QUASI SIDON SETS

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ABSTRACT. We study sets of integers A with $|A-A|$ close to $|A|^2$ and prove that $|A| < \sqrt{n} + \sqrt{|A|^2 - |A-A|}$ for any set $A \subset \{1, \dots, n\}$. For infinite sequences of positive integers $A = (a_n)$ we define $A_n = \{a_1, \dots, a_n\}$ and prove that if $|A_n - A_n| \sim n^2$ then $\limsup_{n\to\infty} a_n/n^2 = \infty$. On the opposite hand we construct, for any positive function $\omega(n) \to \infty$, an infinite sequence A satisfying $|A_n - A_n| \sim n^2$ and $a_n \ll \omega(n)n^2$.

1. Introduction

A Sidon set is a set of integers having the property that all the nonzero differences $a - a'$, $a, a' \in A$ are distinct; i.e. the difference set $A - A = \{a - a' : a, a' \in A\}$ has the maximum possible size: $|A - A| = |A|^2 - |A| + 1$.

Sidon sets have been studied for a long time but we are interested here in those sets with $|A - A|$ close to $|A|^2$ (quasi difference Sidon sets). We prove the inequality

$$|A| < \sqrt{n} + \sqrt{|A|^2 - |A-A|} \tag{1.1}$$

for any set $A \subset \{1, \dots, n\}$. An inmediate consequence of this inequality is that if $|A - A| = |A|^2(1 + o(1))$, then $|A| \le \sqrt{n}(1 + o(1))$, the same asymptotic upper bound we have for Sidon sets. We deduce oher results on Sidon sets from inequality (1.1).

Our main results concern to infinite quasi difference Sidon sequences $A = (a_n)$. Denote $A_n = \{a_1, \dots, a_n\}$. We say that A is a *quasi difference* Sidon sequence if $|A_n - A_n| \sim n^2$. We prove that if A is a quasi difference Sidon sequence then

$$\limsup_{n\to\infty} a_n/n^2 = \infty. \tag{1.2}$$

Erdős proved that $\limsup_{n\to\infty} a_n/(n^2 \log n) > 0$ for Sidon sequences, but conclusion (1.2) is best possible for quasi difference sequences. Indeed we can construct, for any positive function $\omega(n) \to \infty$, an infinite sequence $A = \{a_n\}$ with $|A_n - A_n| \sim n^2$ and $a_n \ll \omega(n)n^2$.

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We cannot decide if (1.2) holds for *quasi sum* Sidon sequences, those sequences with $|A_n + A_n| \sim n^2/2$.

2. Finite quasi Sidon sets

We start with an inequality which is non trivial for those sets A with $|A - A|$ close to $|A|^2$. This inequality simplifies some proofs of known results about Sidon sets. A less precise version of the inequality below has appeared before in [1] and [10].

Theorem 2.1. *If $A \subset \{1, \dots, n\}$ then*

$$|A| < \sqrt{n} + \sqrt{|A|^2 - |A - A|}. \tag{2.1}$$

Theorem 2.1 is a consequence of the following lemma, which is generalization of Theorem 4.2 in [17].

Lemma 2.1. *Let A and B be two subsets of an abelian group G. Then we have*

$$|A|^2 \le |A + B| \left(1 + \frac{|A|^2 - |A - A|}{|B|}\right). \tag{2.2}$$

Proof. As usual we define $r_{A+B}(x) = \#\{(a, b) \in A \times B : \ a + b = n\}$. The following equalities are well known:

i) $|A||B| = \sum_{x \in G} r_{A+B}(x)$
ii) $\sum_{x \in G} r^2_{A+B}(x) = \sum_{x \in G} r_{A-A}(x) r_{B-B}(x)$.

Cauchy inequality and the identities above give the following inequality:

$$\begin{aligned}
(|A||B|)^2 = \left(\sum_{x \in A+B} r_{A+B}(x)\right)^2 &\le |A + B| \sum_x r^2_{A+B}(x) \\
&= |A + B| \sum_{x \in A-A} r_{A-A}(x) r_{B-B}(x) \\
&= |A + B| \left(\sum_{x \in A-A} r_{B-B}(x) + \sum_{x \in A-A} (r_{A-A}(x) - 1) r_{B-B}(x)\right) \\
&\le |A + B| \left(\sum_x r_{B-B}(x) + |B| \sum_{x \in A-A} (r_{A-A}(x) - 1)\right) \\
&\le |A + B| \left(|B|^2 + |B|(|A|^2 - |A - A|)\right).
\end{aligned}$$

□

When A is a Sidon set then $|A-A|=|A|^2-|A|+1$ and Lemma 2.1 gives the following inequality proved by Ruzsa[17]:

$$|A|^2 \le |A+B|\left(1+\frac{|A|-1}{|B|}\right). \tag{2.3}$$

Proof of Theorem 2.1. We consider the set $B=[0,l]\cap\mathbb{Z}$ with

$$l=\lfloor\sqrt{n(|A|^2-|A-A|)}\rfloor.$$

Then $|A+B|\le n+l$ y $|B|=l+1$ and Lemma 3.1 implies that

$$\begin{aligned}|A|^2 &\le (n+l)\left(1+\frac{|A|^2-|A-A|}{l+1}\right)\\ &< n+l+\frac{n(|A|^2-|A-A|)}{l+1}+|A|^2-|A-A|\\ &\le n+2\sqrt{n(|A|^2-|A-A|)}+|A|^2-|A-A|\\ &= (\sqrt{n}+\sqrt{|A|^2-|A-A|})^2\end{aligned}$$

and we get the inequality of the Theorem. □

This inequality has interesting consequences. The first one is the best known upper bound for the size of Sidon sets in intervals [3].

Corollary 2.1. *If $A\subset[1,n]$ is a Sidon set then $|A|<\sqrt{n}+n^{1/4}+1/2$.*

Proof. If A is a Sidon set then $|A-A|=|A|^2-|A|+1$ and the inequality (2.1) implies

$$|A|<\sqrt{n}+\sqrt{|A|-1} \implies (|A|-\sqrt{n})^2<|A|-1.$$

Writting $|A|=\sqrt{n}+cn^{1/4}+1/2$ and putting this expression in the last inequality we obtain

$$c^2n^{1/2}+cn^{1/4}+1/4<n^{1/2}+cn^{1/4}-1/2,$$

which provides a contradiction when $c\ge 1$. □

Ruzsa called weak-Sidon sets those sets having the property that all the sums $a+a',\ a\ne a',\ a,a'\in A$ are distinct. Notice that $2a=a'+a''$ is allowed, so any Sidon set is a weak-Sidon set but the converse is not true. Ruzsa [17] proved that the cardinality of a weak Sidon set $A\subset[1,n]$ is bounded by $\sqrt{n}+4n^{1/4}+11$. P. Mark [15] improved it to $\sqrt{n}+\sqrt{3}n^{1/4}+O(1)$. We give a short proof of this last result.

Corollary 2.2. *If $A\subset[1,n]$ is a weak Sidon set then $|A|<\sqrt{n}+\sqrt{3}n^{1/4}+3/2$.*

Proof. Define the sets

$$(A-A)_1 = \{x: \ x \neq 0, \ r_{A-A}(x) = 1\}$$
$$(A-A)_2 = \{x: \ x \neq 0, \ r_{A-A}(x) = 2\}.$$

It is clear that $r_{A-A}(x) \le 2$ for any $x \neq 0$. Otherwise we would have $x = a-b = c-d$ with $a \neq d$ and $b \neq c$, which is not allowed. On the other hand, if $x \in (A-A)_2$ then there exists $a,b,c \in A$ such that $x = a-b = c-a$ or $x = b-a = a-c$. Thus, each non trivial arithmetic progression of elements of A, say $2a = b+c$ corresponds to two elements of $(A-A)_2$, say $x = a-b$ and $x = b-a$. Thus we have

$$\begin{aligned} |A|^2 &= \sum_x r_{A-A}(x) = |A| + |(A-A)_1| + 2|(A-A)_2| \\ &= |A| + |A-A| - 1 + |(A-A)_2| \\ &= |A| + |A-A| - 1 + 2|P_3|, \end{aligned}$$

where P_3 is the set of non trivial arithmetic progressions in A. Clearly $|P_3| \le |A| - 2$. Thus

$$|A|^2 - |A-A| \le 3|A| - 5.$$

Theorem 2.1 implies that

$$|A| < \sqrt{n} + \sqrt{3|A|-5}.$$

Writing $|A| = \sqrt{n} + cn^{1/4} + 3/2$ an substituying this in $(|A| - \sqrt{n})^2 < 3|A| - 5$ we get

$$c^2\sqrt{n} + 3cn^{1/4} + 9/4 < 3\sqrt{n} + 3cn^{1/4} - 1/2$$

and then $c^2\sqrt{n} + 11/4 < 3\sqrt{n}$, which implies that $c < \sqrt{3}$. □

The B_h sets are sets A with the property that all the sums $a_1 + \cdots + a_h, \ a_1 \le \cdots \le a_h, \ a_i \in A$ are all distinct. The B_2 sets are just the Sidon sets. While there are constructions of B_h sets in $[1,n]$ with $\sim n^{1/h}$ elements, it is unknown the asymptotic estimate for the largest cardinality of a B_h set in $[1,n]$ when $h \ge 3$. The easy counting argument gives the upper bound $(h \cdot h!n)^{1/h}$. A non trivial upper bound was obtained by Lindstrom [14] for B_4 sets, by Jia [13] for B_h sets with h even and by Chen [2] for B_h sequences with h odd. These upper bounds have been improved slightly using deeper methods (see [4] and [11]). We present here a shorter proof of Jia's estimate as consequence of Theorem 2.1.

Corollary 2.3. *If $A \subset [1,n]$ is a B_{2h} set then $|A| \le (h \cdot h!^2 n)^{1/(2h)}(1 + o(1))$.*

Proof. Assume that A is a B_{2h} set. For each $x = a_1 + \cdots + a_h \in hA$ we define the multiset $\overline{x} = \{a_1, \ldots, a_h\}$. We observe that any z has at most one representation of the form $z = x - y$ with $x, y \in hA$ and $\overline{x} \cap \overline{y} = \emptyset$. The reason is that if

$x-y=x'-y'$ then $x+y'=y+x'$ and since A is a B_h set then $\overline{x}\cup\overline{y}'=\overline{x}'\cup\overline{y}$. But since $\overline{x}\cap\overline{y}=\overline{x}'\cap\overline{y}'=\emptyset$ we have that $x=x',\ y=y'$. Thus we get

$$|hA|^2=\sum_{z\in hA-hA} r_{hA-hA}(z)=|hA-hA|+\sum_{z\in hA-hA}(r_{hA-hA}(z)-1)$$

and the last sum is bounded by the number of pairs $(x,y)\in hA\times hA$ with $\overline{x}\cap\overline{y}\neq\emptyset$, which is $O(|A|^{2h-1})$. Since $|hA|^2=\binom{|A|+h-1}{h}^2\gg|A|^{2h}$ we have that

$$|hA-hA|=|hA|^2(1+o(1)).$$

Assuming this we apply the inequality (2.1) to get $|hA|\leq\sqrt{hn}(1+o(1))$. The asymptotic estimate $|hA|\sim|A|^h/h!$ finishes the proof. □

We finish the colection of consequences of Theorem 2.1 with the following Corollary.

Corollary 2.4. *If $A\subset[1,n]$ and $|A-A|\sim|A|^2$ then $|A|\leq\sqrt{n}(1+o(1))$.*

This Corollary follows inmediately from Theorem 2.1, but what it is interesting is that we have not a similar conclusion for sets A with $|A+A|\sim|A|^2/2$. Indeed Erdős and Freud [7, 8, 9] gave an example of a set $A\subset[1,n]$ with $|A+A|\sim|A|^2/2$ of size $|A|\sim\frac{2}{\sqrt{3}}\sqrt{n}$. They considered the set $A=B\cup(n-B)$ where $B\subset[1,n/3)$ is a Sidon set of asymptotic size $\sqrt{n/3}$. It is unknown if the constant $\frac{2}{\sqrt{3}}$ is the largest constant for this problem. Trivially $|A|\leq 2\sqrt{n}(1+o(1))$ if $A\subset[1,n]$ and $|A+A|\sim|A|^2/2$. Erdős and Freud claimed to have a proof of $|A|\leq 1.98\sqrt{n}(1+o(1))$ but Pikhurko [16] has proved that $|A|\leq 1.863\sqrt{n}(1+o(1))$.

Obviously, the set A constructed by Erdős and Freud is an example of a set with $|A+A|\sim|A|^2/2$ but $|A-A|\not\sim|A|^2$. Ruzsa [19] has proved that there exists $c>0$ and sets A with $|A-A|\sim|A|^2$ and $|A+A|\leq|A|^{2-c}$ and sets with $|A+A|\sim|A|^2/2$ and $|A-A|\leq|A|^{2-c}$.

3. Infinite quasi Sidon sequences

A simple counting argument shows that if $A=(a_n)$ is an infinite Sidon sequence then $a_n\gg n^2$. Then, it is a natural question to ask if there is an infinte Sidon sequence A with $a_n\ll n^2$. Erdős (see Theorem 8, Chapter II in [12]) gave a negative answer to this question.

Theorem 3.1 (Erdős)**.** *If A is an infinite Sidon sequence then*

$$\limsup_{n\to\infty}\frac{a_n}{n^2}=\infty. \tag{3.1}$$

Indeed Erdős proved that if A is an infinite Sidon sequence then

$$\limsup_{n\to\infty}\frac{a_n}{n^2\log n}\gg 1. \tag{3.2}$$

We prove here that (3.1) also holds for quasi difference Sidon sequences. The proof of Theorem 3.2 follows the ideas of Erdős to prove (3.2).

Theorem 3.2. *If $A = (a_n)$ is an infinite quasi difference Sidon sequence then*

$$\limsup_{n\to\infty} \frac{a_n}{n^2} = \infty. \tag{3.3}$$

Proof. Denote $A^n = A\cap[1,n]$, so $A(n) = |A^n|$. We observe that (3.3) is equivalent to $\liminf_{n\to\infty} A(n)/\sqrt{n} = 0$. This is what we are proving.

Since A is a quasi difference sequence then $|A^n - A^n| \sim |A^n|^2$, which implies that there exists a positive decreasing function $\epsilon(n) \to 0$ such that

$$|A^n - A^n| \ge |A^n|^2(1-\epsilon(n)). \tag{3.4}$$

We consider the intervals $I_k = ((k-1)n, kn]$, $k = 1,\dots,\omega(n)$ where $\omega(n) = \lceil 1/\sqrt{\epsilon(n)}\rceil$. We denote $D_k = |A\cap I_k|$ and $m = n\omega(n)$. It is clear that

$$\sum_{k\le\omega(n)} \binom{D_k}{2} \le \sum_{1\le x\le n} r_{A^m-A^m}(x) \le n + \sum_{x\in A^m-A^m} (r_{A^m-A^m}(x)-1).$$

On the other hand

$$|A^m|^2 = \sum_{x\in A^m-A^m} r_{A^m-A^m}(x) = |A^m - A^m| + \sum_{x\in A^m-A^m} (r_{A^m-A^m}(x)-1).$$

Then, using (3.4) and $|A^m|^2 \le m(1+o(1))$ we have

$$\begin{aligned}\sum_{k\le\omega(n)} \binom{D_k}{2} &\le n + |A^m|^2 - |A^m - A^m| \\ &\le n + \epsilon(m)|A^m| \\ &\le n + \epsilon(m)m(1+o_m(1)) \\ &\le n + \epsilon(n\omega(n))(\omega(n)n)(1+o_n(1)).\end{aligned}$$

Notice that $\epsilon(n\omega(n))\omega(n) \le \epsilon(n)\omega(n) \ll \sqrt{\epsilon(n)} \to 0$. So,

$$\sum_{k\le\omega(n)} \binom{D_k}{2} \le n(1+o(1)).$$

On the one hand

$$\left(\sum_{k\le\omega(n)} \frac{D_k}{\sqrt{k}}\right)^2 \le \left(\sum_{k\le\omega(n)} \frac{1}{k}\right)\left(\sum_{k\le\omega(n)} D_k^2\right)$$

with

$$\sum_{k\le\omega(n)} \frac{1}{k} \ll \log\omega(n)$$

and

$$\sum_{k\le\omega(n)} D_k^2 = 2\sum_{k\le\omega(n)}\binom{D_k}{2}+\sum_{k\le\omega(n)} D_k$$
$$\le n(1+o(1))+|A^m|$$
$$\le n(1+o(1))+O(\sqrt{n\omega(n)})\le n(1+o(1)).$$

Thus

$$\sum_{k\le\omega(n)}\frac{D_k}{\sqrt{k}}\ll (n\log\omega(n))^{1/2}. \tag{3.5}$$

On the other hand

$$\sum_{k\le\omega(n)}\frac{D_k}{\sqrt{k}}\ge\frac{1}{2}\int_2^{\omega(n)}\frac{\sum_{k\le t}D_k}{t^{3/2}}dt=\frac{1}{2}\int_2^{\omega(n)}\frac{A([t]n)}{t^{3/2}}dt.$$

If $\liminf_{x\to\infty} A(x)/\sqrt{x}>0$ we would have that $A([t]n)\gg\sqrt{[t]n}$ and then

$$\sum_{k\le\omega(n)}\frac{D_k}{\sqrt{k}}\gg\sqrt{n}\int_2^{\omega(n)}\frac{dt}{t}\gg\sqrt{n}\log\omega(n),$$

which is a contradiction with (3.5). □

The following Theorem shows that Theorem 3.2 is sharp. Note that (3.2) does not hold for quasi difference Sidon sequences (take any function $\omega(n)=o(\log n)$ in Theorem 3.3).

Theorem 3.3. *For any positive function $\omega(n)\to\infty$ as $n\to\infty$ it is possible to construct an infinite a sequence $A=\{a_n\}$ satisfying $|A_n-A_n|\sim n^2$ and $a_n\ll\omega(n)n^2$.*

Proof. We can assume that $\omega(n)$ is a non decreasing function. Otherwise we can consider the function $\omega'(n)=\inf_{m\ge n}\omega(n)$.

Lemma 3.1. *For any non decreasing positive function $\omega(n)\to\infty$; there exists a no decreasing function $\omega^*(n)$ satisfying the following conditions:*

i) $\omega^*(n)\le\omega(n)$.
ii) $\omega^*(n+1)\le\omega^*(n)(1+1/n)$.
iii) $\omega^*(n)\to\infty$.

Proof. Define $\omega^*(1)=\omega(1)$ and for $n\ge 1$,

$$\omega^*(n+1)=\min(\omega(n+1),\omega^*(n)(1+1/n)).$$

If $\omega^*(n+1)=\omega^*(n)(1+1/n)$ then it is clear that $\omega^*(n+1)\ge\omega^*(n)$. If $\omega^*(n+1)=\omega(n+1)$ we also have that

$$\omega^*(n+1)\ge\omega(n)\ge\min(\omega(n),\omega^*(n-1)(1+1/(n-1)))=\omega^*(n).$$

Thus, $\omega^*(n)$ is a non decreasing function.

The conditions i) and ii) are trivial consequences from the definition of $\omega^*(n)$. For iii), we distinghis two cases:

a) If $\omega^*(n+1)=\omega^*(n)(1+1/n)$ for $n\ge n_0$ then $\omega^*(n_0+m)=\omega^*(n_0)\prod_{i=0}^{m-1}\left(1+\frac{1}{n_0+i}\right)$ and then $\omega^*(n_0+m)\to\infty$ when $m\to\infty$.
b) If $\omega^*(n+1)=\omega(n+1)$ for infinite many n, then $\limsup\omega^*(n+1)\to\infty$ and then $\omega^*(n+1)\to\infty$ because ω^* is a non decreasing function.

□

Given $\omega(n)$, we construct our sequence A with the following greedy algorithm: Let $a_1=1$ and for $n\ge 1$, define a_{n+1} as the smallest positive integer m, distinct to $a_1,\dots,a_n$ such that

$$|(A_n\cup m)-(A_n\cup m)|\ge(n^2+n)(1-1/\omega^*(n+1)).$$

Thus, the sequence generated by this greedy algorithm satisfies that $|A_n-A_n|\ge(n^2-n)(1-1/\omega^*(n))$. Since $\omega^*(n)\to\infty$ we have that A is a quasi difference Sidon sequence. Hence we have to prove that $a_n\ll\omega(n)n^2$.

The forbidden elements for a_{n+1} are the elements of A_n and the elements m of the set F_n defined by

$$F_n\ =\ \left\{m:\ |(A_n\cup m)-(A_n\cup m)|<(n^2+n)(1-1/\omega^*(n+1))\right\}.$$

Denote

$$T_n(m)=|\{m-a_i\in A_n-A_n:\ i=1,\dots,n\}|.$$

We have

$$\begin{aligned}|(A_n\cup m)-(A_n\cup m)| &\ge |A_n-A_n|+2|\{m-a_i\notin A_n-A_n:\ i=1,\dots,n\}|\\ &\ge (n^2-n)(1-1/\omega^*(n))+2n-2T_n(m).\end{aligned}$$

If $T_n(m)\le\frac{n^2+n}{2\omega^*(n+1)}-\frac{n^2-n}{2\omega^*(n)}$ then

$$|(A_n\cup m)-(A_n\cup m)|\ \ge\ n^2-n+2n-\frac{n^2+n}{\omega^*(n+1)}\ge(n^2+n)\left(1-\frac{1}{\omega^*(n+1)}\right),$$

and $m\notin F_n$. Thus, using the property ii) of $\omega^*(n+1)$ we have

$$\begin{aligned}\sum_m T_n(m) &\ge \left(\frac{n^2+n}{2\omega^*(n+1)}-\frac{n^2-n}{2\omega^*(n)}\right)|F_n|\\ &\ge \left(\frac{n^2+n}{2\omega^*(n)(1+1/n)}-\frac{n^2-n}{2\omega^*(n)}\right)|F_n|\ge\frac{n}{2\omega^*(n)}|F_n|.\end{aligned}$$

On the other hand

$$\sum_m T_n(m)=n|A_n-A_n|\le n(n^2-n+1).$$

It implies that $|F_n| \le 2\omega^*(n)(n^2-n+1)$. Thus the number of forbidden elements for a_{n+1} is at most

$$n + |F_{2,n}| \le n + 2\omega^*(n)(n^2 - n + 1)$$

and a_{n+1} will be an integer less or equal than $\omega^*(n)(2n^2 - 2n + 2) + n + 1 \ll \omega(n)n^2$. □

The greedy algorithm can be modified to get a sequence $A = \{a_n\}$ with $a_n \ll \omega(n)n^2$, which is both, a quasi difference Sidon sequence and a quasi sum Sidon sequence.

We remark that the densest known Sidon sequences $A = \{a_n\}$ have been found by Ruzsa [18] and the author [6] and satisfy $a_n \ll n^{\sqrt{2}+1+o(1)}$.

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