NEW UPPER BOUNDS FOR FINITE B_h SEQUENCES

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Abstract. Let $F_h(N)$ be the maximum number of elements that can be selected from the set $\{1,\dots,N\}$ such that all the sums $a_1+\cdots+a_h$, $a_1\le\cdots\le a_h$ are different. We introduce new combinatorial and analytic ideas to prove new upper bounds for $F_h(N)$. In particular we prove

$$F_3(N)\le\left(\frac{4}{1+\frac{16}{(\pi+2)^4}}N\right)^{1/3}+o(N^{1/3}),$$

$$F_4(N)\le\left(\frac{8}{1+\frac{16}{(\pi+2)^4}}N\right)^{1/4}+o(N^{1/4}).$$

Besides, our techniques have an independent interest for further research in additive number theory.

1.Introduction

Let $h\ge 2$ be an integer. A subset A of integers is called a B_h set if for every positive integer m, the equation

$$m=x_1+\cdots+x_h,\qquad x_1\le\cdots\le x_h,\qquad x_i\in A$$

has, at most, one solution.

Let $F_h(N)$ denote the maximum number of elements that can be selected from the set $\{1,\dots,N\}$ so as to form a B_h set. Bose and Chowla [1] proved that

$$F_h(N)\ge N^{1/h}+o(N^{1/h}).\tag{1.1}$$

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On the other hand, if A is a B_h subset of $\{1,\dots,N\}$, an easy counting argument implies the trivial upper bound

$$F_h(N) \le (hh!N)^{1/h}. \tag{1.2}$$

The reader is refered to [9], [11] and [15] for well written surveys about this topic.

For $h = 2$, it is possible to take advantage of counting differences $x_i - x_j$ instead of sums $x_i + x_j$. In this way, P.Erdős and P.Turán [4] proved that $F_2(N) \le N^{1/2} + O(N^{1/4})$, which is the best possible upper bound except for the estimate of the error term. Unfortunately, a similar argument doesn't work for $h > 2$.

Denote by mA the set $A+\dots+A = \{a_1+\dots+a_m; a_i \in A\}$. In 1969 Lindstrom [14] observed that if A is a B_4 sequence then $2A = A+A$ is nearly a B_2 sequence. Using this idea he obtained the non-trivial upper bound for $F_4(N)$.

$$F_4(N) \le (8N)^{1/4} + O(N^{1/8}). \tag{1.3}$$

X.D.Jia [10] generalized Lindstrom's argument to $h = 2m$ and he obtained

$$F_{2m}(N) \le (m(m!)^2)^{1/2m} N^{1/2m} + O(N^{1/4m}). \tag{1.4}$$

An analytic proof of (1.4) has been given by M.Kolountzakis [12]. A similar upper bound for $F_{2m-1}(N)$ has been proved independently by S.Chen [2] and S.W.Graham [7]:

$$F_{2m-1}(N) \le ((m!)^2)^{1/(2m-1)} N^{1/(2m-1)} + O(N^{1/(4m-2)}). \tag{1.5}$$

For large m ($m > 6^3$), N.Alon (unpublished) has obtained a better upper bound for $F_{2m}(N)$ exploiting the "concentration" of the sums $a_1 + \dots + a_m - a'_1 - \dots - a'_m$ around the value 0:

$$F_{2m}(N) \le (6^{3/2}\sqrt{m}(m!)^2)^{1/2m} N^{1/2m} + o(N^{1/2m}). \tag{1.6}$$

An sketch of the proof of (1.6) is contained in [11].

For $m = 2$, the estimate (1.5) gives $F_3(N) \le (4N)^{1/3} + O(N^{1/6})$. S.W.Graham [7], using an argument due to A.Li [13], made a slight improvement to this upper bound:

$$F_3(N) \le (3.996N)^{1/3} + O(1). \tag{1.7}$$

In this paper we introduce new combinatorial and analytic ideas to improve all these upper bounds. Precisely, our aim is proving the following result.

Theorem 1.1.

$$F_3(N) \leq \left(\frac{4}{1+\frac{16}{(\pi+2)^4}}N\right)^{1/3} + o(N^{1/3}),$$

$$F_4(N) \leq \left(\frac{8}{1+\frac{16}{(\pi+2)^4}}N\right)^{1/4} + o(N^{1/4}).$$

For $3 \leq m < 38$*,*

$$F_{2m-1}(N) \leq \left(\frac{(m!)^2}{1+\cos^{2m}(\pi/m)}N\right)^{1/(2m-1)} + o(N^{1/(2m-1)}),$$

$$F_{2m}(N) \leq \left(\frac{m(m!)^2}{1+\cos^{2m}(\pi/m)}N\right)^{1/2m} + o(N^{1/2m}).$$

For $m \geq 38$*,*

$$F_{2m-1}(N) \leq \left(\frac{5}{2}\left(\frac{15}{4}-\frac{5}{4m}\right)^{1/4}\frac{(m!)^2}{\sqrt{m}}N\right)^{1/(2m-1)} + o(N^{1/(2m-1)}),$$

$$F_{2m}(N) \leq \left(\frac{5}{2}\left(\frac{15}{4}-\frac{5}{4m}\right)^{1/4}\sqrt{m}(m!)^2N\right)^{1/2m} + o(N^{1/2m}).$$

We shall use two different strategies to prove Theorem 1.1. For small m ($m < 38$) we will use that the sums $a_1 + \cdots + a_m$ are not well distributed in the interval $[m, mN]$. It is the most interesting part of this paper. For large m ($m \geq 38$) we will take advantage of the concentration of the sums $a_1 + \cdots + a_m - a'_1 - \cdots - a'_m$ around the value 0.

Section 2 is devoted to prove a combinatorial identity for sequences of integers (Lemma 2.1) that has independent interest in additive number theory. See [4] and [5] for more applications of this identity.

In Section 3, inspired by an idea due to I.Ruzsa, we use Fourier analysis to prove that if A is a B_m sequence contained in $[1,N]$, then the sequence $mA \subset [m, mN]$ is not well distributed in short intervals. A weaker result was used by I. Ruzsa, C.Trujillo and the author in [3] to obtain non trivial upper bounds for $B_h[g]$ sequences.

In Section 4, we shall use the results of the previous sections to prove Theorem 1.1. for the cases $2 \leq m < 38$.

Finally, in Section 5 we prove Theorem 1.1 for $m \geq 38$, using a probabilistic approach.

2. A combinatorial lemma for sequences.

In this section we present a combinatorial identity which works for general sequences of integers.

Lemma 2.1. *Let $A \subset \{1, \dots, N\}$. Then, for any integer $H \geq 1$ we have*

$$2 \sum_{1 \leq h \leq H-1} d_A(h)(H-h) = \frac{H^2|A|^2}{N+H-1} - H|A| + \sum_{n=1}^{N+H-1} \left(A(n) - A(n-H) - \frac{H|A|}{N+H-1} \right)^2,$$

where $d_A(h)$ is number of solutions of $h = a - a'; \quad a, a' \in A$ and $A(n)$ is the counting function of A.

This identity captures valuable information about the sequence A: the size of A, the number of small differences $a - a'$ and a measure of the distribution of the elements of A.

Proof.
$A(n) - A(n-H)$ is the size of the set $I_n = \{i; \quad 0 \leq i \leq H-1, \quad n \in A+i\}$. Then we can write the last sum of the lemma as

$$\sum_{1 \leq n \leq N+H-1} |I_n|^2 - 2\frac{H|A|}{N+H-1} \sum_{1 \leq n \leq N+H-1} |I_n| + \frac{H^2|A|^2}{N+H-1},$$

where

$$\sum_{0 \leq i \leq H-1} |I_n| = \sum_{1 \leq n \leq N+H-1} A(n) - A(n-H) = H|A|.$$

Then we have

$$\sum_{1\le n\le N+H-1}\left(|I_n|-\frac{H|A|}{N+H-1}\right)^2=\sum_{1\le n\le N+H-1}|I_n|^2-\frac{H^2|A|^2}{N+H-1}.$$

We also observe that

$$|I_n|^2=\#\{(i,j);n\in A+i,n\in A+j\}=\#\{(i,j);n\in(A+i)\cap(A+j)\}.$$

Then

$$\sum_{0\le n\le N+H-1}|I_n|^2=\sum_{0\le i,j\le H-1}|(A+i)\cap(A+j)|=$$

$$=H|A|+2\sum_{0\le i<j\le H-1}|(A+i)\cap(A+j)|.$$

It is easy to see that $|(A+i)\cap(A+j)|=|A\cap(A+j-i)|$. Then

$$\sum_{0\le i<j\le H-1}|(A+i)\cap(A+j)|=\sum_{1\le h\le H-1}|A\cap(A+h)|(H-h)=\sum_{1\le h\le H-1}d_A(h)(H-h),$$

and the lemma follows. $\square$

The following corollary is an inmediate consequence of Lema 2.1; however it loses the information about the distribution of the elements of A, which is crucial to improve (1.4) and (1.5).

Corollary 2.1. *Let $A\subset[1,N]$ a sequence of integers. Then for any integer $H\ge 1$ we have*

$$2\sum_{1\le h\le H-1}d_A(h)(H-h)\ge\frac{H^2|A|^2}{N+H-1}-H|A|.$$

S.W. Graham [7] deduced the above inequality from the Van der Corput lemma and applied it to the sequence mA when A is a B_{2m} sequence or a B_{2m-1} sequence to prove (1.4) and (1.5).

A known fact about B_{2m} sequences is that most differences $a_1+\cdots+a_m-a_1'-\cdots-a_m'$ are different up to the order of the a_i and a_i'. We state this fact (see [7] for a proof) in a suitable form to apply it to Lemma 2.1 in Section 4.

Lemma 2.2. *Let $A \subset [1, N]$ be a B_{2m} sequence. Then*

$$2 \sum_{1 \le h \le H-1} d_{mA}(h)(H-h) \le H^2 + O(HN^{(2m-1)/2m}).$$

Lemma 2.3. *Let $A \subset [1, N]$ be a B_{2m-1} sequence. Then*

$$2 \sum_{1 \le h \le H-1} d_{mA}(h)(H-h) \le \frac{H^2|A|}{m} + O(H|A|^{2m-1}).$$

In this paper we will take advantage of the last sum in Lemma 2.1. It should be noted that this information is lost if we apply Corollary 2.1. instead of lemma 2.1.

The sum $\sum_{n=1}^{N+H-1}\left(A(n) - A(n-H) - \frac{H|A|}{N+H-1}\right)^2$ is a measure of the distribution of the elements of A in short intervals. We shall study this question in the next section.

3. The distribution of the elements of sumsets.

The following theorem has an independent interest. In [3] we used it in a weaker version to obtain non trivial upper bounds for $B_h[g]$ sequences. Actually, the upper bounds obtained in [3] follow immediately from Theorem 3.1 by taking $H = 1$ and $\mu = gm!$.

Given a sequence of integers A, we define

$$r_h(n) = \#\{n = a_1 + \cdots + a_h; a_i \in A\} \qquad \text{and} \qquad R_h(N) = \sum_{n \le N} r_h(n).$$

Theorem 3.1. *Let $A \subset [1, N]$ be a sequence of integers and $h \ge 2$ an integer. For any real μ and any positive integer $H = o(N)$ we have*

$$\sum_{h \le n < hN+H} |R_h(n) - R_h(n-H) - \mu| \ge (L_h + o(1))\, H|A|^h,$$

where

$$L_2 = \frac{4}{(\pi+2)^2} \qquad \textit{and} \qquad L_h = \cos^h(\pi/h) \textit{ for } h > 2.$$

Proof. Let $f(t) = \sum_{a \in A} e^{iat}$. Then $f^h(t) = \sum_{h \le n \le hN} r_h(n)e^{int}$, and for any m, $0 \le m \le H-1$ we have

$$f^h(t)e^{imt} = \sum_{h \le n \le hN+H-1} r_h(n-m)e^{int}.$$

Then

$$f^h(t)\sum_{0\le m\le H-1} e^{imt} = \sum_{h\le n\le hN+H-1} (R_h(n)-R_h(n-H))\, e^{int} =$$

$$= \mu \sum_{h\le n\le hN+H-1} e^{int} + \sum_{h\le n\le hN+H-1} (R_h(n)-R_h(n-H)-\mu)\, e^{int}$$

for any real μ.

Using the notation $D_k(t)=\sum_{m=0}^{k} e^{imt}$ we can write

$$\begin{aligned} f^h(t)D_{H-1}(t) =& \mu e^{hit} D_{h(N-1)+H-1}(t) \\ &+ \sum_{h\le n\le hN+H-1} (R_h(n)-R_h(n-H)-\mu)e^{int}. \end{aligned}$$

We known that

$$|D_k(t)| = \left| \frac{\sin(\frac{k+1}{2}t)}{\sin\frac{t}{2}} \right| \qquad \text{for } 0<t<2\pi. \tag{3.1}$$

Let $t_h=\frac{2\pi}{h(N-1)+H}$. Then, for any integer j, $0<j<N$ we have

$$D_{h(N-1)+H-1}(jt_h)=0.$$

and we have

$$f^h(jt_h)D_{H-1}(jt_h) = \sum_{h\le n\le hN+H-1} (R_h(n)-R_h(n-H)-\mu)\, e^{injt_h}$$

for any integer j, $0<j<N$.

Taking absolute values gives

$$|f(jt_h)|^h|D_{H-1}(jt_h)| \le \sum_{h\le n\le hN+H-1} |R_h(n)-R_h(n-H)-\mu| = S.$$

Now we use the fact that $|D_k(t)|$ is a decreasing function for t, $0\le t\le \frac{\pi}{k+1}$ to conclude

$$|f(jt_h)| \le \left(\frac{S}{|D_{H-1}(Jt_h)|}\right)^{1/h} \tag{3.2}$$

for any integer j, $1\le j\le J\le N$, whenever $Jt_h\le \frac{\pi}{H}$.

In the next we will take $J = [(N/H)^{2/3}]$. This choice implies that $HJ \le H^{1/3}N^{2/3} \le (H+2N)/3$, which gives $Jt_h < \pi/H$.

Now we are looking for a lower bound for $|f(jt_h)|$. Since the midpoint of the interval $[1,N]$ is $(N+1)/2$, it will useful to express f as

$$f(jt_h) = e^{\frac{N+1}{2}ijt_h} f^*(jt_h)$$

where

$$f^*(jt_h) = \sum_{a\in A} e^{i(a-\frac{N+1}{2})jt_h}.$$

Suppose we have a function $F(x) = \sum_{1\le j\le J} b_j \cos(jx)$ such that $F(x) \ge 1$ if $|x| \le \frac{\pi}{h}$. Let $C_F = \sum_j |b_j|$. We are looking for a lower and an upper bound for $Re\left(\sum_{j=1}^J b_j f^*(jt_h)\right)$.

$$Re\left(\sum_{j=1}^J b_j f^*(jt_h)\right) = Re\left(\sum_{a\in A}\sum_{j=1}^J b_j e^{i(a-(N+1)/2)jt_h}\right) =$$

$$= \sum_{a\in A}\sum_{j=1}^J b_j \cos((a-(N+1)/2)jt_h) = \sum_{a\in A} F((a-(N+1)/2)t_h) \ge |A|, \qquad (3.3)$$

because $|(a-(N+1)/2)t_h| \le \frac{\pi}{h}$ for any integer $a \in A$.

On the other hand, using (3.2) we obtain

$$Re\left(\sum_{j=1}^J b_j f^*(jt_h)\right) \le \sum_{j=1}^J |b_j||f^*(jt_h)| = \sum_{j=1}^J |b_j||f(jt_h)| \le C_F\left(\frac{S}{D_{H-1}(Jt_h)}\right)^{1/h}.$$

From this estimate and (3.3) we deduce

$$S = \sum_{h\le n\le hN+H-1} |R_h(n) - R_h(n-H) - \mu| \ge \frac{1}{C_F^h}|D_{H-1}(Jt_h)||A|^h.$$

Now we obtain a lower bound for $|D_{H-1}(Jt_h)|$ by noting that

$$|D_{H-1}(Jt_h)| = \left|\frac{\sin(\frac{H}{2}Jt_h)}{\sin\frac{Jt_h}{2}}\right| \ge H\frac{\sin(\frac{H}{2}Jt_h)}{\frac{H}{2}Jt_h} \ge H\left(1 - \frac{\pi^2}{6}\left(\frac{HJ}{N}\right)^2\right),$$

where we have used the estimate $\frac{\sin x}{x} \geq 1 - \frac{x^2}{6}$ for $x > 0$. Then

$$S \geq \left(\frac{1}{C_F^h} + O\left(\left(\frac{H}{N} \right)^{2/3} \right) \right) H|A|^h. \tag{3.4}$$

To finish the theorem we look for a function F such that C_F is as small as possible.

For $h > 2$ we choose $F(x) = \frac{1}{\cos(\pi/h)} \cos x$ with $C_F = \frac{1}{\cos(\pi/h)}$. Then

$$S \geq \left(\cos^h(\pi/h) + O\left(\left(\frac{H}{N} \right)^{2/3} \right) \right) H|A|^h$$

and we prove the theorem for $h > 2$.

For the case $h = 2$, a little more work is required. If we consider the function

$$G(x) = \begin{cases} 1, & |x| \leq \pi/2 \\ 1 + \pi \cos(x), & \pi/2 < |x| \leq \pi \end{cases}$$

it is easy to see that

$$G(x) = \frac{\pi}{2} \cos(x) + 2 \sum_{j=2}^{\infty} \frac{\cos(\pi j/2)}{j^2 - 1} \cos(jx).$$

We need truncate the series. Let

$$G_J(x) = \frac{\pi}{2} \cos(x) + 2 \sum_{j=2}^{J} \frac{\cos(\pi j/2)}{j^2 - 1} \cos(jx).$$

Obviously $|G(x) - G_J(x)| \leq 2 \sum_{j > \frac{J}{2}} \frac{1}{4j^2-1} \leq \frac{2}{(J-1)}$.

We shall prove that the function $F_J(x) = \frac{J-1}{J-3} G_J(x)$ satisfies the suitable conditions.

$$F_J(x) = \frac{J-1}{J-3} G_J(x) = \frac{J-1}{J-3} G(x) + \frac{J-1}{J-3} (G_J(x) - G(x))$$

and

$$\begin{aligned} F_J(x) &\geq \frac{J-1}{J-3} G(x) - \frac{J-1}{J-3} |G_J(x) - G(x)| \geq \\ &\geq \frac{J-1}{J-3} - \frac{J-1}{J-3} \frac{2}{J-1} = 1 \end{aligned}$$

for $|x| \le \pi/2$.

On the other hand,

$$C_{F_J} = \frac{J-1}{J-3}\left(\pi/2 + 2\sum_{1\le j\le J/2}\frac{1}{4j^2-1}\right) \le$$

$$\le \frac{J-1}{J-3}\left(\pi/2 + 2\sum_{j=1}^{\infty}\frac{1}{4j^2-1}\right) = \frac{J-1}{J-3}(\pi/2+1).$$

Substitution in (3.4) with $J = [(\frac{N}{H})^{2/3}]$ gives the theorem for $h = 2$. $\square$

S.W. Graham [8] has observed that the constant $C_F = \pi/2 + 1$ is the best possible constant for $h = 2$. I include here his elegant proof.

Proposition 3.1. (S.W. Graham) *Suppose* $F(x) = \sum_{1\le j\le\infty} b_j \cos(jx)$ *satisfies* $F(x) \ge 1$ *for* $|x| \le \frac{\pi}{2}$. *Then* $C_F = \sum_j |b_j| \ge \pi/2 + 1$.

Proof.

First note that $1 \le F(\pi/2) = \sum_j b_{2j}(-1)^j \le \sum_j |b_{2j}|$. Next, we have

$$0 = \int_{-\pi}^{\pi} F(x)dx = 2\int_0^{\pi/2} F(x)dx + 2\int_{\pi/2}^{\pi} F(x)dx.$$

Since $F(x) \ge 1$ for $|x| \le \pi/2$, it follows that

$$\left|\int_{\pi/2}^{\pi} F(x)dx\right| = \left|\sum_j b_j \frac{\sin(\pi j/2)}{j}\right| \ge \pi/2,$$

and this in turn gives us

$$\sum_{j=0}^{\infty} |b_{2j+1}| \ge \pi/2.$$

So, we have lower bounds for the sum of $|b_j|$, j even and j odd. Adding these bounds together the theorem folows. $\square$

If we apply Theorem 3.1 to B_m sequences we obtain the following corollary:

Corollary 3.1. *If $A \subset [1, N]$ is a B_m sequence then, for any real μ and any integer $H = o(N)$*

$$\sum_{m \le n < mN+H} |(mA)(n) - (mA)(n-H) - \mu| \ge (D_m + o(1))\, H|A|^m,$$

where

$$D_2 = \frac{2}{(\pi+2)^2} \qquad \text{and} \qquad D_m = \frac{\cos^m(\pi/m)}{m!} \qquad \text{for} \qquad m > 2.$$

Proof. Observe that if A is a B_m sequence and $s \in mA$, then $r_m(s) = m!$ except when the unique representation (up to order) of s as a sum of m summands has some repeated summand. Then we can write $r_m(s) = r'_m(s) - r''_m(s)$ where $r'_m(s) = m!$ if $s \in mA$ and $r'_m(s) = 0$ if $s \notin mA$. Obviously $0 \le r''_m(s) \le r'_m(s) \le m!$.

Let $R'_m(n) = \sum_{k \le n} r'(k) = m!(mA)(n)$ and $R''_m(n) = \sum_{k \le n} r''(k)$

Then

$$\sum_{m \le n \le mN+H-1} |R_h(n) - R_h(n-H) - \mu m!| =$$

$$= \sum_{m \le n \le mN+H-1} |m!\,((mA)(n) - (mA)(n-H) - \mu) - (R''_m(n) - R''_m(n-H))| \le$$

$$\le m! \sum_{m \le n \le mN+H-1} |(mA)(n) - (mA)(n-H) - \mu|$$

$$+ \sum_{m \le n \le mN+H-1} |R''_m(n) - R''_m(n-H)|.$$

It should be noted that $|R''_m(mN)|$ counts the elements of mA with some repeated summand. Then

$$\sum_{m \le n \le mN+H-1} |R''_m(n) - R''_m(n-H)| = HR''(mN) \ll H|A|^{m-1}.$$

Now we apply Theorem 3.1 to get

$$\sum_{m \le n \le mN+H-1} |(mA)(n) - (mA)(n-H) - \mu| \ge$$

$$\ge \frac{1}{m!} \sum_{m \le n \le mN+H-1} |R_h(n) - R_h(n-H) - \mu m!| + O(H|A|^{m-1}) \ge$$

$$\ge (D_m + o(1))H|A|^m.$$

□

4. Proof of Theorem 1.1 (Cases $2 \le m < 38$)

Let A be a B_{2m} or a B_{2m-1} sequence contained in $[1, N]$. Now we apply Lemma 2.1 to the sequence mA, which is contained in $[1, mN]$.

$$2 \sum_{1 \le h \le H-1} d_{mA}(h)(H-h) = \frac{H^2|mA|^2}{mN+H-1} - H|mA| +$$

$$+ \sum_{n=1}^{mN+H-1} ((mA)(n) - (mA)(n-H) - \mu)^2 \qquad (4.1.)$$

with $\mu = \frac{H|mA|}{mN+H-1}$. If we apply the Cauchy inequality to the last sum we obtain

$$\sum_{1 \le n \le mN+H-1} ((mA)(n) - (mA)(n-H) - \mu)^2 \ge$$

$$\ge (mN+H-1)^{-1} \left(\sum_{1 \le n < mN+H} |(mA)(n) - (mA)(n-H) - \mu| \right)^2 \ge$$

$$\ge (mN+H-1)^{-1} \left(D_m^2 + o(1) \right) H^2 |A|^{2m}.$$

We have applied Corollary 3.1 in the last inequality.

Now, if $h = 2m$ we use Lemma 2.2 to obtain an upper bound for the left hand side of (4.1) giving

$$H^2 + O(HN^{(2m-1)/2m}) \ge 2 \sum_{1 \le h \le H-1} d_{mA}(h)(H-h) \ge$$

$$\ge \frac{H^2|mA|^2}{mN+H-1} - H|mA| + (mN+H-1)^{-1}(D_m^2 + o(1))H^2|A|^{2m}.$$

Trivially we have that $|A| \ll N^{1/2m}$ and $|mA| \ll N^{1/2}$. Then, if we take $H = [N^{1-1/4m}]$ and divide the inequality by H^2 we can write

$$1 + o(1) \ge \frac{|mA|^2}{mN(1+o(1))} - o(1) + \frac{D_m^2 + o(1)}{mN(1+o(1))} |A|^{2m}.$$

It should be noted that A is "a priori" a B_m sequence. Then

$$|mA| = \binom{|A|+m-1}{m} \ge \frac{|A|^m}{m!}$$

and, consequently,

$$mN(1+o(1)) \geq \frac{|A|^{2m}}{(m!)^2} + (D_m^2 + o(1))|A|^{2m}$$

so, finally,

$$|A| \leq \left(\frac{m}{\frac{1}{(m!)^2} + D_m^2} + o(1) \right)^{1/2m} N^{1/2m}.$$

The numbers D_m are defined in Corollary 3.1 and we obtain the upper bounds

$$F_4(N) \leq \left(\frac{2}{\frac{1}{4} + \frac{4}{(\pi+2)^4}} N \right)^{1/4} + o(N^{1/4}) \leq (7.821N)^{1/4} + o(N^{1/4})$$

and

$$F_{2m}(N) \leq \frac{m^{1/2m}(m!)^{1/m}}{(1+\cos^{2m}(\pi/m))^{1/2m}} N^{1/2m} + o(N^{1/2m}), \qquad m > 2.$$

If $h = 2m-1$ we use Lemma 2.3 to get an upper bound for the left side of (4.1) and we proceed in a similar way.

$$\frac{H^2|A|}{m} + O(H|A|^{2m-1}) \geq 2 \sum_{1 \leq h \leq H-1} d_{mA}(h)(H-h) \geq$$

$$\geq \frac{H^2|mA|^2}{mN+H-1} - H|mA| + (mN+H-1)^{-1}(D_m^2 + o(1))H^2|A|^{2m}.$$

Trivially we have that $|A| \ll N^{1/(2m-1)}$ and $|mA| \ll N^{m/(2m-1)}$. If we take $H = [N^{1/2}|A|^{m-1}]$ and divide the inequality by $H^2|A|$ we can write

$$\frac{1}{m} + o(1) \geq \frac{|mA|^2}{|A|mN(1+o(1))} - o(1) + \frac{D_m^2 + o(1)}{N(1+o(1))}|A|^{2m-1}.$$

Again, it should be noted that A is "a priori" a B_m sequence. Then

$$|mA| = \binom{|A|+m-1}{m} \geq \frac{|A|^m}{m!}$$

and

$$N(1+o(1)) \geq \frac{|A|^{2m-1}}{(m!)^2} + (D_m^2 + o(1))|A|^{2m-1},$$

so, finally,

$$|A| \le \left(\frac{1}{\frac{1}{(m!)^2} + D_m^2} + o(1) \right)^{1/(2m-1)} N^{1/(2m-1)}.$$

Again, the numbers D_m are defined in Corollary 3.1 and we obtain the upper bounds

$$F_3(N) \le \left(\frac{1}{\frac{1}{4} + \frac{4}{(\pi+2)^4}} N \right)^{1/3} + o(N^{1/3}) \le (3.911N)^{1/3} + o(N^{1/3}),$$

$$F_{2m-1}(N) \le \frac{(m!)^{2/2m-1}}{(1+\cos^{2m}(\pi/m))^{1/2m-1}} N^{1/2m-1} + o(N^{1/(2m-1)}), \qquad m > 2. \quad \square$$

5. Proof of Theorem 1.1 (Cases $m \ge 38$). A probabilistic approach.

Suppose that $A \subset [1, N]$ is a B_{2m} sequence. Let the random variable Y be defined by

$$Y = X_1 + \cdots + X_m - X_1' - \cdots - X_m',$$

where the X_j are independent random variables uniformly distributed in A.

We will take advantage of the concentration of Y around the value 0, considering the 4−moment of Y.

We know that

$$E_4 = E(Y^4) \ge \lambda^4 N^4 P(|Y| \ge \lambda N).$$

Then

$$P(|Y| < \lambda N) \ge 1 - \frac{E_4}{\lambda^4 N^4} \tag{5.1}$$

On the other hand we have that

$$P(|Y| < \lambda N) \le |A|^{-2m} \sum_{|n| < \lambda N} r(n),$$

where $r(n) = \#\{n = a_1 + \cdots + a_m - a_1' - \cdots - a_m'; \quad a_i, a_i' \in A\}$.

Let $r(n) = r'(n) + r''(n)$, where $r'(n)$ counts the representations where all the a_i, a_i' are distinct and $r''(n)$ counts the representations with some repeated element.

Then, $\sum r''(n) \ll |A|^{2m-1} \ll N^{\frac{2m-1}{2m}} < o(N)$ because of the trivial estimate $|A| \ll N^{1/2m}$.

Now we observe that if A is a B_{2m} sequence, the representation

$$n = a_1 + \cdots + a_m - a_1' - \cdots - a_m', \quad a_1 < \cdots < a_m, a_1' < \cdots < a_m'$$

is unique. So $r'(n) \le (m!)^2$ due to the $m!$ permutations of the a_i and a_i'. and

$$\sum_{|n|<\lambda N} r(n) \le 2\lambda N(m!)^2 + o(N). \tag{5.2}$$

Now we can write

$$P(|Y| < \lambda N) \le |A|^{-2m}\lambda\left(2(m!)^2 + o(1)\right)N. \tag{5.3}$$

From (5.1) and (5.3) we deduce

$$|A|^{2m} \le \frac{\lambda^5 N^4}{\lambda^4 N^4 - E_4}\left(2(m!)^2 + o(1)\right)N$$

and taking $\lambda = \left(\frac{5E_4}{N^4}\right)^{1/4}$ we have

$$|A|^{2m} \le \frac{5}{2}\left(\frac{5E_4}{N^4}\right)^{1/4}\left((m!)^2 + o(1)\right)N \tag{5.4}$$

The next step is to obtain an upper bound for $E_4 = E(|Y|^4)$.

It is useful to write $Y = Y_1 + \cdots + Y_m$, where $Y_i = X_i - X_i'$. The first observation is that $E(Y_i^j) = 0$ if j is odd. Then

$$E((Y_1 + \cdots + Y_m)^4) = 4! \sum_{\gamma_1+\cdots+\gamma_m=4} \frac{E(Y_1^{\gamma_1})\cdots E(Y_m^{\gamma_m})}{\gamma_1!\cdots\gamma_m!}$$

$$= mE(Y_1^4) + 3m(m-1)E^2(Y_1^2)$$

Observe that $E(Y_1^4) \le N^2 E(Y_1^2)$ and

$$E(Y_1^2) = E((X_1 - X_1')^2) = 2E(X_1^2) - 2E^2(X_1) \le 2(E(X_1)(N - E(X_1))) \le \frac{N^2}{2}.$$

Then

$$E_4 \le N^4\left(\frac{m}{2} + \frac{3m(m-1)}{4}\right) = \frac{3m^2 - m}{4}N^4$$

If we substitute in (5.4) we obtain

$$|A|^{2m} \leq \frac{5}{2}\left(\frac{5}{4}(3-1/m)\right)^{1/4}\left(\sqrt{m}(m!)^2+o(1)\right)N.$$

and we finish the proof of Theorem 1.1. for B_{2m} sequences.

If A is a B_{2m-1} sequence the proof is similar except that in this case we have the following estimate in (5.2):

$$\sum_{|n|<\lambda N} r(n) \leq \frac{|A|}{m}\left(2\lambda(m!)^2+o(1)\right)N$$

and the proof follows in the same way. □

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