THE LEAST COMMON MULTIPLE OF RANDOM SETS OF POSITIVE INTEGERS

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Abstract. We study the typical behavior of the least common multiple of the elements of a random subset $A \subset \{1, \dots, n\}$. For example we prove that $\operatorname{lcm}\{a : a \in A\} = 2^{n(1+o(1))}$ for almost all subsets $A \subset \{1, \dots, n\}$.

1. Introduction

The function $\psi(n) = \log \operatorname{lcm}\{m : 1 \le m \le n\}$ was introduced by Chebyshev in his study on the distribution of the prime numbers. It is a well known fact that the asymptotic relation $\psi(n) \sim n$ is equivalent to the Prime Number Theorem, which was proved finally by Hadamard and de la Vallée Poussin.

In the present paper, instead of considering the whole set $\{1, \dots, n\}$, we study the typical behavior of the quantity $\psi(A) := \log \operatorname{lcm}\{a : a \in A\}$ for a random set A in $\{1, \dots, n\}$ when $n \to \infty$. We consider two natural models.

In the first one, denoted by $B(n, \delta)$, each element in A is chosen independently at random in $\{1, \dots, n\}$ with probability $\delta = \delta(n)$, typically a function of n.

Theorem 1.1. *If $\delta = \delta(n) < 1$ and $\delta n \to \infty$ then*

$$\psi(A) \sim n \frac{\delta \log(\delta^{-1})}{1 - \delta}$$

asymptotically almost surely in $B(n, \delta)$ when $n \to \infty$.

The case $\delta = 1$, which corresponds to the asymptotic estimate for the classical Chebyshev function, appears as the limiting case, as δ tends to 1, in Theorem 1.1, since $\lim_{\delta \to 1} \frac{\delta \log(\delta^{-1})}{1-\delta} = 1$.

When $\delta = 1/2$ all the subsets $A \subset \{1, \dots, n\}$ are chosen with the same probability and Theorem 1.1 gives the following result.

Corollary 1.1. *For almost all sets $A \subset \{1, \dots, n\}$ we have that*

$$\operatorname{lcm}\{a : a \in A\} = 2^{n(1+o(1))}.$$

For a given positive integer $k = k(n)$, again typically a function of n, we consider the second model, where each subset of k elements is chosen uniformly at random among all sets of size k in $\{1, \dots, n\}$. We denote this model by $S(n, k)$.

When $\delta = k/n$ the heuristic suggests that both models are quite similar. Indeed, this is the strategy we follow to prove Theorem 1.2.

Theorem 1.2. *For $k = k(n) < n$ and $k \to \infty$ we have*

$$\psi(A) = k\frac{\log(n/k)}{1-k/n}\left(1 + O(e^{-C\sqrt{\log k}})\right)$$

almost surely in $S(n,k)$ when $n \to \infty$ for some positive constant C.

The case $k = n$, which corresponds to Chebyshev's function, is also obtained as a limiting case in Theorem 1.2 in the sense that $\lim_{k/n\to 1} \frac{\log(n/k)}{1-k/n} = 1$.

This work has been motivated by a result of the first author about the asymptotic behavior of $\psi(A)$ when $A = A_{q,n} := \{q(m) : \ 1 \le q(m) \le n\}$ for a quadratic polynomial $q(x) \in \mathbb{Z}[x]$. We wondered if that behavior was typical among the sets $A \subset \{1, \dots, n\}$ of similar size. We analyze this issue in the last section.

2. Chebyshev's function for random sets in $B(n,\delta)$. Proof of Theorem 1.1

The following lemma provides us with an explicit expression for $\psi(A)$ in terms of the Mangoldt function

$$\Lambda(m) = \begin{cases} \log p & \text{if } m = p^k \text{ for some } k \ge 1 \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.1. *For any set of positive integers A we have $\psi(A) = \sum_m \Lambda(m) I_A(m)$, where Λ denotes the classical Von Mangoldt function and*

$$I_A(m) = \begin{cases} 1 & \text{if } A \cap \{m, 2m, 3m, \dots\} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We observe that for any positive integer l, the number $\log l$ can be written as $\log l = \sum_{p^k | l} \log p$, where the sum is taken over all the powers of primes. Thus, using that $p^k | \operatorname{lcm}\{a : a \in A\}$ if and only if $A \cap \{p^k, 2p^k, 3p^k, \dots\} \neq \emptyset$, we get

$$\log \operatorname{lcm}(a : \ a \in A) = \sum_{p^k|\operatorname{lcm}(a:\ a\in A)} \log p = \sum_{p^k} (\log p) I_A(p^k) = \sum_m \Lambda(m) I_A(m).$$

$\square$

Note that if $A = \{1, \dots, n\}$ then $\psi(A) = \sum_{m \le n} \Lambda(m)$ is the classical Chebychev function $\psi(n)$.

2.1. Expectation.

First of all we give an explicit expression for the expected value of the random variable $X = \psi(A)$ where A is a random set in $B(n,\delta)$.

Proposition 2.1. *For the random variable $X = \psi(A)$ in $B(n,\delta)$ we have*

$$\mathbb{E}(X) = n\frac{\delta \log(\delta^{-1})}{1-\delta} + \delta \sum_{r \ge 1} R\left(\frac{n}{r}\right)(1-\delta)^{r-1},$$

where $R(x) = \psi(x) - x$ denotes the error term in the Prime Number Theorem.

Proof. The ambiguous case $\delta = 1$ must be understood as the limit as $\delta \to 1$, which recovers the equality $\psi(n) = n + R(n)$. In the following we assume that $\delta < 1$. By linearity of the expectation, Lemma 2.1 clearly implies

$$\mathbb{E}(X) = \sum_{m \le n} \Lambda(m)\mathbb{E}(I_A(m)).$$

Since $\mathbb{E}(I_A(m)) = \mathbb{P}(A \cap \{m, 2m, \dots\} \neq \emptyset) = 1 - \prod_{r \le n/m} \mathbb{P}(rm \notin A) = 1 - (1-\delta)^{\lfloor n/m \rfloor}$, we obtain

$$\mathbb{E}(X) = \sum_{m \le n} \Lambda(m) \left(1 - (1-\delta)^{\lfloor n/m \rfloor}\right). \tag{1}$$

We observe that $\lfloor n/m \rfloor = r$ whenever $\frac{n}{r+1} < m \le \frac{n}{r}$, so we split the sum into intervals $J_r = (\frac{n}{r+1}, \frac{n}{r}]$, obtaining

$$\begin{aligned}
\mathbb{E}\left(X\right) &= \sum_{r \ge 1} (1 - (1-\delta)^r) \sum_{m \in J_r} \Lambda(m) \\
&= \sum_{r \ge 1} (1 - (1-\delta)^r) \Big(\psi\Big(\frac{n}{r}\Big) - \psi\Big(\frac{n}{r+1}\Big) \Big) \\
&= \delta \sum_{r \ge 1} \psi\Big(\frac{n}{r}\Big) (1-\delta)^{r-1} \\
&= \delta n \sum_{r \ge 1} \frac{(1-\delta)^{r-1}}{r} + \delta \sum_{r \ge 1} R\left(\frac{n}{r}\right)(1-\delta)^{r-1}. \\
&= n \frac{\delta \log(\delta^{-1})}{1-\delta} + \delta \sum_{r \ge 1} R\left(\frac{n}{r}\right)(1-\delta)^{r-1}.
\end{aligned}$$

□

Corollary 2.1. *If $\delta = \delta(n) < 1$ and $\delta n \to \infty$ then*

$$\mathbb{E}\left(X\right) = n \frac{\delta \log(\delta^{-1})}{1-\delta} \left(1 + O\left(e^{-C\sqrt{\log(\delta n)}}\right)\right).$$

for some constant $C > 0$.

Proof. We estimate the absolute value of sum appearing in Proposition 2.1. For any positive integer T and using that $|R(y)| < 2y$ for all $y > 0$ we have

$$\begin{aligned}
\sum_{r \ge 1} |R\left(n/r\right)|(1-\delta)^{r-1} &= \sum_{1 \le r \le T} |R\left(n/r\right)|(1-\delta)^{r-1} + \sum_{r \ge T+1} |R\left(n/r\right)|(1-\delta)^{r-1} \\
&\le n \sum_{1 \le r \le T} \frac{|R\left(n/r\right)|}{(n/r)} \frac{(1-\delta)^{r-1}}{r} + 2n \sum_{r \ge T+1} \frac{(1-\delta)^{r-1}}{r} \\
&\le n \left(\max_{x \ge n/T} \frac{|R(x)|}{x} \right) \sum_{1 \le r \le T} \frac{(1-\delta)^{r-1}}{r} + 2n \sum_{r \ge T+1} \frac{(1-\delta)^{r-1}}{r} \\
&\le n \frac{\log(\delta^{-1})}{(1-\delta)} \left(\max_{x \ge n/T} \frac{|R(x)|}{x} \right) + \frac{2n}{T+1} \frac{(1-\delta)^T}{\delta}
\end{aligned}$$

Taking into account that $(1-\delta)^T < e^{-\delta T}$ and the known estimate

$$\max_{x > y} \frac{|R(x)|}{x} \ll e^{-C_1\sqrt{\log y}}$$

for the error term in the PNT, we have

$$\sum_{r\geq 1} |R\left(n/r\right)|(1-\delta)^{r-1} \ll n\frac{\log(\delta^{-1})}{(1-\delta)}e^{-C\sqrt{\log(n/T)}} + n\frac{e^{-\delta T}}{\delta T}.$$

Thus we have proved that for any positive integer T we have

$$\mathbb{E}(X) = n\frac{\delta\log(\delta^{-1})}{1-\delta}\left(1 + O\left(e^{-C\sqrt{\log(n/T)}}\right) + O\left(\frac{1-\delta}{\log(\delta^{-1})}\frac{e^{-\delta T}}{\delta T}\right)\right).$$

We take $T \asymp \delta^{-1}\sqrt{\log(\delta n)}$ to minimize the error term. To estimate the first error term we observe that $\log(n/T) \gg \log(\delta n/\sqrt{\log(\delta n)}) \gg \log(\delta n)$, so $e^{-C\sqrt{\log(n/T)}} \ll e^{-C_1\sqrt{\log(\delta n)}}$ for some constant C_1. To bound the second error term we simply observe that $\delta T > 1$ and that $\frac{1-\delta}{\log(\delta^{-1})} \leq 1$ and we get a similar upper bound. $\square$

2.2. **Variance.**

Proposition 2.2. *For the random variable $X = \psi(A)$ in $B(n,\delta)$ we have*

$$V(X) \ll \delta n \log^2 n.$$

Proof. By linearity of expectation we have that

$$\begin{aligned} V(X) &= \mathbb{E}\left(X^2\right) - \mathbb{E}^2\left(X\right) \\ &= \sum_{m,l\leq n} \Lambda(m)\Lambda(l)\left(\mathbb{E}\left(I_A(m)I_A(l)\right) - \mathbb{E}\left(I_A(m)\right)\mathbb{E}\left(I_A(l)\right)\right). \end{aligned}$$

We observe that if $\Lambda(m)\Lambda(l) \neq 0$ then $l \mid m,\ m \mid l$ or $(m,l) = 1$. Let us now study the term $\mathbb{E}(I_A(m)I_A(l))$ in these cases.

(i) If $l \mid m$ then

$$\mathbb{E}(I_A(m)I_A(l)) = 1 - (1-\delta)^{\lfloor n/m\rfloor}.$$

(ii) If $(l,m) = 1$ then

$$\mathbb{E}(I_A(m)I_A(l)) = 1 - (1-\delta)^{\lfloor n/m\rfloor} - (1-\delta)^{\lfloor n/l\rfloor} + (1-\delta)^{\lfloor n/m\rfloor+\lfloor n/l\rfloor-\lfloor n/ml\rfloor}.$$

Both of these relations are subsumed in

$$\mathbb{E}(I_A(m)I_A(l)) = 1 - (1-\delta)^{\lfloor n/m\rfloor} - (1-\delta)^{\lfloor n/l\rfloor} + (1-\delta)^{\lfloor n/m\rfloor+\lfloor n/l\rfloor-\lfloor n(m,l)/ml\rfloor}.$$

Therefore, it follows from (1) that for each term in the sum we have

$$\begin{aligned} &\Lambda(m)\Lambda(l)\left(\mathbb{E}\left(I_A(m)I_A(l)\right) - \mathbb{E}\left(I_A(m)\right)\mathbb{E}\left(I_A(l)\right)\right) \\ &= \Lambda(m)\Lambda(l)(1-\delta)^{\lfloor n/m\rfloor+\lfloor n/l\rfloor-\lfloor n(m,l)/ml\rfloor}\left(1-(1-\delta)^{\lfloor n(m,l)/ml\rfloor}\right). \end{aligned}$$

Finally, by using the inequality $1-(1-x)^r \leq rx$ we have

$$\Lambda(m)\Lambda(l)\left(\mathbb{E}\left(I_A(m)I_A(l)\right) - \mathbb{E}\left(I_A(m)\right)\mathbb{E}\left(I_A(l)\right)\right) \leq \delta n\frac{\Lambda(l)}{l}\frac{\Lambda(m)}{m}(m,l),$$

and therefore:

$$V(X) \leq 2\delta n \sum_{1\leq l\leq m\leq n} \frac{\Lambda(l)}{l}\frac{\Lambda(m)}{m}(m,l).$$

We now split the sum according to $l \mid m$ or $(l,m)=1$ and estimate each one separately.

$$\sum_{\substack{1\le l\le m\le n\\ l|m}} \frac{\Lambda(l)}{l}\frac{\Lambda(m)}{m}(m,l) = \sum_{p\le n}\sum_{1\le j\le i}\frac{\log p}{p^i}\frac{\log p}{p^i}p^j \le \sum_{p\le n}\sum_{1\le i}\frac{i\log^2 p}{p^i} \ll \log^2 n,$$

$$\sum_{\substack{1\le l\le m\le n\\ (l,m)=1}} \frac{\Lambda(l)}{l}\frac{\Lambda(m)}{m}(m,l) \le \left(\sum_{1\le l\le n}\frac{\Lambda(l)}{l}\right)\left(\sum_{1\le m\le n}\frac{\Lambda(m)}{m}\right) \ll \log^2 n,$$

as we wanted to prove. □

We finish the proof of Theorem 1.1 by observing that $V(X)=o(\mathbb{E}(X)^2)$ when $\delta n\to\infty$, so $X\sim\mathbb{E}(X)$ asymptotically almost surely.

3. Chebyshev's function for random sets in $S(n,k)$. Proof of Theorem 1.2

Let us consider again the random variable $X=\psi(A)$, but in the model $S(n,k)$. From now on $\mathbb{E}_k(X)$ and $V_k(X)$ will denote the expected value and the variance of X in this probability space. Clearly, for $s=1,2$ we have

$$\begin{aligned}\mathbb{E}_k(X^s) &= \frac{1}{\binom{n}{k}}\sum_{|A|=k}\psi^s(A)\\ V_k(X) &= \frac{1}{\binom{n}{k}}\sum_{|A|=k}\left(\psi(A)-\mathbb{E}_k(X)\right)^2\end{aligned}$$

Lemma 3.1. *For $s=1,2$ and $1\le j<k$ we have that*

$$\mathbb{E}_j(X^s)\le\mathbb{E}_k(X^s)\le\mathbb{E}_j(X^s)+(k^s-j^s)\log^s n.$$

Proof. Suppose $j<k$. There are $\binom{n-j}{k-j}$ ways to add $k-j$ new elements to a set $A\in\binom{[n]}{j}$ in order to obtain a subset of $\binom{[n]}{k}$. Observe that the function ψ is monotone with respect to inclusion, i.e. $\psi(A\cup A')\ge\psi(A)$ for any sets A,A'. Therefore it is clear that, for $s=1,2$, we have

$$\psi^s(A)\le\binom{n-j}{k-j}^{-1}\sum_{\substack{A\cap A'=\emptyset\\ |A'|=k-j}}\psi^s(A\cup A'),$$

and then

$$\begin{aligned}\sum_{|A|=j}\psi^s(A) &\le \binom{n-j}{k-j}^{-1}\sum_{\substack{A\cap A'=\emptyset\\ |A|=j,\ |A'|=k-j}}\psi^s(A\cup A')\\
&= \binom{n-j}{k-j}^{-1}\sum_{|A''|=k}\ \sum_{\substack{A\cup A'=A''\\ |A|=j,\ |A'|=k-j}}\psi^s(A'')\\
&= \binom{n-j}{k-j}^{-1}\sum_{|A''|=k}\psi^s(A'')\sum_{\substack{A\cup A'=A''\\ |A|=j,\ |A'|=k-j}}1\\
&= \frac{\binom{n}{j}}{\binom{n}{k}}\sum_{|A''|=k}\psi^s(A''),\end{aligned}$$

and the first inequality follows.

For the second inequality we observe that for any set $A \in \binom{[n]}{k}$ and any partition into two sets $A = A' \cup A''$ with $|A'| = j$, $|A''| = k-j$ we have that $\psi(A) \le \psi(A') + \psi(A'') \le \psi(A') + (k-j)\log n$. Similarly,

$$\begin{aligned} \psi^2(A) &\le (\psi(A') + (k-j)\log n)^2 \\ &= \psi^2(A') + 2\psi(A')(k-j)\log n + (k-j)^2 \log^2 n \\ &\le \psi^2(A') + 2j(k-j)\log^2 n + (k-j)^2 \log^2 n \\ &= \psi^2(A') + (k^2 - j^2)\log^2 n. \end{aligned}$$

Thus, for $s = 1, 2$ we have

$$\begin{aligned} \psi^s(A) &\le \binom{k}{j}^{-1} \sum_{\substack{A' \subset A \\ |A'|=j}} (\psi^s(A') + (k^s - j^s)\log^s n) \\ &\le \binom{k}{j}^{-1} \Big(\sum_{\substack{A' \subset A \\ |A'|=j}} \psi^s(A') \Big) + (k^s - j^s)\log^s n. \end{aligned}$$

Then,

$$\begin{aligned} \sum_{|A|=k} \psi^s(A) &\le \binom{k}{j}^{-1} \sum_{|A|=k} \sum_{\substack{A' \subset A \\ |A'|=j}} \psi^s(A') + \binom{n}{k}(k^s - j^s)\log^s n \\ &= \binom{k}{j}^{-1} \sum_{|A'|=j} \psi^s(A') \sum_{\substack{A' \subset A \\ |A|=k}} 1 + \binom{n}{k}(k^s - j^s)\log^s n \\ &= \binom{k}{j}^{-1} \binom{n-j}{k-j} \sum_{|A'|=j} \psi^s(A') + \binom{n}{k}(k^s - j^s)\log^s n \\ &= \frac{\binom{n}{k}}{\binom{n}{j}} \sum_{|A'|=j} \psi^s(A') + \binom{n}{k}(k^s - j^s)\log^s n, \end{aligned}$$

and the second inequality holds. $\square$

Proposition 3.1. *For $s = 1, 2$ we have that*

$$\mathbb{E}_k(X^s) = \mathbb{E}(X^s) + O(k^{s-1/2}\log^s n)$$

where $\mathbb{E}(X^s)$ denotes the expectation of X^s in $B(n, k/n)$ and $\mathbb{E}_k(X^s)$ the expectation in $S(n, k)$.

Proof. Observe that for $s = 1, 2$ we have

$$\begin{aligned} \mathbb{E}(X^s) - \mathbb{E}_k(X^s) &= -\mathbb{E}_k(X^s) + \sum_{j=0}^{n} \left(\frac{k}{n}\right)^j \left(1 - \frac{k}{n}\right)^{n-j} \sum_{|A|=j} \psi^s(A) \\ &= -\mathbb{E}_k(X^s) + \sum_{j=0}^{n} \left(\frac{k}{n}\right)^j \left(1 - \frac{k}{n}\right)^{n-j} \binom{n}{j} \mathbb{E}_j(X^s) \\ &= \sum_{j=0}^{n} \left(\frac{k}{n}\right)^j \left(1 - \frac{k}{n}\right)^{n-j} \binom{n}{j} \left(\mathbb{E}_j(X^s) - \mathbb{E}_k(X^s)\right), \end{aligned}$$

for $s = 1, 2$. Using Lemma 3.1 we get

$$|\mathbb{E}_k(X^s) - \mathbb{E}(X^s)| \le \log^s n \sum_{j=0}^{n} \left(\frac{k}{n}\right)^j \left(1 - \frac{k}{n}\right)^{n-j} \binom{n}{j} |j^s - k^s|. \tag{2}$$

The sum in (2) for $s = 1$ is $\mathbb{E}(|Y - \mathbb{E}(Y)|)$, where $Y \sim \mathrm{Bin}(n, k/n)$ is the binomial distribution of parameters n and k/n. Chauchy-Schwarz inequality for the expectation implies that this quantity is bounded by the standard deviation of the binomial distribution.

$$\sum_{j=0}^{n} \left(\frac{k}{n}\right)^j \left(1 - \frac{k}{n}\right)^{n-j} \binom{n}{j} |j - k| \le \sqrt{n(k/n)(1 - k/n)} \le \sqrt{k}, \tag{3}$$

which proves Proposition 3.1 for $s = 1$.

To estimate the sum in (2) for $s = 2$, we split the expression in two terms: the sum indexed by $j \le 2k$ and the one with $j > 2k$. We use (3) to get

$$\begin{aligned} \sum_{j \le 2k} \left(\frac{k}{n}\right)^j \left(1 - \frac{k}{n}\right)^{n-j} \binom{n}{j} |j^2 - k^2| &\le 3k \sum_{j=0}^{n} \left(\frac{k}{n}\right)^j \left(1 - \frac{k}{n}\right)^{n-j} \binom{n}{j} |j - k| \\ &\le 3k^{3/2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\sum_{j > 2k} \left(\frac{k}{n}\right)^j \left(1 - \frac{k}{n}\right)^{n-j} \binom{n}{j} |j^2 - k^2| \\ \le\ & \sum_{l \ge 2} (l+1)^2 k^2 \sum_{lk < j \le (l+1)k} \left(\frac{k}{n}\right)^j \left(1 - \frac{k}{n}\right)^{n-j} \binom{n}{j} \\ \le\ & \sum_{l \ge 2} (l+1)^2 k^2 \ \mathbb{P}(Y > lk) \end{aligned}$$

where, once again, $Y \sim \mathrm{Bin}(n, k/n)$. Chernoff's Theorem implies that for any $\epsilon > 0$ we have

$$\mathbb{P}(Y > (1+\epsilon)k) \le e^{-\epsilon^2 k/3}.$$

Applying this inequality to $\mathbb{P}(Y > lk)$ we get

$$\begin{aligned} &\sum_{j > 2k} \left(\frac{k}{n}\right)^j \left(1 - \frac{k}{n}\right)^{n-j} \binom{n}{j} |j^2 - k^2| \\ \le\ & \sum_{l \ge 2} (l+1)^2 k^2 e^{-(l-1)^2 k/3} \ll k^2 e^{-k/3} \ll k^{3/2}. \end{aligned}$$

□

The next corollary proves the first part of Theorem 1.2.

Corollary 3.1. *If $k = k(n) < n$ and $k \to \infty$ then*

$$\mathbb{E}_k(X) = k \frac{\log(n/k)}{1 - k/n} \left(1 + O\left(e^{-C\sqrt{\log k}}\right)\right)$$

Proof. Proposition 3.1 for $s = 1$ and Corollary 2.1 imply that

$$\mathbb{E}_k(X) = k \frac{\log(n/k)}{1 - k/n} \left(1 + O\left(e^{-C\sqrt{\log k}}\right) + O\left(k^{-1/2}\right)\right)$$

and clearly $k^{-1/2} = o\left(e^{-C\sqrt{\log k}}\right)$ when $k \to \infty$. $\square$

To conclude the proof of Theorem 1.2 we combine Proposition 2.2 and Proposition 3.1 to estimate the variance $V_k(X)$ in $S(n,k)$:

$$\begin{aligned} V_k(X) &= \mathbb{E}_k(X^2) - \mathbb{E}_k^2(X) \\ &= V(X) + \left(\mathbb{E}_k(X^2) - \mathbb{E}(X^2)\right) + \left(\mathbb{E}(X) - \mathbb{E}_k(X)\right)\left(\mathbb{E}(X) + \mathbb{E}_k(X)\right) \\ &\ll k\log^2 n + \left(k^{1/2}\log n\right)(k\log n) \\ &\ll k^{3/2}\log^2 n. \end{aligned}$$

The second assertion of Theorem 1.2 is a consequence of the estimate $V_k(X) = o\left(\mathbb{E}_k^2(X)\right)$ when $k \to \infty$.

3.1. **The case when k is constant.** The case when k is constant and $n \to \infty$ is not relevant for our original motivation but we give a brief analysis for the sake of the completeness. In this case Fernandez and Fernandez [3] have been proved that $\mathbb{E}_k(\psi(A)) = k\log n + C_k + o(1)$ where $C_k = -k + \sum_{j=2}^{k}\binom{k}{j}(-1)^j\frac{\zeta'(j)}{\zeta(j)}$. Actually they consider the probabilistic model with k independent choices in $\{1,\dots,n\}$, but when k is fixed it does not make big differences because the probability of a repetition between the k choices is tiny.

It is easy to prove that with probability $1-o(1)$ we have that $\psi(A) \sim k\log n$. To see this we observe that $a_1\cdots a_k\prod_{i<j}(a_i,a_j)^{-1} \le \operatorname{lcm}(a_1,\dots,a_k) \le a_1\cdots a_k \le n^k$, so $\sum_{i=1}^{k}\log a_i - \sum_{i<j}\log(a_i,a_j) \le \psi(A) \le k\log n$.

Now notice that $\mathbb{P}(a_i \le n/\log n \text{ for some } i = 1,\dots,k) \le k/\log n$. and that $\mathbb{P}((a_i,a_j) \ge \log n) \le \sum_{d>\log n}\mathbb{P}(d \mid a_i,\ d \mid a_j) \le \sum_{d>\log n}\frac{1}{d^2} < \frac{1}{\log n}$. These observations imply that with probability at least $1 - \frac{k+\binom{k}{2}}{\log n}$ we have that

$$k\log n\left(1 - O\left(\log\log n/\log n\right)\right) \le \psi(A) \le k\log n.$$

The analysis in the model $B(n,\delta)$ when $\delta n \to c$ can be done using again Proposition 2.1.

$$\mathbb{E}\left(\psi(A)\right) = n\frac{\delta\log(\delta^{-1})}{1-\delta} + \delta\sum_{r<n/\log n} R\left(\frac{n}{r}\right)(1-\delta)^{r-1} + \delta\sum_{n/\log n\le r\le n} R\left(\frac{n}{r}\right)(1-\delta)^{r-1}$$

We use the estimate $R(x) \ll x/\log x$ in the first sum and the estimate $R(x) \ll x$ in the second one. We have

$$\begin{aligned} \mathbb{E}\left(\psi(A)\right) &= c\log n + O(1) + O\left(\frac{c}{\log\log n}\sum_{r<\frac{n}{\log n}}\frac{(1-\delta)^{r-1}}{r}\right) + O\left(c\sum_{\frac{n}{\log n}\le r\le n}\frac{(1-\delta)^{r-1}}{r}\right) \\ &= c\log n + O\left(\frac{c\log\delta}{\log\log n}\right) + O\left(c\log\log n\right) \\ &= c\log n(1+o(1)). \end{aligned}$$

Of course in this model we have not concentration around the expected value because the probability that A is the empty set tends to a positive constant: $\mathbb{P}(A = \emptyset) \to e^{-c}$.

4. The least common multiple of the values of a polynomial

Chebyshev's function could be also generalized to

$$\psi_q(n) = \log \operatorname{lcm}\{q(k): \ 1 \le k, \ 1 \le q(k) \le n\}$$

for a given polynomial $q(x) \in \mathbb{Z}[x]$ and it is natural to try to obtain the asymptotic behavior for $\psi_q(n)$. Some progress has been made in this direction. While the Prime Number Theorem is equivalent to the asymptotic $\psi_q(n) \sim n$ for $q(x) = x$, the Prime Number Theorem for arithmetic progressions can be exploited [1] to obtain the asymptotic estimate when $q(x) = a_1x + a_0$ is a linear polynomial:

$$\psi_q(n) \sim \frac{n}{a_1}\frac{m}{\phi(m)} \sum_{\substack{1 \le l \le m \\ (l,m)=1}} \frac{1}{l},$$

where $m = a_1/(a_1, a_0)$. The first author [2] has extended this result to quadratic polynomials. For a given irreducible quadratic polynomial $q(x) = a_2x^2 + a_1x + a_0$ with $a_2 > 0$ the following asymptotic estimate holds:

$$\psi_q(n) = \frac{1}{2}(n/a_2)^{1/2} \log(n/a_2) + B_q (n/a_2)^{1/2} + o(n^{1/2}), \tag{4}$$

where the constant B_q depends only on q. In the particular case of $q(x) = x^2 + 1$, he got $\psi_q(n) = \frac{1}{2}n^{1/2}\log n + B_q n^{1/2} + o(n^{1/2})$ with

$$B_q = \gamma - 1 - \frac{\log 2}{2} - \sum_{p \ne 2} \frac{\left(\frac{-1}{p}\right)\log p}{p-1},$$

where γ is the Euler constant, $\left(\frac{-1}{p}\right)$ is the Legendre's symbol and the sum is considered over all odd prime numbers. It has recently been proved in [4] that the error term in the previous expression is $O\left(n^{1/2}(\log n)^{-4/9+\epsilon}\right)$ for each $\epsilon > 0$. When $q(x)$ is a reducible polynomial the behavior is, however, different. In this case it is known (see Theorem 3 in [2]) that:

$$\psi_q(n) \sim cn^{1/2}$$

where c is an explicit constant depending only on q.

The asymptotic behavior of $\psi_q(n)$ remains unknown for irreducible polynomials of degree $d \ge 3$, but it is conjectured in [2] that this should be given by

$$\psi_q(n) \sim (1 - 1/d)(n/a_d)^{1/d} \log(n/a_d), \tag{5}$$

where $a_d > 0$ is the coefficient of x^d. For example, this conjecture would imply $\psi_q(n) \sim \frac{2}{3}n^{1/3}\log n$ for $q(x) = x^3 + 2$.

We observe that $\psi_q(n) = \psi(A_{q,n})$ where $A_{q,n} := \{q(k): \ 1 \le k, \ 1 \le q(k) \le n\}$ and it is natural to wonder whether for a given polynomial $q(x)$ the asymptotic $\mathbb{E}_k(X) \sim \psi_q(n)$ holds when $n \to \infty$ where $k = |A_{q,n}|$ and $X = \psi(A)$ for a random set A of k elements in $\{1, \dots, n\}$. This question was the original motivation of this work. Theorem 1.2 applied to $k = |A_{q,n}| = \sqrt{n/a_2} + O(1)$ gives

$$\mathbb{E}_k(X) = k\frac{\log(n/k)}{1 - k/n}\left(1 + O\left(e^{-C\sqrt{\log k}}\right)\right) = \frac{1}{2}(n/a_2)^{1/2}\log(n/a_2) + o\left(n^{1/2}\right).$$

This shows that the asymptotic behavior of $\psi_q(n)$ is the expected of a random set of the same size when $q(x)$ is an irreducible quadratic polynomial. Theorem 1.2 also supports the analogous conjecture 5 for any $q(x) = a_d x^d + \cdots + a_0$ irreducible polynomial of degree $d \geq 3$.

Nevertheless, there are some differences in the second term. For example, if $q(x) = x^2 + 1$, we have

$$\psi_q(n) = \frac{1}{2} n^{1/2} \log n + B_q n^{1/2} + o(n^{1/2}),$$

for $B = -0.06627563...$ On the other hand, Theorem 1.2 implies that in corresponding model $S(n,k)$ with $k = |A_{q,n}| = \lfloor \sqrt{n-1} \rfloor$ we have that

$$\psi(A) = \frac{1}{2} n^{1/2} \log n + o(n^{1/2})$$

almost surely. In other words, when $q(x)$ is an irreducible quadratic polynomial, the asymptotic behavior of $\psi_q(n)$ is the same that $\psi(A)$ in the corresponding model $S(n,k)$. But, the second term is not typical unless $B_q = 0$. Probably $B_q \neq 0$ for any irreducible quadratic polynomial $q(x)$ but we have not found a proof.

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