RATIO SETS OF RANDOM SETS

JAVIER CILLERUELO AND JORGE GUIJARRO-ORDÓÑEZ

Abstract. We study the typical behavior of the size of the ratio set A/A for a random subset $A \subset \{1, \dots, n\}$. For example, we prove that $|A/A| \sim \frac{2\mathrm{Li}_2(3/4)}{\pi^2} n^2$ for almost all subsets $A \subset \{1, \dots, n\}$. We also prove that the proportion of visible lattice points in the lattice $A_1 \times \cdots \times A_d$, where A_i is taken at random in $[1, n]$ with $\mathbb{P}(m \in A_i) = \alpha_i$ for any $m \in [1, n]$, is asymptotic to a constant $\mu(\alpha_1, \dots, \alpha_d)$ that involves the polylogarithm of order d.

1. Introduction

Given a set of positive integers A, we say that a lattice point $P \in \mathbb{N} \times \mathbb{N}$ is visible in the lattice $A \times A$ if the line conecting $(0, 0)$ and P does not contain more lattice points of $A \times A$. We denote by visible$(A \times A)$ the set of the visible lattice points in $A \times A$ and denote by A/A the ratio set $A/A = \{a/a' : \ a, a' \in A\}$. Each visible lattice point in the lattice $A \times A$ can be identified with an element of A/A and then we have that $|\text{visible}(A \times A)| = |A/A|$. It is well known that the set of visible lattice points in the plane has density $6/\pi^2$ so if we write $I_n = \{1, \dots, n\}$, we have that $|\text{visible}(I_n \times I_n)| = |I_n/I_n| \sim \frac{6}{\pi^2} n^2$.

In the present paper we study the typical size of visible$(A \times A)$ for a random set A in $\{1, \dots, n\}$ when $n \to \infty$ or, equivalently, the typical size of the ratio set A/A. We consider two natural probabilistic models.

In the first one, denoted by $B(n, \alpha)$, each element in A is chosen independently at random in $\{1, \dots, n\}$ with probability α. Then we have the following

Theorem 1.1. *Let $\alpha \in (0, 1)$ and consider a random subset A in $B(n, \alpha)$. Then with probability $1 - o(1)$ when $n \to \infty$,*

$$|A/A| \sim \mu_2(\alpha) n^2,$$

where

$$\mu_2(\alpha) = \frac{\alpha^2}{\zeta(2)} \frac{\mathrm{Li}_2(1 - \alpha^2)}{1 - \alpha^2}$$

and $\mathrm{Li}_2(z) = \sum_{k \geq 1} \frac{z^k}{k^2}$ is the dilogarithm function.

Notice that the case $\alpha = 1$ corresponds to take $A = I_n$ and its asymptotic estimate appears as the limiting case, as α tends to 1, in Theorem 1.1, since $\lim_{\alpha \to 1} \mu_2(\alpha) = \zeta^{-1}(2) = 6/\pi^2$.

Furthermore, when $\alpha = 1/2$, all the subsets $A \subset \{1, \dots, n\}$ are chosen with the same probability and Theorem 1.1 gives the following result.

Corollary 1.1. *We have that*

$$\frac{1}{2^n} \sum_{A \subset \{1, \dots, n\}} |A/A| \sim \mu_2(1/2) n^2.$$

Furthermore, for almost all sets $A \subset \{1, \dots, n\}$ we have that

$$|A/A| \sim \mu_2(1/2) n^2.$$

A strong convergence version of Theorem 1.1 is also possible.

Corollary 1.2. *Let A be a random infinite sequence of positive integers where all the events $m \in A$ are independent and $\mathbb{P}(m \in A) = \alpha$ for any positive integer m. Let A_n be the random variable $A \cap [1, n]$. Then we have*

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{|A_n/A_n|}{n^2} = \mu_2(\alpha)\right) = 1.$$

For a given positive integer $k = k(n)$, typically $k \asymp n$, we consider the second model, where each subset of k elements is chosen uniformly at random among all sets of size k in $\{1, \ldots, n\}$. We denote this model by $S(n, k)$.

When $k/n \sim \alpha$, the heuristic suggests that both models, $S(n, k)$ and $B(n, \alpha)$, are quite similar. Indeed, this is the strategy we follow to prove Theorem 1.2.

Theorem 1.2. *Let $k \asymp n$ and consider a random subset A in $S(n, k)$. Then with probability $1 - o(1)$ when $n \to \infty$,*

$$|A/A| \sim \mu_2(k/n)n^2.$$

The case $k = n$, which corresponds to the classical result when $A = I_n$, is also obtained as a limiting case in Theorem 1.2 in the sense that $\lim_{k/n\to 1} \mu_2(k/n) = \zeta^{-1}(2) = 6/\pi^2$.

The next theorem deals with the size of A/A in the special case in which A is an arithmetic progression:

Theorem 1.3. *Let A be the set of integers congruent to $a \pmod q$ in $\{1, \ldots, n\}$. We have*

$$|A/A| \sim c_q \frac{n^2}{q^2}$$

when $n \to \infty$, where

$$c_q = \frac{6}{\pi^2} \prod_{p|q} \left(1 - \frac{1}{p^2}\right)^{-1} \sum_{\substack{1\le t\le q \\ (t,q)=1}} \frac{1}{t^2}.$$

Note that, when $q = 1$, we recover once more the classical result for the plane. In the opposite case, it is easy to see that $c_q \to 1$ when $q \to \infty$. Moreover, if we let $\alpha = 1/q$ in Theorem 1.1, we realize that arithmetic progressions are an *atypical* set for this problem.

The results above might suggest that we always have that $|A/A| \asymp |A|^2$ when $A \subset \{1, \ldots, n\}$ has positive density; or, equivalently, that $|A/A| \gg |A|^2$ since the bound $|A/A| \le |A|^2$ is trivial. However, this intuition is false, as the next theorem shows.

Theorem 1.4. *For any $\epsilon > 0$, there exists $\alpha > 0$ such that, for all sufficiently large n, there exists a subset A of the integers in $[1, n]$ satisfying $|A| \ge \alpha n$ and $|A/A| < \epsilon |A|^2$.*

This was proved in a different setting in [2], but here we give an alternative proof, which is simpler and more compact. It is true, however, that $|A/A| \gg_\delta \alpha^\delta |A|^2$ for all $\delta > 0$ and for all $A \subset \{1, \ldots, n\}$ of density $\alpha > 0$, as proved in [2].

Finally, we might ask what happens with visible lattice points in multidimensional spaces, considering sets of not necessarily the same density in the two models above. The answer to this last question is given by Theorem 1.5. Notice that $|A_1/A_2| = |\text{visible}(A_1 \times A_2)|$, but there is no ratio set version for $d \ge 3$. This is the reason why we state Theorem 1.5 in terms of visible lattice points.

Theorem 1.5. *Let $\alpha_1, \ldots, \alpha_d \in (0, 1)$ and consider random subsets A_i in $B(n, \alpha_i)$, $i = 1, \ldots, d$. Then*

$$|\text{visible}(A_1 \times \cdots \times A_d)| \sim \mu(\alpha_1, \ldots, \alpha_d)n^d$$

with probability $1 - o(1)$ when $n \to \infty$, where

$$\mu(\alpha_1, \ldots, \alpha_d) = \frac{\alpha_1 \cdots \alpha_d}{\zeta(d)} \frac{\mathrm{Li}_d(1 - \alpha_1 \cdots \alpha_d)}{1 - \alpha_1 \cdots \alpha_d}$$

and $\mathrm{Li}_d(z) = \sum_{k\ge 1} \frac{z^k}{k^d}$ *is the polylogarithm of order* d.

In particular, this provides yet another instance in which polylogarithms occur. See [5] for more applications of the polylogarithms. Moreover, when $\alpha_i = 1$ for all $1 \le i \le d$ we recover the classical result ([4]) that the probability that d positive integers are relatively prime is $1/\zeta(d)$, since $\lim_{\alpha_1\cdots\alpha_d\to 1}\mu(\alpha_1,\dots,\alpha_d) = 1/\zeta(d)$.

Theorem 1.5 also works for the probabilistic models $S(n,k_i),\ i=1,\dots,d$. However, we have decided to omit its proof for the sake of brevity, since the ideas involved are those in the proof of Theorem 1.2.

There is also a strong convergence version similar to that of Corollary 1.2: *If* $A_1,\dots,A_d$ *are infinite random sequences such that all the events* $m \in A_i$ *are independent and* $\mathbb{P}(m\in A_i) = \alpha_i,\ i=1,\dots,d$, *the random variable* $X_n = |\mathrm{visible}(A_1\times\cdots\times A_d \cap [1,n]^d)|$ *satisfies that*

$$\mathbb{P}\left(\lim_{n\to\infty} X_n = \mu(\alpha_1,\dots,\alpha_d)\right) = 1.$$

The proof is similar to that of Corollary 1.2, so details will be omitted.

One last remark is in order: Theorem 1.5 is not a generalization of Theorem 1.1. Note that, in Theorem 1.1, we consider the random variable $|\mathrm{visible}(A\times A)|$ where A is a random set in $S(n,\alpha)$; whereas, in Theorem 1.5, when $d=2$ and $\alpha_1=\alpha_2=\alpha$ we deal with the random variable $|\mathrm{visible}(A_1\times A_2)|$ where A_1, A_2 are random sets in $S(n,\alpha)$. However, the natural generalization of Theorem 1.1 also holds, even in the strong convergence version:

Let A *be an infinite random sequence of positive integers where all the events* $m\in A$ *are independent and* $\mathbb{P}(m\in A)=\alpha$, *and let* $X_n = |\mathrm{visible}(A\times \overset{d}{\cdots} \times A)\cap[1,n]^d|$. *Then*

$$\mathbb{P}\left(\lim_{n\to\infty} X_n = \mu_d(\alpha)\right) = 1,$$

where $\mu_d(\alpha) = \mu(\alpha, \overset{d}{\dots}, \alpha)$.

Again the proof follows the same steps that those of Theorem 1.1 and Corollary 1.2.

We now pass to the proofs of the five previous theorems, organized accordingly in five different sections.

2. The size of A/A for random sets in $B(n,\alpha)$. Proof of Theorem 1.1

2.1. **Expectation.** First of all, we give an explicit expression for the expected value of the random variable $X = |A/A|$, where A is a random set in $B(n,\alpha)$.

Proposition 2.1. *For the random variable* $X=|A/A|$ *in* $B(n,\alpha)$ *we have*

$$\mathbb{E}(X) = \frac{6}{\pi^2}(n\alpha)^2\frac{\mathrm{Li}_2(1-\alpha^2)}{1-\alpha^2} + O(n\log^2 n).$$

Proof. Linearity of expectation and the symmetry with respect to the line $r=s$ give the equality

$$\mathbb{E}(X) = \sum_{\substack{r,s\le n\\ (r,s)=1}} \mathbb{P}(r/s\in A/A) = 2\sum_{\substack{r<s\le n\\ (r,s)=1}} \mathbb{P}(r/s\in A/A) + 1. \tag{1}$$

Moreover,

$$\mathbb{P}(r/s\notin A/A) = \mathbb{P}\left(\bigcap_{t\le n/s}\{(rt,st)\notin A\times A\}\right) = \mathbb{P}\left(\bigcap_{t\le n/s} E_t^c\right),$$

where E_t stands for the event $\{rt, st\in A\}$.

Clearly, these events are independent if and only if there do not exist $t,t'\le n/s$ such that $rt'=st$. Since $(r,s)=1$, the former condition implies that $s|t'$, so $s\le t'$. Thus, the inequality $s>\sqrt{n}$ entails the independence of the events, because otherwise we are led to the contradiction $\sqrt{n} \leq s \leq t' \leq n/s < \sqrt{n}$. Hence, if $s > \sqrt{n}$,

$$\mathbb{P}(r/s \notin A/A) = \mathbb{P}\left(\bigcap_{t \leq n/s} E_t^c\right) = \prod_{t \leq n/s} \big(1 - \mathbb{P}(rt \in A)\mathbb{P}(st \in A)\big) = (1-\alpha^2)^{[n/s]}. \tag{2}$$

We consequently split the sum in (1) in two parts:

$$\mathbb{E}(X) = 2 \sum_{\substack{\sqrt{n} < s \leq n \\ r < s, (r,s)=1}} \mathbb{P}(r/s \in A/A) + 2 \sum_{\substack{r < s \leq \sqrt{n} \\ (r,s)=1}} \mathbb{P}(r/s \in A/A) + 1.$$

In the first one, we have independent events and equation (2) holds. In the second one, we simply bound the probabilities by 1. Therefore,

$$\mathbb{E}(X) = 2 \sum_{\substack{\sqrt{n} < s \leq n \\ r < s,\ (r,s)=1}} \left(1 - (1-\alpha^2)^{[n/s]}\right) + O(n) = 2 \sum_{s \leq n} \varphi(s) \left(1 - (1-\alpha^2)^{[n/s]}\right) + O(n),$$

where φ stands for Euler's totient function.

Now we observe that $k \leq n/s < k+1$ if and only if $n/k \geq s > n/(k+1)$. Denoting $\Phi(x) := \sum_{n \leq x} \varphi(n)$ and summing by parts, we obtain

$$\begin{aligned}
\mathbb{E}(X) &= 2 \sum_{k=1}^{n-1} \sum_{\frac{n}{k+1} < s \leq \frac{n}{k}} \varphi(s) \left(1 - (1-\alpha^2)^k\right) + O(n) \\
&= 2 \sum_{k=1}^{n-1} \left(\Phi\left(\frac{n}{k}\right) - \Phi\left(\frac{n}{k+1}\right)\right) \left(1 - (1-\alpha^2)^k\right) + O(n) \\
&= 2 \sum_{k=1}^{n-1} \Phi\left(\frac{n}{k}\right) \left(\left(1 - (1-\alpha^2)^k\right) - \left(1 - (1-\alpha^2)^{k-1}\right)\right) + O(n) \\
&= 2\alpha^2 \sum_{k=1}^{n-1} \Phi\left(\frac{n}{k}\right) (1-\alpha^2)^{k-1} + O(n).
\end{aligned}$$

The classical estimate $\Phi(x) = \frac{3x^2}{\pi^2} + O(x \log x)$ finishes the proof:

$$\begin{aligned}
\mathbb{E}(X) &= 2\alpha^2 \sum_{k=1}^{n-1} \left(\frac{3n^2}{\pi^2 k^2} + O\left(\frac{n}{k} \log(n/k)\right)\right) (1-\alpha^2)^{k-1} + O(n) \\
&= \frac{6\alpha^2 n^2}{\pi^2(1-\alpha^2)} \left(\sum_{k=1}^{\infty} \frac{(1-\alpha^2)^k}{k^2} - \sum_{k=n}^{\infty} \frac{(1-\alpha^2)^k}{k^2}\right) + O(n \log^2 n) \\
&= \frac{6\alpha^2 n^2}{\pi^2(1-\alpha^2)} \sum_{k=1}^{\infty} \frac{(1-\alpha^2)^k}{k^2} + O(n \log^2 n) \\
&= \frac{6}{\pi^2} (n\alpha)^2 \frac{\mathrm{Li}_2(1-\alpha^2)}{1-\alpha^2} + O(n \log^2 n).
\end{aligned}$$

□

The ambiguous case $\alpha = 1$ must be understood as the limit as $\alpha \to 1$, which recovers the equality $|I_n/I_n| = \frac{6}{\pi^2} n^2 + O(n \log n)$.

2.2. **Variance.** Our next step to prove Theorem 1.1 is to estimate the deviation of X from its expected value. To accomplish this, we obtain a bound for its variance:

Proposition 2.2. *For the random variable $X = |A/A|$ in $B(n, \alpha)$ we have*

$$\mathrm{Var}(X) \ll n^3 \log^2 n.$$

Proof. Since $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X)$ and $\mathbb{E}(X)$ was explicitly computed in the previous proposition, it suffices to estimate $\mathbb{E}(X^2)$. Linearity of expectation gives the equality

$$\mathbb{E}(X^2) = \sum_{\substack{r,s,r',s' \le n \\ (r,s)=(r',s')=1}} \mathbb{P}\left(\{r/s \in A/A\} \cap \{r'/s' \in A/A\}\right), \tag{3}$$

so we are led to study the independence of the previous events. Given r, s, r', s', the events

$$\{r/s \in A/A\} = \bigvee_{t \le n/\max(r,s)} \{rt, st \in A\} \quad \text{and} \quad \{r'/s' \in A/A\} = \bigvee_{t' \le n/\max(r',s')} \{r't', s't' \in A\}$$

are dependent if and only if there are $t \le n/\max(r,s)$ and $t' \le n/\max(r',s')$ such that $rt = r't'$ or $rt = s't'$ or $st = r't'$ or $st = s't'$. Now $rt = r't'$ if and only if $tr/(r,r') = t'r'/(r,r')$, so coprimality implies the existence of positive integers u, u' such that $t = ur'/(r,r')$ and $t' = u'r/(r,r')$. We observe that

$$\frac{n}{r} \ge t = \frac{ur'}{(r,r')} \ge \frac{r'}{(r,r')};$$

so condition $n < rr'/(r,r')$ guarantees that there are no $t \le n/\max(r,s), t' \le n/\max(r',s')$ such that $rt = r't'$. Similarly, conditions $n < rr'/(r,r'), sr'/(s,r'), rs'/(r,s'), ss'/(s,s')$ ensure that $\{r/s \in A/A\}$ and $\{r'/s' \in A/A\}$ are independent events.

Consequently, we split the sum in (3) in two parts: the terms which satisfy the former conditions and the terms which do not satisfy them. In the first one, all the events are independent, so their sum is bounded by

$$\left(\sum_{\substack{r,s \le n \\ (r,s)=1}} \mathbb{P}\left(\{r/s \in A/A\}\right)\right)\left(\sum_{\substack{r',s' \le n \\ (r',s')=1}} \mathbb{P}\left(\{r'/s' \in A/A\}\right)\right) = \mathbb{E}(X)^2. \tag{4}$$

In the second one, we simply bound the probabilities by 1. Hence, we are led to count the number of irreducible fractions $r/s, r'/s'$ whose denominators and numerators are bounded by n and which satisfy one of the following conditions: $rr'/(r,r'), sr'/(s,r'), rs'/(r,s'), ss'/(s,s') \le n$. By symmetry, it suffices to treat the first case. Firstly, we observe that

$$\sum_{\substack{r,s,r',s' \le n \\ (r,s)=(r',s')=1 \\ rr'/(r,r') \le n}} 1 \ll n^2 \sum_{\substack{r,r' \le n \\ rr'/(r,r') \le n}} 1.$$

Then, defining $l := (r,r')$ and writing $r = lm, r' = lm'$ for certain integers m, m', the expression above is bounded by

$$n^2 \sum_{\substack{l \le n \\ m,m' \le n/l \\ mm' \le n/l}} 1 \ll n^2 \sum_{\substack{l \le n \\ k \le n/l}} d(k),$$

where we denote by $d(k)$ the number of divisors of k. Finally, the classical estimates for the average order of $d(k)$ and for the harmonic numbers imply that the former quantity is

$$\ll n^2 \sum_{l \le n} \frac{n}{l} \log\left(\frac{n}{l}\right) \ll n^3 \log n \sum_{l \le n} \frac{1}{l} \ll n^3 \log^2 n. \tag{5}$$

Substituting the estimates for (4) and (11) into (3), we obtain

$$\mathbb{E}(X^2) \le \mathbb{E}(X)^2 + O(n^3 \log^2 n).$$

Thus,

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \ll n^3 \log^2 n,$$

as we wished to show. $\square$

Now we are in a position to prove Theorem 1.1, since the estimate $\mathrm{Var}(X) = o(\mathbb{E}(X)^2)$ and Chebyshev's inequality imply that, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{X}{\mathbb{E}(X)} - 1\right| \geq \varepsilon\right) = 0.$$

Hence, $X \sim \mathbb{E}(X)$ with probability $1 - o(1)$ as $n \to \infty$, and Theorem 1.1 is obtained.

Proof of Corollary 1.2. Using Chebychev's inequality and Proposition 2.2 we have

$$\sum_M \mathbb{P}\left(\left|\frac{|A_{M^3}/A_{M^3}|}{M^6} - \mathbb{E}\left(\frac{|A_{M^3}/A_{M^3}|}{M^6}\right)\right| > 1/\sqrt{M}\right) \leq \sum_M \frac{\mathrm{Var}\left(\frac{|A_{M^3}/A_{M^3}|}{M^6}\right)}{(1/\sqrt{M})^2} \ll \sum_M \frac{\log^2 M}{M^2}.$$

Since the former sum is convergent, the Borel-Cantelli lemma implies that

$$\left|\frac{|A_{M^3}/A_{M^3}|}{M^6} - \mathbb{E}\left(\frac{|A_{M^3}/A_{M^3}|}{M^6}\right)\right| \ll 1/\sqrt{M}$$

almost surely.

Given a positive integer n, let M such that $M^3 \leq n < (M+1)^3$. It is clear that

$$|A_n/A_n| = |A_{M^3}/A_{M^3}| + O(M^5).$$

We have that, almost surely,

$$\begin{aligned}\left|\frac{|A_n/A_n|}{n^2} - \mu_2(\alpha)\right| &\leq \left|\frac{|A_n/A_n|}{n^2} - \frac{|A_{M^3}/A_{M^3}|}{M^6}\right| + \left|\frac{|A_{M^3}/A_{M^3}|}{M^6} - \mathbb{E}\left(\frac{|A_{M^3}/A_{M^3}|}{M^6}\right)\right| \\ &+ \left|\mathbb{E}\left(\frac{|A_{M^3}/A_{M^3}|}{M^6}\right) - \mu_2(\alpha)\right| \\ &\ll \frac{1}{M} + \frac{1}{\sqrt{M}} + \left|\mathbb{E}\left(\frac{|A_{M^3}/A_{M^3}|}{M^6}\right) - \mu_2(\alpha)\right| \to 0\end{aligned}$$

as $n \to \infty$. In other words, with probability 1 we have that $\lim_{n\to\infty} \frac{|A_n/A_n|}{n^2} = \mu_2(\alpha)$, and Corollary 1.2 is obtained. □

3. Random sets in $S(n,k)$. Proof of Theorem 1.2

We follow the same strategy used in [3]. Let us consider again the random variable $X = |A/A|$, but in the model $S(n,k)$. From now on $\mathbb{E}_k(X)$ and $V_k(X)$ will denote the expected value and the variance of X in this probability space. Clearly, for $s = 1, 2$ we have

$$\begin{aligned}\mathbb{E}_k(X^s) &= \frac{1}{\binom{n}{k}} \sum_{|A|=k} |A/A|^s \\ V_k(X) &= \frac{1}{\binom{n}{k}} \sum_{|A|=k} \left(|A/A| - \mathbb{E}_k(X)\right)^2\end{aligned}$$

Lemma 3.1. *For $s = 1, 2$ and $1 \leq j < k$ we have that*

$$\mathbb{E}_j(X^s) \leq \mathbb{E}_k(X^s) \leq \mathbb{E}_j(X^s) + k^{2s} - j^{2s}.$$

Proof. In order to prove the lower bound it is enough to consider the case $j = k-1$. Observe that $|A/A|$ is monotone with respect to inclusion, i.e. $|\,(A \cup \{a\})\,/\,(A \cup \{a\})\,| \geq |A/A|$ for any $A, \{a\} \subseteq [n]$. Using this we get

$$\begin{aligned}
\mathbb{E}_{k-1}(X^s) &= \frac{1}{\binom{n}{k-1}} \sum_{|A|=k-1} |A/A|^s \\
&\leq \frac{1}{\binom{n}{k-1}} \sum_{|A|=k-1} \frac{1}{n-k+1} \sum_{a\in[n]\setminus A} |\,(A\cup\{a\})\,/\,(A\cup\{a\})\,|^s \\
&= \frac{1}{\binom{n}{k-1}} \frac{k}{(n-k+1)} \sum_{|A'|=k} |A'/A'|^s \\
&= \frac{1}{\binom{n}{k}} \sum_{|A'|=k} |A'/A'|^s = \mathbb{E}_k(X^s).
\end{aligned}$$

For the second inequality we observe that for any set $A \in \binom{[n]}{k}$ and any partition into two sets $A = A' \cup A''$ with $|A'| = j,\ |A''| = k-j$ we have that $|A/A| \leq |A'/A'| + 2|A'||A''| + |A''|^2 = |A'/A'| + k^2 - j^2$. Similarly,

$$\begin{aligned}
|A/A|^2 &\leq (|A'/A'| + k^2 - j^2)^2 \\
&= |A'/A'|^2 + 2|A'/A'|(k^2 - j^2) + (k^2 - j^2)^2 \\
&\leq |A'/A'|^2 + 2j^2(k^2 - j^2) + (k^2 - j^2)^2 \\
&= |A'/A'|^2 + k^4 - j^4.
\end{aligned}$$

Thus, for $s = 1, 2$ we have

$$\begin{aligned}
|A/A|^s &\leq \binom{k}{j}^{-1} \sum_{\substack{A'\subset A\\ |A'|=j}} \left(|A'/A'|^s + k^{2s} - j^{2s}\right) \\
&\leq \binom{k}{j}^{-1} \Big(\sum_{\substack{A'\subset A\\ |A'|=j}} |A'/A'|^s \Big) + k^{2s} - j^{2s}.
\end{aligned}$$

Then,

$$\begin{aligned}
\sum_{|A|=k} |A/A|^s &\leq \binom{k}{j}^{-1} \sum_{|A|=k} \sum_{\substack{A'\subset A\\ |A'|=j}} |A'/A'|^s + \binom{n}{k}(k^{2s} - j^{2s}) \\
&= \binom{k}{j}^{-1} \sum_{|A'|=j} |A'/A'|^s \sum_{\substack{A'\subset A\\ |A|=k}} 1 + \binom{n}{k}(k^{2s} - j^{2s}) \\
&= \binom{k}{j}^{-1} \binom{n-j}{k-j} \sum_{|A'|=j} |A'/A'|^s + \binom{n}{k}(k^{2s} - j^{2s}) \\
&= \frac{\binom{n}{k}}{\binom{n}{j}} \sum_{|A'|=j} |A'/A'|^s + \binom{n}{k}(k^{2s} - j^{2s}),
\end{aligned}$$

and the second inequality holds. □

Proposition 3.1. *For $s = 1, 2$ we have that*

$$\mathbb{E}_k(X^s) = \mathbb{E}(X^s) + O(k^{2s-1/2}).$$

where $\mathbb{E}(X^s)$ denotes the expectation of X^s in $B(n, k/n)$ and $\mathbb{E}_k(X^s)$ the expectation in $S(n, k)$.

Proof. Observe that for $s=1,2$ we have

$$\begin{aligned}
\mathbb{E}(X^s)-\mathbb{E}_k(X^s) &= -\mathbb{E}_k(X^s)+\sum_{j=0}^{n}\left(\frac{k}{n}\right)^j\left(1-\frac{k}{n}\right)^{n-j}\sum_{|A|=j}|A/A|^s \\
&= -\mathbb{E}_k(X^s)+\sum_{j=0}^{n}\left(\frac{k}{n}\right)^j\left(1-\frac{k}{n}\right)^{n-j}\binom{n}{j}\mathbb{E}_j(X^s) \\
&= \sum_{j=0}^{n}\left(\frac{k}{n}\right)^j\left(1-\frac{k}{n}\right)^{n-j}\binom{n}{j}\left(\mathbb{E}_j(X^s)-\mathbb{E}_k(X^s)\right),
\end{aligned}$$

for $s=1,2$. Using Lemma 3.1 we get

$$|\mathbb{E}_k(X^s)-\mathbb{E}(X^s)|\le\sum_{j=0}^{n}\left(\frac{k}{n}\right)^j\left(1-\frac{k}{n}\right)^{n-j}\binom{n}{j}|j^{2s}-k^{2s}|. \tag{6}$$

To estimate the sum above we observe that

$$|j^{2s}-k^{2s}|\le 4|j-k|(\max(j,k))^{2s-1}. \tag{7}$$

We also consider $\mathbb{E}(|Y-\mathbb{E}(Y)|)$, where $Y\sim \mathrm{Bin}(n,k/n)$ is the binomial distribution of parameters n and k/n. Cauchy–Schwarz inequality for the expectation implies that this quantity is bounded by the standard deviation of the binomial distribution.

$$\sum_{j=0}^{n}\left(\frac{k}{n}\right)^j\left(1-\frac{k}{n}\right)^{n-j}\binom{n}{j}|j-k|=\mathbb{E}(|Y-\mathbb{E}(Y)|)\le\sqrt{\mathrm{Var}(Y)}=\sqrt{n(k/n)(1-k/n)}\le\sqrt{k}. \tag{8}$$

To estimate the sum in (6) we split the expression in two terms: the sum indexed by $j\le 2k$ and the one with $j>2k$. We use (7) and (8) to get

$$\begin{aligned}
\sum_{j\le 2k}\left(\frac{k}{n}\right)^j\left(1-\frac{k}{n}\right)^{n-j}\binom{n}{j}|j^{2s}-k^{2s}| &\le 4(2k)^{2s-1}\sum_{j=0}^{n}\left(\frac{k}{n}\right)^j\left(1-\frac{k}{n}\right)^{n-j}\binom{n}{j}|j-k| \\
&\le 32k^{2s-1/2}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{j>2k}\left(\frac{k}{n}\right)^j\left(1-\frac{k}{n}\right)^{n-j}\binom{n}{j}|j^{2s}-k^{2s}| \\
\le\ &\sum_{j>2k}\left(\frac{k}{n}\right)^j\left(1-\frac{k}{n}\right)^{n-j}\binom{n}{j}j^{2s} \\
\le\ &\sum_{l\ge 2}(l+1)^{2s}k^{2s}\sum_{lk<j\le(l+1)k}\left(\frac{k}{n}\right)^j\left(1-\frac{k}{n}\right)^{n-j}\binom{n}{j} \\
\le\ &\sum_{l\ge 2}(l+1)^{2s}k^{2s}\ \mathbb{P}(Y>lk)
\end{aligned}$$

where, once again, $Y\sim\mathrm{Bin}(n,k/n)$. Chernoff's Theorem implies that for any $\epsilon>0$ we have

$$\mathbb{P}(Y>(1+\epsilon)k)\le e^{-\epsilon^2k/3}.$$

Applying this inequality to $\mathbb{P}(Y>lk)$ we get

$$\begin{aligned}
&\sum_{j>2k}\left(\frac{k}{n}\right)^j\left(1-\frac{k}{n}\right)^{n-j}\binom{n}{j}|j^{2s}-k^{2s}| \\
\le\ &\sum_{l\ge 2}(l+1)^{2s}k^{2s}e^{-(l-1)^2k/3}\ll k^{2s}e^{-k/3}\ll k^{2s-1/2}.
\end{aligned}$$

□

The next corollary proves the first part of Theorem 1.2.

Corollary 3.1. *If $k \asymp n$ then*

$$\mathbb{E}_k(X) = \frac{6}{\pi^2} k^2 \frac{Li_2(1-(k/n)^2)}{1-(k/n)^2}\left(1+O\left(n^{-1/2}\right)\right).$$

Proof. Proposition 3.1 for $s = 1$ and Proposition 2.1 imply that

$$\begin{aligned}\mathbb{E}_k(X) &= \mathbb{E}(X) + O(n^{3/2}) \\ &= \frac{6}{\pi^2} k^2 \frac{\mathrm{Li}_2(1-(k/n)^2)}{1-(k/n)^2} + O(n\log^2 n) + O(k^{3/2}) \\ &= \frac{6}{\pi^2} k^2 \frac{\mathrm{Li}_2(1-(k/n)^2)}{1-(k/n)^2}\left(1+O\left(n^{-1/2}\right)\right).\end{aligned}$$

□

To conclude the proof of Theorem 1.2 we combine Proposition 2.2 and Proposition 3.1 to estimate the variance $V_k(X)$ in $S(n,k)$:

$$\begin{aligned}V_k(X) &= \mathbb{E}_k(X^2) - \mathbb{E}_k^2(X) \\ &= V(X) + \left(\mathbb{E}_k(X^2) - \mathbb{E}(X^2)\right) + \left(\mathbb{E}(X) - \mathbb{E}_k(X)\right)\left(\mathbb{E}(X) + \mathbb{E}_k(X)\right) \\ &\ll n^3 \log^2 n + k^{5/2} + \left(k^{3/2}\right)\left(k^2\right) \\ &\ll k^{7/2}.\end{aligned}$$

The assertion of Theorem 1.2 is a consequence of the estimate $V_k(X) = o\left(\mathbb{E}_k^2(X)\right)$ when $k \to \infty$.

4. The size of the ratio set A/A when A is an arithmetic progression.

Proof of Theorem 1.3. Without loss of generality, we may assume that $(a,q) = 1$ by dividing all the coordinates of the lattice points by (a,q).

If $(r,s) = 1$, note that $r/s \in A/A$ if and only if $(rt, st) \in A \times A$ for certain positive integer t, which occurs if and only if $rt \equiv st \equiv a \pmod q$ for some t with $rt, st \leq n$. A necessary, but not sufficient, condition for this is that $r \equiv s \pmod q$. Indeed, $rt \equiv st \equiv a \pmod q$ and $(a,q) = 1$ imply that $(t,q) = 1$, so $r \equiv s \equiv t^{-1}a \pmod q$.

We consequently classify r, s according to their remainder l modulo q. Thus, if $r \equiv s \equiv l\ (q)$ then $r/s \in A/A$ if and only if there exists $t = t(l)$ such that $lt \equiv a \pmod q$ and $rt, st \leq n$. For the first condition, it is enough that $(l,q) = 1$; and it is necessary for, if such a t exists, then $(l,q)|a$ and $(a,q) = 1$. Hence

$$|A/A| = \sum_{\substack{1\leq l\leq q\\ (l,q)=1}} |S(l)|,$$

where $S(l) := \{r, s \in \mathbb{N} : (r,s) = 1, r \equiv s \equiv l \pmod q,\ rt(l), st(l) \leq n\}$.

We begin by estimating the cardinality of these sets. Observe that

$$|S(l)| = \sum_{\substack{r,s\leq n/t(l)\\ r\equiv s\equiv l \pmod q\\ (r,s)=1}} 1 = \sum_{\substack{r,s\leq n/t(l)\\ r\equiv s\equiv l \pmod q}} \sum_{d|(r,s)} \mu(d), \tag{9}$$

and that, for $d|(r,s)$, we may write $r = dr', s = ds'$, where $r', s' \leq n/t(l)d$. Since $(q,l) = 1$ and $(d,q)|l$ because $dr' = r \equiv l \pmod q$, we must have $(d,q) = 1$. Thus, $r \equiv s \equiv l \pmod q$ is equivalent to

$r' \equiv s' \equiv d^{-1}l \pmod q$, and we may rewrite (9) as

$$|S(l)| = \sum_{\substack{d \le n/t(l) \\ (d,q)=1}} \mu(d) \sum_{\substack{r',s' \le n/t(l)d \\ r' \equiv s' \equiv d^{-1}l \pmod q}} 1 = \sum_{\substack{d \le n/t(l) \\ (d,q)=1}} \mu(d) \left(\frac{n}{t(l)dq} + O(1) \right)^2,$$

since the number of positive integers less or equal than a certain m and congruent to a certain a modulo b is $m/b + O(1)$. Thus,

$$|S(l)| = \frac{n^2}{q^2 t(l)^2} \sum_{\substack{d \le n/t(l) \\ (d,q)=1}} \frac{\mu(d)}{d^2} + O\left(\frac{n}{qt(l)} \log(n/t(l)) \right).$$

Now we split the former sum in two parts

$$\sum_{\substack{d \le n/t(l) \\ (d,q)=1}} \frac{\mu(d)}{d^2} = \sum_{(d,q)=1} \frac{\mu(d)}{d^2} - \sum_{\substack{d > n/t(l) \\ (d,q)>1}} \frac{\mu(d)}{d^2}.$$

The second one may be bounded as follows

$$\sum_{\substack{d > n/t(l) \\ (d,q)>1}} \frac{\mu(d)}{d^2} \ll \sum_{d > n/t(l)} \frac{1}{d^2} = O\left(\frac{t(l)}{n} \right).$$

And, for the first one, note that

$$\sum_{(d,q)=1} \frac{\mu(d)}{d^2} = \prod_{p \nmid q} \left(1 - \frac{1}{p^2} \right) = \frac{6}{\pi^2} \prod_{p \mid q} \left(1 - \frac{1}{p^2} \right)^{-1}.$$

Hence

$$|S(l)| = \frac{n^2}{q^2 t(l)^2} \frac{6}{\pi^2} \prod_{p \mid q} \left(1 - \frac{1}{p^2} \right)^{-1} + O\left(\frac{n \log n}{q} \right).$$

Finally, we have to add the previous quantities for all $1 \le l \le q$ such that $(l,q) = 1$. But $t(l)$ was the least positive integer congruent to $l^{-1}a$ modulo q, with $(a,q) = 1$, and $\psi : l \mapsto l^{-1}a$ is a set automorphism in $(\mathbb{Z}/q\mathbb{Z})^\times$. Therefore, $\{l \in \mathbb{Z} : 1 \le l \le q, (q,l) = 1\}$ and $\{t(l) : 1 \le l \le q, (q,l) = 1\}$ coincide. Thus,

$$|A/A| = \sum_{\substack{l \le q \\ (l,q)=1}} S(l) = \frac{n^2}{q^2} \frac{6}{\pi^2} \prod_{p \mid q} \left(1 - \frac{1}{p^2} \right)^{-1} \sum_{\substack{1 \le t \le q \\ (t,q)=1}} \frac{1}{t^2} + O(n \log n),$$

and Theorem 1.3 is proved. □

To prove that $c_q \to 1$ when $q \to \infty$, observe that

$$\begin{aligned} \sum_{\substack{1 \le t \le q \\ (t,q)=1}} \frac{1}{t^2} &= \sum_{t=1}^{q} \sum_{d \mid (t,q)} \frac{\mu(d)}{t^2} = \sum_{d \mid q} \frac{\mu(d)}{d^2} \sum_{t' \le q/d} \frac{1}{t'^2} = \sum_{d \mid q} \frac{\mu(d)}{d^2} \left(\frac{\pi^2}{6} + O\left(\frac{d}{q} \right) \right) \\ &= \frac{\pi^2}{6} \prod_{p \mid q} \left(1 - \frac{1}{p^2} \right) + O\left(\frac{1}{q} \sum_{d \mid q} \frac{1}{d} \right). \end{aligned}$$

We observe that $\frac{1}{q} \sum_{d \mid q} \frac{1}{d} \ll \frac{\tau(q)}{q}$ where $\tau(q)$ is the number of divisors of q, so

$$c_q = \frac{6}{\pi^2} \prod_{p \mid q} \left(1 - \frac{1}{p^2} \right)^{-1} \sum_{\substack{1 \le t \le q \\ (t,q)=1}} \frac{1}{t^2} = 1 + O\left(\frac{\tau(q)}{q} \right) \xrightarrow{q \to \infty} 1,$$

since $\frac{6}{\pi^2} \prod_{p \mid q} \left(1 - \frac{1}{p^2} \right)^{-1} = \prod_{p \nmid q} \left(1 - \frac{1}{p^2} \right) \le 1$.

5. Bounds for $|A/A|$ when A has positive density

We now pass to the proof of Theorem 1.4.

Proof. Let m be a sufficiently large integer. We shall denote by $\mathcal{P}$ the set $\{p_i\}_{i=1}^{2m}$ of the first $2m$ prime numbers, and by S_m the set of all the products of m different elements of $\mathcal{P}$. Furthermore, given a large integer $n > m$, we consider the set $A := \{sr \leq n : s \in S_m, r \in \mathbb{N}, (r, \mathcal{P}) = 1\}$. Our theorem is an immediate corollary of the following three facts:

(1) For any $\epsilon > 0$, $|S_m/S_m| \leq \epsilon |S_m|^2$ for sufficiently large m.

(2) $|A/A| \leq |S_m/S_m||A|^2/|S_m|^2$.

(3) $\alpha := |A|/n \geq \prod_{i=1}^{2m}(1 - 1/p_i)/(p_1 \cdots p_m)$.

Let us prove separately each one of these assertions:

(1) Clearly, $|S_m| = \binom{2m}{m}$. Moreover $S_m/S_m = \{p_1^{\epsilon_1} \cdots p_m^{\epsilon_m},\ \epsilon_i \in \{-1, 0, 1\}\}$. Then

$$|S_m/S_m| = 3^m = 3^m \frac{|S_m|^2}{\binom{2m}{m}^2} \leq (2m+1)^2 \left(\frac{3}{4}\right)^m |S_m|^2.$$

The statement then follows from the fact that $(2m+1)^2 \left(\frac{3}{4}\right)^m$ tends to 0 when $m \to \infty$.

(2) By coprimality of r and s, each element of A may be written in a unique fashion as rs. Thus, letting $B := \{r \leq n/\max(S) : (r, \mathcal{P}) = 1\}$, the maps $A/A \to S_m/S_m \times B/B : sr/s'r' \mapsto (s/s', r/r')$ and $B \times S_m \to A : (r, s) \mapsto rs$ are injective. Hence

$$|A/A| \leq |S_m/S_m||B/B| \leq |S_m/S_m||B|^2 \leq |S_m/S_m||A|^2/|S_m|^2.$$

(3) Since $p_1 \cdots p_m \in S_m$, $A' := \{r(p_1 \cdots p_m) \leq n : (r, p_1 \cdots p_{2m}) = 1\} \subseteq A$. Thus, $|A| \geq |A'|$, where $|A'|$ is the number of positive integers less or equal than $N := \lfloor n/(p_1 \cdots p_m) \rfloor$ that are not divisible by any p_i, for $1 \leq i \leq 2m$. Consider the sets $B := \{1, \ldots, N\}$ and $B_i := \{r \leq N : p_i | r\}$, for $1 \leq i \leq 2m$. Since $|A'| = |B \setminus \cup_{i=1}^{2m} B_i| = |B| - |\cup_{i=1}^{2m} B_i|$, the inclusion-exclusion principle gives

$$\begin{aligned}|A'| &= |B| + \sum_{1 \leq j \leq 2m} (-1)^j \sum_{1 \leq i_1 < \ldots < i_j \leq 2m} |B_{i_1} \cap \ldots \cap B_{i_j}| \\ &= N + \sum_{1 \leq j \leq 2m} (-1)^j \left(\frac{N}{p_{i_1} \cdots p_{i_j}} + O(1)\right) = N \prod_{j=1}^{2m}\left(1 - \frac{1}{p_j}\right) + O(2m).\end{aligned}$$

The asserted inequality follows from the definition of N and the fact that $\alpha := |A|/n \geq |A'|/n$.

This concludes the proof of the theorem and leads us into the last section of this paper. □

6. Visible lattice points in multidimensional spaces. Proof of Theorem 1.5

In order to prove Theorem 1.5, it is convenient to introduce a certain generalization of Euler's totient function.

Definition 1. *Given $k, m \in \mathbb{N}$, the Jordan's totient function $J_k(m)$ is defined as the cardinality of the set $\{(a_1, ..., a_k) \in \mathbb{N}^k : a_1 \leq ... \leq a_k \leq m, \gcd(a_1, ..., a_k, m) = 1\}$.*

Plainly, $J_1 = \varphi$. Further, it is known [1] that

$$\Phi_k(n) := \sum_{m \leq n} J_k(m) = \frac{n^{k+1}}{(k+1)!\zeta(k+1)} + O(n^k). \tag{10}$$

The proof of Theorem 1.5 is now similar to that of Theorem 1.1, so we shall only indicate the appropriate modifications.

Proof. Let X denote the random variable $|\text{visible}(A_1 \times \cdots \times A_d)|$, where each element of A_i is chosen independently at random in $\{1, \ldots, n\}$ with probability α_i, for $1 \le i \le d$. By symmetry,

$$\mathbb{E}(X) = \sum_{\substack{a_1,\ldots,a_d \le n \\ (a_1,\ldots,a_d)=1}} \mathbb{P}\left(\{(a_1 t, \ldots, a_d t) \in A_1 \times \cdots \times A_d \text{ for some } t \le n/\max(a_i)\}\right)$$

$$= d! \sum_{\substack{a_1<\ldots<a_d \le n \\ (a_1,\ldots,a_d)=1}} \mathbb{P}\left(\{(a_1 t, \ldots, a_d t) \in A_1 \times \cdots \times A_d \text{ for some } t \le n/a_d\}\right) + O(n^{d-1}), \tag{11}$$

where $O(n^{d-1})$ bounds the number of points having at least two identical coordinates. If $a_1 < \cdots < a_d$, we have

$$\begin{aligned}
&\mathbb{P}\left(\{(a_1 t, \ldots, a_d t) \in A_1 \times \cdots \times A_d \text{ for some } t \le n/a_d\}\right) \\
= \;& 1 - \mathbb{P}\left(\{(a_1 t, \ldots, a_d t) \notin A_1 \times \cdots \times A_d \text{ for any } t \le n/a_d)\}\right) \\
= \;& 1 - \prod_{t \le n/a_d} \mathbb{P}\left(\{(a_1 t, \ldots, a_d t) \notin A_1 \times \cdots \times A_d\}\right) \\
= \;& 1 - \prod_{t \le n/a_d} \left(1 - \mathbb{P}\left(\{(a_1 t, \ldots, a_d t) \in A_1 \times \cdots \times A_d\}\right)\right) \\
= \;& 1 - (1 - \alpha_1 \cdots \alpha_d)^{[n/a_d]}.
\end{aligned}$$

Thus, we may rewrite (11) as

$$\begin{aligned}
\mathbb{E}(X) &= d! \sum_{\substack{a_1<\cdots<a_d \le n \\ (a_1,\ldots,a_d)=1}} \left(1 - (1 - \alpha_1 \cdots \alpha_d)^{[n/a_d]}\right) + O(n^{d-1}) \\
&= d! \sum_{a_d \le n} J_{d-1}(a_d) \left(1 - (1 - \alpha_1 \cdots \alpha_d)^{[n/a_d]}\right) + O(n^{d-1}).
\end{aligned}$$

Finally, the explicit expression for $\mathbb{E}(X)$ follows by proceeding as in the last part of the proof of Proposition 2.1. and by applying (10).

$$\begin{aligned}
\mathbb{E}(X) &= d! \sum_{k=1}^{n-1} \sum_{\frac{n}{k+1} < a_d \le \frac{n}{k}} J_{d-1}(a_d) \left(1 - (1 - \alpha_1 \cdots \alpha_d)^k\right) + O(n^{d-1}) \\
&= d! \sum_{k=1}^{n-1} \left(\Phi_{d-1}\left(\frac{n}{k}\right) - \Phi_{d-1}\left(\frac{n}{k+1}\right)\right) \left(1 - (1 - \alpha_1 \cdots \alpha_d)^k\right) + O(n^{d-1}) \\
&= d! \sum_{k=1}^{n-1} \Phi_{d-1}\left(\frac{n}{k}\right) \left(\left(1 - (1 - \alpha_1 \cdots \alpha_d)^k\right) - \left(1 - (1 - \alpha_1 \cdots \alpha_d)^{k-1}\right)\right) + O(n^{d-1}) \\
&= d! \alpha_1 \cdots \alpha_d \sum_{k=1}^{n-1} \Phi_{d-1}\left(\frac{n}{k}\right) (1 - \alpha_1 \cdots \alpha_d)^{k-1} + O(n^{d-1}) \\
&= d! \alpha_1 \cdots \alpha_d \sum_{k=1}^{n-1} \left(\frac{n^d}{d! \zeta(d) k^d} + O((n/k)^{d-1})\right) (1 - \alpha_1 \cdots \alpha_d)^{k-1} + O(n^{d-1}) \\
&= n^d \frac{\alpha_1 \cdots \alpha_d}{\zeta(d)} \sum_{k=1}^{n-1} \frac{(1 - \alpha_1 \cdots \alpha_d)^{k-1}}{k^d} + O(n^{d-1} \log n),
\end{aligned}$$

obtaining

$$\mathbb{E}(X) = n^d \frac{\alpha_1 \cdots \alpha_d}{\zeta(d)} \frac{\mathrm{Li}_d(1 - \alpha_1 \cdots \alpha_d)}{1 - \alpha_1 \cdots \alpha_d} + O(n^{d-1} \log n).$$

To estimate the variance of $X = |\text{visible}(A \times \cdots A_d)|$ we compute

$$\begin{aligned}
\mathbb{E}(X^2) &= \sum_{\substack{a_1,\dots,a_d,a'_1,\dots,a'_d\le n\\ (a_1,\dots,a_d)=(a'_1,\dots,a'_d)=1}} \mathbb{P}\begin{pmatrix}(ta_1,\dots,ta_d)\in A_1\times\cdots\times A_d \text{ and } (t'a'_1,\dots,t'a'_d)\in A_1\times\cdots\times A_d)\\ \text{for some } t,t'\end{pmatrix}\\
&= (d!)^2 \sum_{\substack{a_1<\cdots<a_d\le n\\ a'_1<\cdots<a'_d\le n\\ (a_1,\dots,a_d)\\ =(a'_1,\dots,a'_d)=1}} \mathbb{P}\begin{pmatrix}(ta_1,\dots,ta_d)\in A_1\times\cdots\times A_d, \text{ for some } t\le \frac{n}{a_d}\\ (t'a'_1,\dots,t'a'_d)\in A_1\times\cdots\times A_d \text{ for some } t'\le\frac{n}{a'_d}\end{pmatrix} + O(n^{2d-1})\\
&= (d!)^2 \sum_{\substack{a_1<\cdots<a_d\le n\\ a'_1<\cdots<a'_d\le n\\ (a_1,\dots,a_d)\\ =(a'_1,\dots,a'_d)=1}} \mathbb{P}\left(ta_i, t'a'_i\in A_i,\ i=1,\dots,d,\ \ \text{for some } t\le\frac{n}{a_d}, t'\le \frac{n}{a'_d}\right) + O(n^{2d-1}).
\end{aligned}$$

As we did in the proof of Theorem 1.1, we can check that the condition $n < a_ia'_i/(a_i,a'_i)$ implies that there are no $t\le n/a_i, t'\le n/a'_i$ such that $ta_i = t'a'_i$. Hence, we split the sum above in two parts. In the first one, we sum over all $a_1,\dots,a_d,a'_1,\dots,a'_d\le n$ with $(a_1,\dots,a_d)=(a'_1,\dots,a'_d)=1$ and $a_ia'_i/(a_ia'_i) > n$; and, by independence, this part is bounded by

$$\begin{aligned}
&(d!)^2 \sum_{\substack{a_1<\cdots<a_d\le n\\ (a_1,\dots,a_d)=1}} \mathbb{P}\left(ta_i\in A_i,\ i=1,\dots,d,\ \ \text{for some } t\le\frac{n}{a_d}\right)\\
&\times \sum_{\substack{a'_1<\cdots<a'_d\le n\\ (a'_1,\dots,a'_d)=1}} \mathbb{P}\left(t'a'_i\in A_i,\ i=1,\dots,d,\ \ \text{for some } t'\le\frac{n}{a'_d}\right) + O(n^{2d-1})\\
&= \mathbb{E}^2(X) + O(n^{2d-1}).
\end{aligned}$$

In the second one, we sum over all $a_1,\dots,a_d,a'_1,\dots,a'_d\le n$ with $a_ia'_i/(a_i,a'_i)\le n$ for some $i=1,\dots,n$. This is clearly bounded by

$$d\sum_{\substack{a_1,\dots,a_d\le n\\ a'_1,\dots,a'_d\le n\\ a_1a'_1/(a_1a'_1)\le n}} 1 \le dn^{2d-2} \sum_{\substack{a_1,a'_1\le n\\ a_1a'_1/(a_1,a'_1)\le n}} 1.$$

On the other hand,

$$\sum_{\substack{a_1,a'_1\le n\\ a_1a'_1/(a_1,a'_1)\le n}} 1 = \sum_{l\le n}\sum_{\substack{a_1,a'_1\le n\\ (a_1,a'_1)=l\\ a_1a_1\le ln}} 1 \le \sum_{l\le n}\sum_{\substack{b_1,b'_1\le n/l\\ b_1b'_1\le n/l}} 1 \le \sum_{l\le n}\sum_{m\le n/l}\tau(m) \ll \sum_{l\le n}\frac{n}{l}\log(n/l) \ll n\log^2 n.$$

Thus, the second sum is $O\left(n^{2d-1}\log^2 n\right)$ so we have that

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) \ll n^{2d-1}\log^2 n.$$

Since $\mathrm{Var}(X) = o(\mathbb{E}^2(X))$, then $X\sim\mathbb{E}(X)$ with probability $1-o(1)$ as $n\to\infty$ and the proof is concluded. □

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J. Cilleruelo: Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM) and Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain

E-mail address: franciscojavier.cilleruelo@uam.es

J. Guijarro-Ordóñez: Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM) and Universidad Complutense de Madrid, 28040 Madrid, Spain

E-mail address: jorge.guijarro.ord@hotmail.com