Let $k \geq 1$ be an integer. A set $A \subset \mathbb{Z}$ is a $k$-fold Sidon set if $A$ has only trivial solutions to each equation of the form $c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0$ where $0 \leq |c_i| \leq k$, and $c_1 + c_2 + c_3 + c_4 = 0$. We prove that for any integer $k \geq 1$, a $k$-fold Sidon set $A \subset [N]$ has at most $(N/k)^{1/2} + O((Nk)^{1/4})$ elements. Indeed we prove that given any $k$ positive integers $c_1 < \cdots < c_k$, any set $A \subset [N]$ that contains only trivial solutions to $c_i(x_1 - x_2) = c_j(x_3 - x_4)$ for each $1 \leq i \leq j \leq k$, has at most $(N/k)^{1/2} + O((c_k^2N/k)^{1/4})$ elements. On the other hand, for any $k \geq 2$ we can exhibit $k$ positive integers $c_1, \ldots, c_k$ and a set $A \subset [N]$ with $|A| \geq \left(\frac{1}{k} + o(1)\right)N^{1/2}$, such that $A$ has only trivial solutions to $c_i(x_1 - x_2) = c_j(x_3 - x_4)$ for each $1 \leq i \leq j \leq k$.

1 Introduction

Let $\Gamma$ be an abelian group. A set $A \subset \Gamma$ is a Sidon set if $a + b = c + d$ and $a, b, c, d \in A$ implies $\{a, b\} = \{c, d\}$. Sidon sets in $\mathbb{Z}$ and in the group $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ have been studied extensively. Erdős and Turán [5] proved that a Sidon set $A \subset [N]$ has at most $N^{1/2} + O(N^{1/4})$ elements. Constructions of Singer [10], Bose and Chowla [2], and Ruzsa [9] show that this upper bound is asymptotically best possible. It is a prize problem of Erdős [4] to determine whether or not the error term is bounded. For more on Sidon sets we recommend O’Bryant’s survey [8].

Let

$$c_1x_1 + \cdots + c_rx_r = 0$$

be an integer equation where $c_i \in \mathbb{Z}\setminus\{0\}$, and $c_1 + \cdots + c_r = 0$. Call such an equation an invariant equation. A solution $(x_1, \ldots, x_r) \in \mathbb{Z}^r$ to (1) is trivial if there is a partition of $\{1, \ldots, r\}$ into nonempty sets $T_1, \ldots, T_m$ such that for every $1 \leq i \leq m$, we have $\sum_{j \in T_i} c_j = 0$, and $x_{j_1} = x_{j_2}$ whenever $j_1, j_2 \in T_i$. A natural extremal problem is to determine the maximum size of a set $A \subset [N]$ with only trivial solutions to (1). This
problem was investigated in detail by Ruzsa [9]. One of the important open problems from [9] is the genus problem. Given an invariant equation \( E : c_1 x_1 + \cdots + c_r x_r = 0 \), the \textit{genus} \( g(E) \) is the largest integer \( m \) such that there is a partition of \( \{1, \ldots, r\} \) into nonempty sets \( T_1, \ldots, T_m \), such that \( \sum_{j \in T_i} c_j = 0 \) for \( 1 \leq i \leq m \). Ruzsa proved that if \( E \) is an invariant equation and \( A \subset [N] \) has only trivial solutions to \( E \), then \( |A| \leq c_E N^{1/g(E)} \). Here \( c_E \) is a positive constant depending only on the equation \( E \). Determining if there are sets \( A \subset [N] \) with \( |A| = N^{1/g(E) - o(1)} \) and having only trivial solutions to \( E \) is open for most equations. In particular, the genus problem is open for the equation \( 2x_1 + 2x_2 = 3x_3 + x_4 \). This equation has genus 1 but the best known construction [9] gives a set \( A \) with \( |A| = cN^{1/2} \) where \( c > 0 \) is a positive constant. More generally, Ruzsa showed that for any four variable equation \( E : c_1 x_1 + c_2 x_2 = c_3 x_3 + c_4 x_4 \) with \( c_1 = c_2 = c_3 + c_4 \) and \( c_i \in \mathbb{N} \), there is a set \( A \subset [N] \) with only trivial solutions to \( E \) and \( |A| \geq c_E N^{1/2 - o(1)} \). In this paper we consider special types of four variable invariant equations.

Let \( k \geq 1 \) be an integer. A set \( A \subset \mathbb{Z} \) is a \textit{k-fold Sidon} set if \( A \) has only trivial solutions to each equation of the form

\[
c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 = 0
\]

where \( 0 \leq |c_i| \leq k \), and \( c_1 + c_2 + c_3 + c_4 = 0 \). A 1-fold Sidon set is a Sidon set. A 2-fold Sidon set has only trivial solutions to each of the equations

\[
x_1 + x_2 - x_3 - x_4 = 0, \quad 2x_1 + x_2 - 2x_3 - x_4 = 0, \quad 2x_1 - x_2 - x_3 = 0.
\]

One can also define \( k \)-fold Sidon sets in \( \mathbb{Z}_N \). We must add the condition that \( N \) is relatively prime to all integers in the set \( \{1, 2, \ldots, k\} \). The reason for this is that if a coefficient \( c_i \in \{1, 2, \ldots, k\} \) has a common factor with \( N \), then in \( \mathbb{Z}_N \) one could have \( c_i(a_1 - a_2) = 0 \) with \( a_1 \neq a_2 \). In this case, if \( |A| \geq 3 \), we can choose \( a_3 \in A \setminus \{a_1, a_2\} \), and obtain the nontrivial solution \( (x_1, x_2, x_3, x_4) = (a_1, a_2, a_3, a_3) \) to the equation \( c_i(x_1 - x_2) + x_3 - x_4 = 0 \).

Lazebnik and Verstraëte [6] were the first to define \( k \)-fold Sidon sets. They conjectured the following.

**Conjecture 1.1 (Lazebnik, Verstraëte [6])** For any integer \( k \geq 3 \), there is a positive constant \( c_k > 0 \) such that for all integers \( N \geq 1 \), there is a \( k \)-fold Sidon set \( A \subset [N] \) with \( |A| \geq c_k N^{1/2} \).

This conjecture is still open. Lazebnik and Verstraëte proved that for infinitely many \( N \), there is a 2-fold Sidon set \( A \subset \mathbb{Z}_N \) with \( |A| \geq \frac{1}{2} N^{1/2} - 3 \). Axenovich [1] and Verstraëte (unpublished) observed that one can adapt Ruzsa’s construction for four variable equations (Theorem 7.3, [9]) to construct \( k \)-fold Sidon sets \( A \subset [N] \) or \( A \subset \mathbb{Z}_N \) with \( |A| \geq c_k N^{1/2} e^{-c_k \sqrt{\log N}} \) for any \( k \geq 3 \). An affirmative answer to Conjecture 1.1, even in the case when \( k = 3 \), would have applications to hypergraph Turán problems [6] and extremal graph theory [11].

Since any \( k \)-fold Sidon set is a Sidon set, the trivial upper bound \( |A| \leq \sqrt{N - 3/4} + 1/2 \) for a Sidon set \( A \subset \mathbb{Z}_N \), and the Erdős-Turán bound \( |A| \leq N^{1/2} + O(N^{1/4}) \) for any
under the assumption that 

Taking \( c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0 \) with \( c_1 + c_2 + c_3 + c_4 = 0 \), we will take advantage only of the equations of the form

\[ c_1(x_1 - x_2) = c_2(x_3 - x_4). \]

For any \( c_1, \ldots, c_k \) with \( (c_i, N) = 1 \), if \( A \subset \mathbb{Z}_N \) contains only trivial solutions to \( c_i(x_1 - x_2) = c_j(x_3 - x_4) \) for each \( 1 \leq i \leq j \leq k \), then

\[ |A| \leq \sqrt{\frac{N - 1}{k}} + \frac{1}{4} + \frac{1}{2}. \tag{2} \]

To see this, consider all elements of the form \( c_i(x - y) \) where \( 1 \leq i \leq k \), and \( x \neq y \) are elements of \( A \). All of these elements are distinct and nonzero. Therefore, \( k|A|(|A| - 1) \leq N - 1 \) which is equivalent to (2).

The short counting argument used to obtain (2) does not work in \( \mathbb{Z} \). Using a more sophisticated argument, we can show that a bound similar to (2) does hold in \( \mathbb{Z} \).

**Theorem 1.2** Let \( k \geq 1 \) be an integer and \( 1 \leq c_1 < c_2 < \cdots < c_k \) be a set of \( k \) distinct integers. If \( A \subset \mathbb{Z}_N \) is a set with only trivial solutions to \( c_i(x_1 - x_2) = c_j(x_3 - x_4) \) for each \( 1 \leq i \leq j \leq k \), then

\[ |A| \leq \left( \frac{N}{k} \right)^{1/2} + O\left( \left( \frac{c_k^2N}{k} \right)^{1/4} \right). \]

Taking \( c_j = j \) for \( 1 \leq j \leq k \), we have the following corollary.

**Corollary 1.3** If \( k \geq 1 \) is an integer and \( A \subset [N] \) is a \( k \)-fold Sidon set, then

\[ |A| \leq \left( \frac{N}{k} \right)^{1/2} + O((kN)^{1/4}). \]

It is natural to ask if we can improve Corollary 1.3 if we make full use of the assumption that \( A \) is a \( k \)-fold Sidon set. For example, the bound \( |A| \leq (N/3)^{1/2} + O(N^{1/4}) \) holds under the assumption that \( A \subset [N] \) has only trivial solutions to \( c_1(x_1 - x_2) = c_2(x_3 - x_4) \) for each \( 1 \leq c_1 \leq c_2 \leq 3 \). A 3-fold Sidon set additionally has only trivial solutions to \( 2x_1 + 2x_2 = 3x_3 + x_4 \). Our argument does not capture this property. It is not known if this additional assumption would improve the upper bound \( |A| \leq (N/3)^{1/2} + O(N^{1/4}) \).

The method used by Lazebnik and Verstraëtte to construct 2-fold Sidon sets is rather robust. Using this method, we prove the following theorem.

**Theorem 1.4** There exist \( k \) distinct integers \( c_1, \ldots, c_k \) and infinitely many \( N \), such that there is a set \( A \subset \mathbb{Z}_N \) with

\[ |A| \geq \frac{N^{1/2}}{k} (1 - o(1)) \]

and having only trivial solutions to \( c_i(x_1 - x_2) = c_j(x_3 - x_4) \) for each \( 1 \leq i \leq j \leq k \).

The next section contains the proof of Theorem 1.2. Section 3 contains the proof of Theorem 1.4.
2 Proof of Theorem 1.2

For finite sets $B, C \subset \mathbb{Z}$, define

$$r_{B-C}(x) = |\{(b,c) : b - c = x, b \in B, c \in C\}|.$$

The following useful lemma has appeared in the literature (see [3] or [9]).

Lemma 2.1 For any finite sets $B, C \subset \mathbb{Z}$,

$$\frac{(|B||C|)^2}{|B + C|} \leq |B||C| + \sum_{x \neq 0} r_{B-B}(x)r_{C-C}(x). \quad (3)$$

Proof. By Cauchy-Schwarz,

$$\frac{(|B||C|)^2}{|B + C|} = \frac{(\sum_{x \in B+C} r_{B+C}(x))^2}{|B + C|} \leq \sum_x r_{B+C}^2(x)$$

$$= \sum_x r_{B-B}(x)r_{C-C}(x) = |B||C| + \sum_{x \neq 0} r_{B-B}(x)r_{C-C}(x).$$

Proof of Theorem 1.2. Let $1 \leq c_1 < c_2 < \cdots < c_k$ be $k$ distinct integers. Let $A \subset \{N\}$ be a set with only trivial solutions to $c_i(x_1 - x_2) = c_j(x_3 - x_4)$ for each $1 \leq i \leq j \leq k$. Let

$$B_{r,i} = \{x : c_r x + i \in A\}$$

for $1 \leq r \leq k$ and $0 \leq i \leq c_r - 1$. Therefore,

$$|A| = \sum_{i=0}^{c_r-1} |\{a \in A : a \equiv i \pmod{c_r}\}| = \sum_{i=0}^{c_r-1} |B_{r,i}|$$

so by Cauchy-Schwarz,

$$|A|^2 = \left(\sum_{i=0}^{c_r-1} |B_{r,i}|\right)^2 \leq c_r \sum_{i=0}^{c_r-1} |B_{r,i}|^2. \quad (4)$$

For any $y \neq 0$,

$$\sum_{r=1}^{k} \sum_{i=0}^{c_r-1} r_{B_{r,i} - B_{r,i'}}(y) \leq 1. \quad (5)$$

To see this, suppose

$$y = x_1 - x_2 = x_3 - x_4$$

where $x_1, x_2 \in B_{r,i}$ and $x_3, x_4 \in B_{r',i'}$ for some $1 \leq r, r' \leq k$, $1 \leq i \leq c_r - 1$, and $1 \leq i' \leq c_{r'} - 1$. There are elements $a_1, a_2, a_3, a_4 \in A$ such that

$$c_r x_1 + i = a_1, \ c_r x_2 + i = a_2, \ c_{r'} x_3 + i' = a_3, \ \text{and} \ c_{r'} x_4 + i' = a_4.$$
Then (6) implies
\[ \frac{1}{c_r}(a_1 - i) - \frac{1}{c_r}(a_2 - i) = \frac{1}{c_r}(a_3 - i') - \frac{1}{c_r}(a_4 - i'), \]
thus \( c_r(a_1 - a_2) = c_r(a_3 - a_4). \) Since \( y \neq 0, \) we have \( a_1 \neq a_2 \) and \( a_3 \neq a_4 \) and the we would have a non trivial solution of the equation.

Let \( C = \{0, 1, \ldots, m - 1\}. \) For any \( 1 \leq r \leq k \) and \( 0 \leq i \leq c_r - 1, \) the set \( B_{r,i} + C \) is contained in the interval \( \{0, 1, \ldots, N/c_r + m - 1\}. \) This gives the trivial estimate \( |B_{r,i} + C| \leq N/c_r + m. \) By Lemma 2.1,
\[ \frac{|B_{r,i}|^2 m^2}{N/c_r + m} \leq |B_{r,i}| m + \sum_{y \neq 0} r_{B_{r,i} - B_r}(y) r_{C-C}(y). \]
We sum this inequality over all \( 1 \leq r \leq k \) and \( 0 \leq i \leq c_r - 1 \) to get
\[ m^2 \sum_{r=1}^{k} \frac{1}{N/c_r + m} \sum_{i=0}^{c_r-1} |B_{r,i}|^2 \leq \sum_{r=1}^{k} \sum_{i=0}^{c_r-1} |B_{r,i}| m + \sum_{y \neq 0} \sum_{r=1}^{k} \sum_{i=0}^{c_r-1} r_{B_{r,i} - B_r}(y) r_{C-C}(y) \]
\[ \leq k |A| m + \sum_{y \neq 0} r_{C-C}(y) \]
\[ \leq m(k|A| + m). \]

From (4) we deduce
\[ m^2 |A|^2 \sum_{r=1}^{k} \frac{1}{N + c_k m} \leq m(k|A| + m). \tag{7} \]
The left hand side of (7) is at least \( \frac{|A|^2 k m^2}{N + c_k m}. \) Therefore, \( \frac{|A|^2 k m^2}{N + c_k m} \leq k|A| + m, \) and
\[ |A|^2 k m \leq (N + c_k m)(m + k|A|). \]
From this inequality, we obtain
\[ \left( |A| - \left( \frac{N}{2m} + \frac{c_k}{2} \right) \right)^2 \leq \frac{N}{k} + \frac{c_k m}{k} + \left( \frac{N}{2m} + \frac{c_k}{2} \right)^2 \]
\[ \leq \frac{N}{k} + \frac{c_k m}{k} + \frac{N^2}{2m^2} + \frac{c_k^2}{2} \]
\[ = \frac{N}{k} \left( 1 + \frac{c_k m}{N} + \frac{N k}{2m^2} + \frac{k c_k^2}{2N} \right). \]

Upon solving for \( |A|, \) we get
\[ |A| \leq \left( \frac{N}{k} \right)^{1/2} \left( 1 + \frac{c_k m}{N} + \frac{N k}{2m^2} + \frac{k c_k^2}{2N} \right) + \frac{N}{2m} + \frac{c_k}{2} \]
\[ \leq \left( \frac{N}{k} \right)^{1/2} + \frac{c_k m}{k^{1/2} N^{1/2}} + \frac{N^{3/2} k^{1/2}}{2m^2} + \frac{k^{1/2} c_k^2}{2N^{1/2}} + \frac{N}{2m} + \frac{c_k}{2}. \]
Take \( m = \lceil (N^{3/4}k^{1/4})/c_k^{1/2} \rceil \) to get \(|A| \leq \left( \frac{N}{k} \right)^{1/2} + O((c_k^2 N/k)^{1/4}) \). This completes the proof of Theorem 1.2.

\[ \]

3 Proof of Theorem 1.4

Let \( k \geq 2 \) be an integer. Let \( p \) be a prime, and let \( M \geq 1 \) be a large integer. Let \( r \) be any prime with \( r > Mk \). Let \( i \geq 1 \) be an integer, and set \( t = r^i \) and \( q = p^i \).

We will prove that for \( c_j = p^{j-1} \) for \( j = 1, \ldots, k \) there exists a set \( A \subset \mathbb{Z}_{q^2-1} \) with \(|A| \geq \frac{q}{k} \left( 1 - \frac{1}{M} \right) - (p^4 - 1)(M - 1) \) and having only trivial solutions to

\[ x_1 - x_2 = p^{j-1}(x_3 - x_4) \]

for \( 1 \leq j \leq k \). This proves Theorem 1.4 because as \( i \) tends to infinity, the term \( \frac{q}{k} \left( 1 - \frac{1}{M} \right) \) is the dominant term. \( M \) can be taken as large as we want, and \((p^4 - 1)(M - 1)\) is constant with respect to \( i \).

Let \( \theta \) be a generator of the cyclic group \( \mathbb{F}_{q^2}^* \). Bose and Chowla [2] proved that the set

\[ C(q, \theta) = \{ a \in \mathbb{Z}_{q^2-1} : \theta^a - \theta \in \mathbb{F}_q \} \]

is a Sidon set in \( \mathbb{Z}_{q^2-1} \). Lindström [7] proved

\[ B(q, \theta) = \{ b \in \mathbb{Z}_{q^2-1} : \theta^b + \theta^{qb} = 1 \} \]

is a translate of \( C(q, \theta) \) and is therefore a Sidon set.

**Lemma 3.1** The map \( x \mapsto px \) is an injection from \( \mathbb{Z}_{q^2-1} \) to \( \mathbb{Z}_{q^2-1} \) that maps \( B(q, \theta) \) to \( B(q, \theta) \).

**Proof.** The map \( x \mapsto px \) is 1-to-1 since \( p \) is relatively prime to \( q^2 - 1 \). If \( b \in B(q, \theta) \), then

\[ 1 = (\theta^b + \theta^{qb})^p = \theta^{pb} + \theta^{q^{p}(b)} \]

so \( pb \in B(q, \theta) \).

Let \( \pi : B(q, \theta) \to B(q, \theta) \) be the permutation \( \pi(b) = pb \). As in [6], we use the cycles of \( \pi \) to define \( A \). Let \( \sigma = (b_1, \ldots, b_m) \) be a cycle of \( \pi \). If \( m < k \), then remove all elements of \( \sigma \) from \( B(q, \theta) \). If \( m \geq k \), then remove all \( b_j \) in \( \sigma \) for which \( j \) is not divisible by \( k \). Do this for each cycle of \( \pi \). Let \( A \) be the resulting subset of \( B(q, \theta) \).

**Lemma 3.2** For each \( c \in \{1, p, p^2, \ldots, p^{k-1}\} \), \( A \) has only trivial solutions to

\[ x_1 - x_2 = c(x_3 - x_4) \]
Proof. Suppose $a_1, a_2, a_3, a_4 \in A$ and $a_1 - a_2 = p^j(a_3 - a_4)$ for some $0 \leq j \leq k - 1$. By Lemma 3.1, there are elements $b_3, b_4 \in B(q, \theta)$ such that $p^j a_3 = b_3$ and $p^j a_4 = b_4$. This gives $a_1 - a_2 = b_3 - b_4$. Since $B(q, \theta)$ is a Sidon set, either $a_1 = a_2$, $b_3 = b_4$ or $a_1 = b_3$, $a_2 = b_4$.

If $a_1 = a_2$ and $b_3 = b_4$, then $a_3 = a_4$ and the solution $(a_1, a_2, a_3, a_4)$ is trivial. Suppose $a_1 = b_3$ and $a_2 = b_4$. This implies $b_3 \in A$, so both $p^j a_3$ and $a_3$ are in $A$. This contradicts the way in which $A$ was constructed. 

Lemma 3.3 $|A| \geq \frac{q}{k} (1 - \frac{1}{M}) - (p^4 - 1)(M - 1)$.

Proof. In order to obtain a lower bound on $|A|$, we need to estimate the number of cycles of $\pi$ that are short. For instance, if all cycles of $\pi$ have length less than $k$, then $|A| = 0$. For a cycle $\sigma$ of $\pi$ with length $mk \geq Mk$, we delete at most $m(k - 1)$ elements from $B(q, \theta)$ and keep at least $m - 1$ elements.

We estimate the number of cycles of length at most $Mk - 1$. Let $\sigma = (b, pb, \ldots, p^{e-1}b)$ be a cycle of $\pi$ of length $e$ where $e \leq Mk - 1$. The integer $e$ is the smallest positive integer such that $p^e b \equiv b (\mod q^2 - 1)$. This is the same as saying that the order of $p$ in the multiplicative group of units $\mathbb{Z}_n^*$ is $e$ where $n = \frac{q^2 - 1}{\gcd(b, q^2 - 1)}$. Since $p^{4t} - 1 = (p^{2t} - 1)(p^{2t} + 1) = (q^2 - 1)(p^{2t} + 1)$ we have $p^{4t} \equiv 1 (\mod q^2 - 1)$, so $e$ must divide $4t = 4r^i$. Since $r$ is prime and $r \geq Mk$, $e$ cannot divide $r$, so $e$ must divide $4$. To count the number of cycles of length at most $Mk - 1$, it is enough to count the elements $x \in \mathbb{Z}_{q^2 - 1}\setminus\{0\}$ such that $p^r x \equiv x (\mod q^2 - 1)$. This follows from the fact that if $e \in \{1, 2\}$ and $p^e x \equiv x (\mod q^2 - 1)$, then $p^4 x \equiv x (\mod q^2 - 1)$. The number of solutions to this congruence is $\gcd(p^4 - 1, q^2 - 1) \leq p^4 - 1$. Therefore, there are at most $p^4 - 1$ cycles of $\pi$ of length at most $Mk - 1$. For a cycle of length at least $Mk$, the proportion of elements of the cycle that are put into $A$ is at least $\frac{M - 1}{Mk}$ (the function $f(x) = \frac{x - 1}{xk}$ is increasing provided $k > 0$). Since $|B(q, \theta)| = q$,

$$|A| \geq (q - (p^4 - 1)Mk) \left(\frac{M - 1}{Mk}\right) = \frac{q}{k} \left(1 - \frac{1}{M}\right) - (p^4 - 1)(M - 1).$$

Theorem 1.4 follows from Lemmas 3.2 and 3.3.

4 Concluding Remarks

The most important open problem concerning $k$-fold Sidon sets is an answer to Conjecture 1.1. The case $k = 3$ is particularly interesting. A 3-fold Sidon set $A \subset [N]$ with $|A| \geq cN^{1/2}$ is known to imply the existence of a graph with $c_1 N$ vertices, $c_2 N^{3/2}$ edges, and every edge is in exactly one cycle of length four [11].
Another problem is to determine the maximum size of a 2-fold Sidon set in $\mathbb{Z}_N$ or $[N]$. Let $S_k(N)$ be the maximum size of a $k$-fold Sidon set in $\mathbb{Z}_N$. For any integer $t \geq 1$, there are 2-fold Sidon sets $A \subset \mathbb{Z}_N$, $N = 2^{2t+1} + 2^t + 1$, with $|A| \geq \frac{1}{2}N^{1/2} - 3$ (see [6]). Theorem 1.2 gives an upper bound of $(N/2)^{1/2} + O(N^{1/4})$ so

$$\frac{1}{2} \leq \limsup_{N \to \infty} \frac{S_2(N)}{N^{1/2}} \leq \frac{1}{21/2}.$$  

It would be interesting to determine the above limit. In the case of Sidon sets, we have $\limsup_{N \to \infty} \frac{S_1(N)}{N^{1/2}} = 1$ by [5] and [10].

References