^{c_r-1} r_{B_{r,i}-B_{r,i}}(y) r_{C-C}(y) \\ &\leq k|A|m + \sum_{y \neq 0} r_{C-C}(y) \\ &\leq m(k|A| + m). \end{aligned}$$

From (4) we deduce

$$m^2 |A|^2 \sum_{r=1}^{k} \frac{1}{N + c_r m} \leq m(k|A| + m). \tag{7}$$

The left hand side of (7) is at least $\frac{|A|^2 k m^2}{N + c_k m}$. Therefore, $\frac{|A|^2 km}{N + c_k m} \leq k|A| + m$, and

$$|A|^2 km \leq (N + c_k m)(m + k|A|).$$

From this inequality, we obtain

$$\begin{aligned} \left( |A| - \left( \frac{N}{2m} + \frac{c_k}{2} \right) \right)^2 &\leq \frac{N}{k} + \frac{c_k m}{k} + \left( \frac{N}{2m} + \frac{c_k}{2} \right)^2 \\ &\leq \frac{N}{k} + \frac{c_k m}{k} + \frac{N^2}{2m^2} + \frac{c_k^2}{2} \\ &= \frac{N}{k} \left( 1 + \frac{c_k m}{N} + \frac{Nk}{2m^2} + \frac{k c_k^2}{2N} \right). \end{aligned}$$

Upon solving for $|A|$, we get

$$\begin{aligned} |A| &\leq \left( \frac{N}{k} \right)^{1/2} \left( 1 + \frac{c_k m}{N} + \frac{Nk}{2m^2} + \frac{k c_k^2}{2N} \right) + \frac{N}{2m} + \frac{c_k}{2} \\ &\leq \left( \frac{N}{k} \right)^{1/2} + \frac{c_k m}{k^{1/2} N^{1/2}} + \frac{N^{3/2} k^{1/2}}{2m^2} + \frac{k^{1/2} c_k^2}{2N^{1/2}} + \frac{N}{2m} + \frac{c_k}{2}. \end{aligned}$$

Take $m = \lceil (N^{3/4}k^{1/4})/c_k^{1/2} \rceil$ to get $|A| \leq \left(\frac{N}{k}\right)^{1/2} + O((c_k^2 N/k)^{1/4})$. This completes the proof of Theorem 1.2.

■

# 3 Proof of Theorem 1.4

Let $k \geq 2$ be an integer. Let $p$ be a prime, and let $M \geq 1$ be a large integer. Let $r$ be any prime with $r > Mk$. Let $i \geq 1$ be an integer, and set $t = r^i$ and $q = p^t$.

We will prove that for $c_j = p^{j-1}$ for $j = 1, \dots k$ there exists a set $A \subset \mathbb{Z}_{q^2-1}$ with $|A| \geq \frac{q}{k}\left(1 - \frac{1}{M}\right) - (p^4 - 1)(M - 1)$ and having only trivial solutions to

$$x_1 - x_2 = p^{j-1}(x_3 - x_4)$$

for $1 \leq j \leq k$. This proves Theorem 1.4 because as $i$ tends to infinity, the term $\frac{q}{k}\left(1 - \frac{1}{M}\right)$ is the dominant term. $M$ can be taken as large as we want, and $(p^4-1)(M-1)$ is constant with respect to $i$.

Let $\theta$ be a generator of the cyclic group $\mathbb{F}_{q^2}^*$. Bose and Chowla [2] proved that the set

$$C(q, \theta) = \{a \in \mathbb{Z}_{q^2-1} : \theta^a - \theta \in \mathbb{F}_q\}$$

is a Sidon set in $\mathbb{Z}_{q^2-1}$. Lindström [7] proved

$$B(q, \theta) = \{b \in \mathbb{Z}_{q^2-1} : \theta^b + \theta^{qb} = 1\}$$

is a translate of $C(q, \theta)$ and is therefore a Sidon set.

**Lemma 3.1** *The map $x \mapsto px$ is an injection from $\mathbb{Z}_{q^2-1}$ to $\mathbb{Z}_{q^2-1}$ that maps $B(q, \theta)$ to $B(q, \theta)$.*

**Proof.** The map $x \mapsto px$ is 1-to-1 since $p$ is relatively prime to $q^2 - 1$. If $b \in B(q, \theta)$, then

$$1 = (\theta^b + \theta^{qb})^p = \theta^{pb} + \theta^{q(pb)}$$

so $pb \in B(q, \theta)$. ■

Let $\pi : B(q, \theta) \to B(q, \theta)$ be the permutation $\pi(b) = pb$. As in [6], we use the cycles of $\pi$ to define $A$. Let $\sigma = (b_1, \dots, b_m)$ be a cycle of $\pi$. If $m < k$, then remove all elements of $\sigma$ from $B(q, \theta)$. If $m \geq k$, then remove all $b_j$ in $\sigma$ for which $j$ is not divisible by $k$. Do this for each cycle of $\pi$. Let $A$ be the resulting subset of $B(q, \theta)$.

**Lemma 3.2** *For each $c \in \{1, p, p^2, \dots, p^{k-1}\}$, $A$ has only trivial solutions to*

$$x_1 - x_2 = c(x_3 - x_4).$$

**Proof.** Suppose $a_1, a_2, a_3, a_4 \in A$ and $a_1 - a_2 = p^j(a_3 - a_4)$ for some $0 \le j \le k-1$. By Lemma 3.1, there are elements $b_3, b_4 \in B(q,\theta)$ such that $p^j a_3 = b_3$ and $p^j a_4 = b_4$. This gives $a_1 - a_2 = b_3 - b_4$. Since $B(q,\theta)$ is a Sidon set, either $a_1 = a_2$, $b_3 = b_4$ or $a_1 = b_3$, $a_2 = b_4$.

If $a_1 = a_2$ and $b_3 = b_4$, then $a_3 = a_4$ and the solution $(a_1, a_2, a_3, a_4)$ is trivial. Suppose $a_1 = b_3$ and $a_2 = b_4$. This implies $b_3 \in A$, so both $p^j a_3$ and $a_3$ are in $A$. This contradicts the way in which $A$ was constructed. ■

**Lemma 3.3** $|A| \ge \frac{q}{k}\left(1 - \frac{1}{M}\right) - (p^4 - 1)(M-1)$.

**Proof.** In order to obtain a lower bound on $|A|$, we need to estimate the number of cycles of $\pi$ that are short. For instance, if all cycles of $\pi$ have length less than $k$, then $|A| = 0$. For a cycle $\sigma$ of $\pi$ with length $mk \ge Mk$, we delete at most $m(k-1)$ elements from $B(q,\theta)$ and keep at least $m-1$ elements.

We estimate the number of cycles of length at most $Mk-1$. Let $\sigma = (b, pb, \ldots, p^{e-1}b)$ be a cycle of $\pi$ of length $e$ where $e \le Mk - 1$. The integer $e$ is the smallest positive integer such that $p^e b \equiv b (\text{mod } q^2 - 1)$. This is the same as saying that the order of $p$ in the multiplicative group of units $\mathbb{Z}_n^*$ is $e$ where $n = \frac{q^2-1}{\gcd(b, q^2-1)}$. Since

$$p^{4t} - 1 = (p^{2t} - 1)(p^{2t} + 1) = (q^2 - 1)(p^{2t} + 1)$$

we have $p^{4t} \equiv 1 (\text{mod } q^2 - 1)$, so $e$ must divide $4t = 4r^i$. Since $r$ is prime and $r \ge Mk$, $e$ cannot divide $r$, so $e$ must divide 4. To count the number the number of cycles of $\pi$ with length at most $Mk - 1$, it is enough to count the elements $x \in \mathbb{Z}_{q^2-1} \backslash \{0\}$ such that $p^4 x \equiv x (\text{mod } q^2 - 1)$. This follows from the fact that if $e \in \{1, 2\}$ and $p^e x \equiv x (\text{mod } q^2 - 1)$, then $p^4 x \equiv x (\text{mod } q^2 - 1)$. The number of solutions to this congruence is $\gcd(p^4 - 1, q^2 - 1) \le p^4 - 1$. Therefore, there are at most $p^4 - 1$ cycles of $\pi$ of length at most $Mk - 1$. For a cycle of length at least $Mk$, the proportion of elements of the cycle that are put into $A$ is at least $\frac{M-1}{Mk}$ (the function $f(x) = \frac{x-1}{xk}$ is increasing provided $k > 0$). Since $|B(q,\theta)| = q$,

$$|A| \ge \left(q - (p^4 - 1)Mk\right)\left(\frac{M-1}{Mk}\right) = \frac{q}{k}\left(1 - \frac{1}{M}\right) - (p^4 - 1)(M - 1).$$

■

Theorem 1.4 follows from Lemmas 3.2 and 3.3.

# 4 Concluding Remarks

The most important open problem concerning $k$-fold Sidon sets is an answer to Conjecture 1.1. The case $k = 3$ is particularly interesting. A 3-fold Sidon set $A \subset [N]$ with $|A| \ge cN^{1/2}$ is known to imply the existence of a graph with $c_1 N$ vertices, $c_2 N^{3/2}$ edges, and every edge is in exactly one cycle of length four [11].

Another problem is to determine the maximum size of a 2-fold Sidon set in $\mathbb{Z}_N$ or $[N]$. Let $S_k(N)$ be the maximum size of a $k$-fold Sidon set in $\mathbb{Z}_N$. For any integer $t \geq 1$, there are 2-fold Sidon sets $A \subset \mathbb{Z}_N$, $N = 2^{2^{t+1}} + 2^{2^t} + 1$, with $|A| \geq \frac{1}{2}N^{1/2} - 3$ (see [6]). Theorem 1.2 gives an upper bound of $(N/2)^{1/2} + O(N^{1/4})$ so

$$\frac{1}{2} \leq \limsup_{N \to \infty} \frac{S_2(N)}{N^{1/2}} \leq \frac{1}{2^{1/2}}.$$

It would be interesting to determine the above limit. In the case of Sidon sets, we have $\limsup_{N\to\infty} \frac{S_1(N)}{N^{1/2}} = 1$ by [5] and [10].

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