A NOTE ON PRODUCT SETS OF RATIONALS

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Abstract. Bourgain, Konyagin and Shparlinski obtained a lower bound for the size of the product set AB when A and B are sets of positive rational numbers with numerator and denominator less or equal than Q. We extend and slightly improve that lower bound using a different approach.

1. Introduction

Bourgain, Konyagin and Shparlinsky [1] obtained a lower bound for the size of the product of two sets of rational numbers

$$A, B \subset \mathcal{F}_Q = \{q/q' : \ 1 \le q, q' \le Q\}$$

and they applied it to the study of the distribution of elements of multiplicative groups in residue rings. See [3] and [2] for related results and more applications of this useful inequality.

Theorem A (BKSh). *If $A, B \subset \mathcal{F}_Q$ then*

$$|AB| \ge |A||B| \exp\left(-(9+o(1)) \log Q / \sqrt{\log\log Q}\right), \tag{1}$$

where $o(1) \to 0$ when $Q \to \infty$.

For any real numbers $Q, Q' \ge 1$ let $\mathcal{F}_{Q,Q'}$ denotes the set of rational numbers

$$\mathcal{F}_{Q,Q'} = \{q/q' : \ 1 \le q \le Q, \ 1 \le q' \le Q'\}.$$

We give the following result which extends and slightly improves Theorem A.

Theorem 1. *If $A, B \subset \mathcal{F}_{Q,Q'}$ then*

$$|A/B| \ge |A||B| \exp\left(-(2\sqrt{\log 2} + o(1)) \log(QQ') / \sqrt{\log\log(QQ')}\right),$$

where $o(1) \to 0$ when $QQ' \to \infty$.

Taking $Q' = Q$ and the set $1/B = \{b^{-1} : \ b \in B\}$ instead of B we improve the constant in (1).

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Corollary 1. *If $A, B \in \mathcal{F}_Q$, then*

$$|AB| \geq |A||B| \exp\left(-(4\sqrt{\log 2} + o(1)) \log Q/\sqrt{\log\log Q}\right).$$

2. Proof of Theorem 1

For any pair of sets $A, B \subset \mathcal{F}_{Q,Q'}$ and $\gcd(r,s) = 1$ we define the sets

$$\begin{aligned} \mathcal{M}(A \times B, r/s) &= \{(a/a', b/b') \in A \times B : \gcd(a,b) = r, \ \gcd(a',b') = s\} \\ A_{r/s} &= \{a/a' \in A, \ r \mid a, \ s \mid a'\} \\ B_{r/s} &= \{b/b' \in B, \ r \mid b, \ s \mid b'\}. \end{aligned}$$

It is clear that $\mathcal{M}(A \times B, r/s) \subset A_{r/s} \times B_{r/s}$, so we have

$$|\mathcal{M}(A \times B, r/s)| \leq |A_{r/s}||B_{r/s}|. \tag{2}$$

We claim that each $c/d \in A/B$ (assume that $\gcd(c,d) = 1$) has at most $\tau(c)\tau(d)$ representation as

$$\frac{c}{d} = \frac{a/a'}{b/b'} \tag{3}$$

with $(a/a', b/b') \in \mathcal{M}(A \times B, r/s)$. Indeed we observe that (3) implies $\frac{c}{d} = \frac{a_0 b_0'}{b_0 a_0'}$ where $a_0 = a/r$, $b_0 = b/r$, $a_0' = a_0/s$, $b_0' = b_0/s$. Since $\gcd(c,d) = 1$ and $\gcd(a_0 b_0', a_0' b_0) = 1$ then $c = a_0 b_0'$ and $d = a_0' b_0$, which proves the claim.

Note that $c = a_0 b_0' \leq QQ'$ and $d = a_0' b_0 \leq QQ'$, thus the claim implies the inequality

$$|\mathcal{M}(A, B, r/s)| \leq T^2 |A/B|, \tag{4}$$

where $T = T(QQ')$ and $T(x)$ is the function

$$T(x) = \max_{m \leq x} \tau(m).$$

Using (2), (4) and the well known inequality

$$\sum_{\substack{1 \leq r,s \\ rs \leq x}} 1 \leq x(1 + \log x)$$

we get

$$\begin{aligned} |A||B| &= \sum_{\substack{rs \leq x \\ (r,s)=1}} |\mathcal{M}(A, B, r/s)| + \sum_{\substack{rs > x \\ (r,s)=1}} |\mathcal{M}(A, B, r/s)| \\ &\leq T^2 |A/B| x(1 + \log x) + \sum_{\substack{rs > x \\ (r,s)=1}} |A_{r/s}||B_{r/s}| \end{aligned} \tag{5}$$

for any real number $x \geq 1$. If x is such that the last sum is less than $|A||B|/2$ then we get

$$|A/B| \geq \frac{|A||B|}{2T^2 x(1+\log x)}. \tag{6}$$

Now we are ready to prove the key Lemma.

Lemma 2. *For any $n \geq 1$ and for any $A, B \in \mathcal{F}_{Q,Q'}$ with real numbers $Q, Q' \geq 1$, we have*

$$|A/B| \geq \frac{|A||B|}{(4T)^{n+1}(QQ')^{1/n}(1+\log(QQ'))} \tag{7}$$

where $T = \max_{m \leq QQ'} \tau(m)$.

Proof. We proceed by induction on n: trivially, since $|B| \leq QQ'$ we have

$$|A/B| \geq |A| \geq \frac{|A||B|}{QQ'},$$

which proves (7) for $n = 1$. Suppose that Lemma 2 is true for some $n \geq 1$.

If there is r/s such that

$$|A_{r/s}||B_{r/s}| \geq \frac{(QQ')^{\frac{1}{n(n+1)}}}{4T(rs)^{1/n}}|A||B| \tag{8}$$

we use induction for the sets $A_{r/s}, B_{r/s} \subset \mathcal{F}_{Q/r,Q'/s}$. By observing that the function $T(x) = \max_{m \leq x} \tau(m)$ is a non decreasing function we have

$$\begin{aligned} |A/B| &\geq |A_{r/s}/B_{r/s}| \\ \text{(by induction hypothesis)} &\geq \frac{|A_{r/s}||B_{r/s}|}{(4T)^{n+1}((Q/r)(Q'/s))^{1/n}(1+\log((Q/r)(Q'/s)))} \\ \text{(by (8))} &\geq \frac{|A||B|}{(4T)^{n+2}(QQ')^{1/(n+1)}(1+\log(QQ'))}. \end{aligned}$$

Thus, we assume that

$$|A_{r/s}||B_{r/s}| < \frac{(QQ')^{\frac{1}{n(n+1)}}}{4T(rs)^{1/n}}|A||B|$$

for any r/s, $(r,s) = 1$. In this case we have

$$\begin{aligned} \sum_{rs>x} |A_{r/s}||B_{r/s}| &\leq \max_{rs>x}(|A_{r/s}||B_{r/s}|)^{1/2} \sum_{rs>x} |A_{r/s}|^{1/2}|B_{r/s}|^{1/2} \\ &\leq \frac{(QQ')^{\frac{1}{2n(n+1)}}}{2T^{1/2}x^{\frac{1}{2n}}}(|A||B|)^{1/2}\left(\sum_{r,s}|A_{r/s}|\right)^{1/2}\left(\sum_{r,s}|B_{r/s}|\right)^{1/2}. \end{aligned} \tag{9}$$

To estimate the sums in the brackets we have

$$\sum_{r,s}|A_{r/s}| = \sum_{q/q' \in A}\sum_{\substack{r,s \\ r|q,\ s|q'}} 1 \leq \sum_{q/q' \in A}\tau(qq') \leq |A|T. \tag{10}$$

Putting in (9) the estimate (10) and the analogous for $\sum_{r,s}|B_{r/s}|$ we have

$$\sum_{rs>x}|A_{r/s}||B_{r/s}|\le |A||B|\frac{T^{1/2}(QQ')^{\frac{1}{2n(n+1)}}}{2x^{\frac{1}{2n}}}.$$

Taking $x=T^n(QQ')^{\frac{1}{n+1}}$ we get

$$\sum_{rs>x}|A_{r/s}||B_{r/s}|\le |A||B|/2.$$

Then (6) applies and noting that $\log x\le \log((QQ')^{n+\frac{1}{n+1}})\le 2n\log(QQ')$ we get

$$\begin{aligned}|A/B| &\ge \frac{|A||B|}{2T^2x(1+\log x)}\\ &\ge \frac{|A||B|}{2T^{n+2}(QQ')^{\frac{1}{(n+1)}}(1+2n\log(QQ'))}\\ &\ge \frac{|A||B|}{(4T)^{n+2}(QQ')^{\frac{1}{(n+1)}}(1+\log(QQ'))}\times\frac{2^{2n+3}(1+\log(QQ'))}{1+2n\log(QQ')}\\ &\ge \frac{|A||B|}{(4T)^{n+2}(QQ')^{\frac{1}{(n+1)}}(1+\log(QQ'))}.\end{aligned}$$

□

The well known upper bound for the divisor function,

$$\tau(m)\le \exp((\log 2+o(1))\log m/\log\log m)$$

implies

$$T\le \exp((\log 2+o(1))\log(QQ')/\log\log(QQ')).$$

Thus, an optimal choice of n in Lemma 2 is $n\sim\sqrt{\frac{\log\log(QQ')}{\log 2}}$, from where Theorem 1 follows.

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References

[1] J. Bourgain, S. Konyagin, I. Shparlinski, Product sets of Rationals, Multiplicative Translates of Subgroups in Residue Rings, and Fixed Points of the Discrte Logarithm. *International Mathematical Research Notices*, vol 2008. 1–29.

[2] J. Cilleruelo and M. Garaev, Congruences involving product of intervals and sets with small multiplicative doublings modulo a prime. *Preprint.*

[3] J. Cilleruelo, S. Ramana and O. Ramaré, The number of rational numbers determined by large sets of integers. *Bull. of the London Math. Soc.* vol 42, n3 (2010)

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