EVERY POSITIVE INTEGER IS A SUM OF THREE PALINDROMES

JAVIER CILLERUELO AND FLORIAN LUCA

Abstract. For integer $g \geq 5$, we prove that any positive integer can be written as a sum of three palindromes in base g.

1. Introduction

Let $g \geq 2$ be a positive integer. Any nonnegative integer n has a unique base g representation namely

$$n = \sum_{j \geq 0} \delta_j g^j, \quad \text{with} \quad 0 \leq \delta_j \leq g - 1.$$

The numbers δ_i are called the *digits of n in base g*. If l is the number of digits of n, we use the notation

$$n = \delta_{l-1} \cdots \delta_0, \tag{1.1}$$

where we assume that $\delta_{l-1} \neq 0$.

Definition 1.1. We say that n is a base g palindrome whenever $\delta_{l-i} = \delta_{i-1}$ holds for all $i = 1, \ldots, m = \lfloor l/2 \rfloor$.

There are many problems and results concerning the arithmetic properties of base g palindromes. For example, in [2] it is shown that almost all base g palindromes are composite. In [4], it is shown that for every large L, there exist base g palindromes n with exactly L digits and many prime factors (at least $(\log \log n)^{1+o(1)}$ of them as $L \to \infty$). The average value of the Euler function over binary (that is, with $g = 2$) palindromes n with a fixed even number of digits was investigated in [3]. In [7] (see also [9]), it is shown that the set of numbers n for which F_n, the nth Fibonacci number, is a base g palindrome has asymptotic density zero as a subset of all positive integers, while in [6] it was shown that base g palindromes which are perfect powers (of some integer exponent $k \geq 2$) form a thin set as a subset of all base g palindromes. In [10], the authors found all positive integers n such that $10^n \pm 1$ is a base 2 palindrome, result which was extended in [5].

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Recently, Banks [1] started the investigation of the additive theory of palindromes by proving that every positive integer can be written as a sum of at most 49 base 10 palindromes. A natural question to ask would be how optimal is the number 49 in the above result. In this respect, we prove the following results.

Theorem 1.2. *Let $g \geq 5$. Then any positive integer can be written as a sum of three base g palindromes.*

The case $g = 10$ of Theorem 1.2 is a folklore conjecture which has been around for some time. The paper [8] attributes a stronger conjecture to John Hoffman, namely that every positive integer n can be written in base $g = 10$ as a sum of three palindromes where one of them is the maximal palindrome less than or equal to n itself. This was refuted in [11] which provided infinitely many examples of positive integers n which are not a sum of two decimal palindromes.

However, we prove that "many" positive integers are a sum of two palindromes.

Theorem 1.3. *Let $g \geq 2$. There exists a positive constant c_1 depending on g such that*

$$|\{n \leq x: \ n = p_1 + p_2, \ p_1, \ p_2 \text{ base } g \text{ palindromes}\}| \geq x^{1-\frac{c_1}{\sqrt{\log x}}}$$

for all $x \geq 2$.

On the other hand the set of integers which are not sum of two palindromes has positive density.

Theorem 1.4. *For any $g \geq 3$ there exists a constant $c < 1$ such that*

$$|\{n \leq x : n = p_1 + p_2, \ p_1, p_2 \text{ base } g \text{ palindromes } \}| \leq cx$$

for x large enough.

It makes sense to ask whether the set of positive integers which are sum of two base g palindromes has positive density.

We set forward the following conjecture.

Conjecture 1.5. *The set of positive integers n which are the sum of two base g palindromes has positive density.*

It would be interesting to extend Theorem 1.2 to the missing bases $g \in \{2, 3, 4\}$. Throughout this paper, we use the Landau symbols O and o as well as the Vinogradov symbols $\ll$ and $\gg$ with their usual meaning. These are used only in the proof of Theorem 1.3.

2. The algorithms

The proof of Theorem 1.2 is algorithmic. Indeed one can program the following proof to input a positive integer n and obtain a representation of n as a sum of three palindromes in base $g \geq 5$. We assume through the proof that $g \geq 5$.

For ease of notation, and using a convention introduced by Banks [1], we consider that 0 is a base g palindrome as well. For any integer a, we write $D(a)$ for that unique $d \in \{0, \dots, g-1\}$ such that $d \equiv a \pmod{g}$.

As in (1.1), we write the base g representation of n as

$$n = \delta_{l-1} \dots\dots\dots \delta_1 \delta_0.$$

As before, $\delta_{l-1} \neq 0$.

2.1. **Small cases.** To present a clear algorithm, those integers with less than 7 digits are considered separately in Section §4.

So, the algorithm starts by counting the number of digits of n. If n has less than 7 digits, then Proposition 4.1 from Section §4 shows how to write n as a sum of three palindromes. If n has 7 or more digits then we apply the general algorithm that we present in the next pages.

2.2. **The starting point.** For those integers with at least 7 digits, the starting point consists in assigning a *type* to n according to the following classification. The type will define the lengths and the first digits (so, also the last) of the three palindromes p_1, p_2, p_3 that we will use to represent n.

Type A:

A.1) $\delta_{l-2} \neq 0, 1, 2, \quad z_1 = D(\delta_0 - \delta_{l-1} - \delta_{l-2} + 1) \neq 0.$

n	δ_{l-1}	δ_{l-2}	*	*	*	*	*	*	*	*	*	*	*	*	δ_0
p_1	δ_{l-1}	.	.	.	.	.	.	.	.	.	.	.	.	.	δ_{l-1}
p_2		$\delta_{l-2}-1$	.	.	.	.	.	.	.	.	.	.	.	.	$\delta_{l-2}-1$
p_3			z_1	.	.	.	.	.	.	.	.	.	.	.	z_1

A.2) $\delta_{l-2} \neq 0, 1, 2, \quad D(\delta_0 - \delta_{l-1} - \delta_{l-2} + 1) = 0.$

n	δ_{l-1}	δ_{l-2}	*	*	*	*	*	*	*	*	*	*	*	*	δ_0
p_1	δ_{l-1}	.	.	.	.	.	.	.	.	.	.	.	.	.	δ_{l-1}
p_2		$\delta_{l-2}-2$	.	.	.	.	.	.	.	.	.	.	.	.	$\delta_{l-2}-2$
p_3			1	.	.	.	.	.	.	.	.	.	.	.	1

A.3) $\delta_{l-2} = 0, 1, 2, \quad \delta_{l-1} \neq 1, \quad z_1 = D(\delta_0 - \delta_{l-1} + 2) \neq 0.$

n	δ_{l-1}	δ_{l-2}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	δ_0
p_1	$\delta_{l-1} - 1$	.	.	.	.	.	.	.	.	.	.	.	.	.	$\delta_{l-1} - 1$
p_2		$g - 1$	.	.	.	.	.	.	.	.	.	.	.	.	$g - 1$
p_3			z_1	.	.	.	.	.	.	.	.	.	.	.	z_1

A.4) $\delta_{l-2} = 0, 1, 2 \quad \delta_{l-1} \neq 1, \quad D(\delta_0 - \delta_{l-1} + 2) = 0.$

n	δ_{l-1}	δ_{l-2}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	δ_0
p_1	$\delta_{l-1} - 1$	.	.	.	.	.	.	.	.	.	.	.	.	.	$\delta_{l-1} - 1$
p_2		$g - 2$	.	.	.	.	.	.	.	.	.	.	.	.	$g - 2$
p_3			1	.	.	.	.	.	.	.	.	.	.	.	1

A.5) $\delta_{l-1} = 1, \quad \delta_{l-2} = 0, \ \delta_{l-3} \leq 3, \quad z_1 = D(\delta_0 - \delta_{l-3}) \neq 0.$

n	1	0	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	δ_0
p_1		$g - 1$	.	.	.	.	.	.	.	.	.	.	.	.	$g - 1$
p_2			$\delta_{l-3} + 1$	.	.	.	.	.	.	.	.	.	.	.	$\delta_{l-3} + 1$
p_3				z_1	.	.	.	.	.	.	.	.	.	.	z_1

A.6) $\delta_{l-1} = 1, \quad \delta_{l-2} = 0, \ \delta_{l-3} \leq 3, \quad D(\delta_0 - \delta_{l-3}) = 0.$

n	1	0	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	δ_0
p_1		$g - 1$	.	.	.	.	.	.	.	.	.	.	.	.	$g - 1$
p_2			$\delta_{l-3} + 2$	.	.	.	.	.	.	.	.	.	.	.	$\delta_{l-3} + 2$
p_3				$g - 1$	.	.	.	.	.	.	.	.	.	.	$g - 1$

Type B

B.1) $\delta_{l-1} = 1, \quad \delta_{l-2} \leq 2, \quad \delta_{l-3} \geq 4, \quad z_1 = D(\delta_0 - \delta_{l-3}) \neq 0.$

n	1	δ_{l-2}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	δ_0
p_1	1	δ_{l-2}	.	.	.	.	.	.	.	.	.	.	.	δ_{l-2}	1
p_2			$\delta_{l-3} - 1$	.	.	.	.	.	.	.	.	.	.	.	$\delta_{l-3} - 1$
p_3				z_1	.	.	.	.	.	.	.	.	.	.	z_1

B.2) $\delta_{l-1} = 1, \quad \delta_{l-2} \leq 2, \quad \delta_{l-3} \geq 4, \quad D(\delta_0 - \delta_{l-3}) = 0.$

n	1	δ_{l-2}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	δ_0
p_1	1	δ_{l-2}	.	.	.	.	.	.	.	.	.	.	.	δ_{l-2}	1
p_2			$\delta_{l-3} - 2$	.	.	.	.	.	.	.	.	.	.	.	$\delta_{l-3} - 2$
p_3				1	.	.	.	.	.	.	.	.	.	.	1

B.3) $\delta_{l-1} = 1, \quad \delta_{l-2} = 1, 2, \quad \delta_{l-3} = 0, 1, \quad \delta_0 = 0.$

n	1	δ_{l-2}	δ_{l-3}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	δ_0
p_1	1	$\delta_{l-2}-1$	.	.	.	.	.	.	.	.	.	.	.	$\delta_{l-2}-1$	1
p_2			$g-2$	.	.	.	.	.	.	.	.	.	.	.	$g-2$
p_3				1	.	.	.	.	.	.	.	.	.	.	1

B.4) $\delta_{l-1} = 1, \quad \delta_{l-2} = 1, 2, \quad \delta_{l-3} = 2, 3, \quad \delta_0 = 0.$

n	1	δ_{l-2}	δ_{l-3}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	δ_0
p_1	1	δ_{l-2}	.	.	.	.	.	.	.	.	.	.	.	δ_{l-2}	1
p_2			1	.	.	.	.	.	.	.	.	.	.	.	1
p_3				$g-2$	.	.	.	.	.	.	.	.	.	.	$g-2$

B.5) $\delta_{l-1} = 1, \quad \delta_{l-2} = 1, 2, \quad \delta_{l-3} = 0, 1, 2, \quad z_1 = \delta_0 \neq 0.$

n	1	δ_{l-2}	δ_{l-3}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	δ_0
p_1	1	$\delta_{l-2}-1$	.	.	.	.	.	.	.	.	.	.	.	$\delta_{l-2}-1$	1
p_2			$g-1$	.	.	.	.	.	.	.	.	.	.	.	$g-1$
p_3				z_1	.	.	.	.	.	.	.	.	.	.	z_1

B.6) $\delta_{l-1} = 1, \quad \delta_{l-2} = 1, 2, \quad \delta_{l-3} = 3, \quad z_1 = D(\delta_0 - 3) \neq 0.$

n	1	δ_{l-2}	3	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	δ_0
p_1	1	δ_{l-2}	.	.	.	.	.	.	.	.	.	.	.	δ_{l-2}	1
p_2			2	.	.	.	.	.	.	.	.	.	.	.	2
p_3				z_1	.	.	.	.	.	.	.	.	.	.	z_1

B.7) $\delta_{l-1} = 1, \quad \delta_{l-2} = 1, 2, \quad \delta_{l-3} = 3, \quad \delta_0 = 3.$

n	1	δ_{l-2}	3	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	δ_0
p_1	1	δ_{l-2}	.	.	.	.	.	.	.	.	.	.	.	δ_{l-2}	1
p_2			1	.	.	.	.	.	.	.	.	.	.	.	1
p_3				1	.	.	.	.	.	.	.	.	.	.	1

Notice that all the digits appearing in the clasification are valid digits; i.e. $0 \leq \delta \leq g-1$. We observe also that when n if of type B, the digit of p_1 below δ_{l-3}, which will be denoted by x_2, takes the values $0, 1, 2$ or 3.

2.3. **The algorithms.** Once we have asssigned the type to n we have to check if n is a *special number* or not.

Definition 2.1. We say that n is a *special number* if the palindrome p_1 corresponding to n according the classification in types above has an even number of digits, say $l = 2m$, and at least one of the digits δ_{m-1} or δ_m is equal to 0. Otherwise we say that n is a *normal number.*

We use five distinct algorithms. We use Algorithms I, II, II and IV for *normal numbers* and Algorithm V for *special numbers.*

Algorithm I: To be applied to integers such that the associated palindromes p_1, p_2, p_3 have $2m+1$, $2m$, $2m-1$ digits respectively for some $m \geq 3$. In other words, those of type A1, A2, A3 and A4 when $l = 2m+1$ and those of type A5 and A6 when $l = 2m+2$. The cases $m \leq 2$ correspond to the *small cases.*

Algorithm II: To be applied to integers such that the associated palindromes p_1, p_2, p_3 have $2m$, $2m-1$, $2m-2$ digits respectively for some $m \geq 3$ and such that $\delta_{m-1} \neq 0$ and $\delta_m \neq 0$. In other words, those of type A1, A2, A3 and A4 when $l = 2m$ and $\delta_{m-1} \neq 0$ and $\delta_m \neq 0$ and those of type A5 and A6 when $l = 2m+1$ and $\delta_{m-1} \neq 0$ and $\delta_m \neq 0$. The cases $m \leq 2$ correspond to the *small cases.*

Algorithm III: To be applied to integers such that the associated palindromes p_1, p_2, p_3 have $2m+1, 2m-1, 2m-2$ digits respectively for some $m \geq 3$. In other words, those of type B with $l = 2m+1$. The cases $m \leq 2$ correspond to the *small cases.*

Algorithm IV: To be applied to integers such that the associated palindromes p_1, p_2, p_3 have $2m, 2m-2, 2m-3$ digits respectively for some $m \geq 4$. In other words, those of type B with $l = 2m$ and with $\delta_m \neq 0$ and $\delta_{m-1} \neq 0$. The cases $m \leq 3$ correspond to the *small cases.*

Algorithm V: To be applied to *special numbers* that are not covered by the *small cases.*

2.4. **Algorithm I.** Asssume $m \geq 3$. The *initial configuration* when we apply Algorithm I is one of the following configurations:

δ_{2m}	δ_{2m-1}	δ_{2m-2}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	δ_1	δ_0
x_1	.	.	.	.	.	.	.	.	.	.	.	.	.	x_1
	y_1	.	.	.	.	.	.	.	.	.	.	.	.	y_1
		z_1	.	.	.	.	.	.	.	.	.	.	.	z_1

1	δ_{2m}	δ_{2m-1}	δ_{2m-2}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	δ_1	δ_0
	x_1	.	.	.	.	.	.	.	.	.	.	.	.	.	x_1
		y_1	.	.	.	.	.	.	.	.	.	.	.	.	y_1
			z_1	.	.	.	.	.	.	.	.	.	.	.	z_1

Algorithm I in either case is the following:

Step 1: We choose x_1, y_1, z_1 according to the configurations described in the starting point. Define $c_1 = (x_1 + y_1 + z_1)/g$, which is the carry of the column 1.

Step 2**:** Define the digits

$$\begin{aligned}
x_2 &= \begin{cases} D(\delta_{2m-1} - y_1) & \text{if} \quad z_1 \leq \delta_{2m-2} - 1; \\ D(\delta_{2m-1} - y_1 - 1) & \text{if} \quad z_1 \geq \delta_{2m-2}; \end{cases} \\
y_2 &= D(\delta_{2m-2} - z_1 - 1); \\
z_2 &= D(\delta_1 - x_2 - y_2 - c_1); \\
c_2 &= (x_2 + y_2 + z_2 + c_1 - \delta_1)/g \quad \text{(the carry from column } 2).
\end{aligned}$$

Step $i,\ 3 \leq i \leq m$**:** Define the digits

$$\begin{aligned}
x_i &= \begin{cases} 1 & \text{if} \quad z_{i-1} \leq \delta_{2m-i} - 1; \\ 0 & \text{if} \quad z_{i-1} \geq \delta_{2m-i}; \end{cases} \\
y_i &= D(\delta_{2m-i} - z_{i-1} - 1); \\
z_i &= D(\delta_{i-1} - x_i - y_i - c_{i-1}); \\
c_i &= (x_i + y_i + z_i + c_{i-1} - \delta_{i-1})/g \quad \text{(the carry from column } i).
\end{aligned}$$

Step $m+1$**:** Define

$$x_{m+1} = 0.$$

The diagram below represents the configuration after step i:

...	δ_{2m-i+1}	δ_{2m-i}	δ_{2m-i-1}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	δ_{i-1}	...
...	x_i	.	.	.	.	.	.	.	.	.	.	.	.	x_i	...
...	y_{i-1}	y_i	.	.	.	.	.	.	.	.	.	.	.	y_i	...
...	z_{i-2}	z_{i-1}	z_i	.	.	.	.	.	.	.	.	.	.	z_i	...

A few words to explain what is behind the algorithm:

The digit y_i is defined to adjust the digit δ_{2m-i} from the left side once we know the digit z_{i-1} and assuming a possible carry from the previous column (the -1 in the definition of y_i takes into account this possible carry). The z_i is defined to adjust the digit δ_{i-1} in the right side once we know x_i, y_i and c_{i-1}, the carry from the previous column. Now we go again to the left side. If $z_i \geq \delta_{2m-i-1}$ we will get the possible carry we had assumed and then we define $x_{i+1} = 0$. If $z_i \leq \delta_{2m-i-1} - 1$ we do not get any carry and then we define $x_{i+1} = 1$, which has the same effect that the carry we expected.

After the last step the configuration that we obtain is the following:

δ_{2m}	δ_{2m-1}	δ_{2m-2}	$*$	$*$	$*$	δ_m	δ_{m-1}	δ_{m-2}	$*$	$*$	$*$	$*$	$*$	δ_1	δ_0
x_1	$*$	$*$	$*$	$*$	x_m	0	x_m	$*$	$*$	$*$	$*$	$*$	$*$	$*$	x_1
	y_1	$*$	$*$	$*$	y_{m-1}	y_m	y_m	y_{m-1}	$*$	$*$	$*$	$*$	$*$	$*$	y_1
		z_1	$*$	$*$	$*$	z_{m-1}	z_m	z_{m-1}	$*$	$*$	$*$	$*$	$*$	$*$	z_1

We call *temporary configuration* the configuration we get after the last step. We have drawn a vertical line where both sides of the algorithm collide. It is not true in general that n is equal to the sum of the three palindromes we obtain in the temporary configuration.

If Δ_m is the digit we obtain in column $m+1$ when we sum the three palindromes, we observe that

$$\Delta_m \equiv y_m + z_{m-1} + c_m \equiv \delta_m + c_m - 1 \pmod g.$$

If $c_m = 1$ then $\Delta_m = \delta_m$ and we obtain the correct digit in column $m+1$ and, as consequence of Proposition 2.2, we obtain the correct digit in all the columns. In this case n is equal to the sum of the three palindromes of the temporary configuration so the *temporary configuration* is also the *final configuration*.

If $c_m \neq 1$, then we need an extra adjustment.

2.5. **The adjustment step.**

For $i = 0, \ldots, 2m$, we denote by Δ_i the digit we obtain in column $i+1$ when we sum the three palindromes that we have obtained after the last step. Of course we want that $\Delta_i = \delta_i$ for all i, $0 \leq i \leq 2m$. Unfortunately, this is not always true but it is almost true. The following proposition shows that we obtain the correct digits on the left side (thanks to the z_i's) and that we obtain the correct digit in a column of the right side if the digit we obtain in the previous column is also the correct digit.

Proposition 2.2. *Let $g \geq 5$ and $m \geq 3$. We have that $\Delta_i = \delta_i$ for all $0 \leq i \leq m-1$. Furhthermore, for any $0 \leq i \leq m-1$, if $\Delta_{m+i} = \delta_{m+i}$, then $\Delta_{m+i+1} = \delta_{m+i+1}$.*

Proof. The first statement of the proposition is clear because of the way we have defined the z_i's. As for the second statement, we prove it separately for $i = 0$, for $1 \leq i \leq m-3$, for $i = m-2$ and for $i = m-1$.

i) $i = 0$. We have

$$\begin{aligned}\Delta_{m+1} &\equiv x_m + y_{m-1} + z_{m-2} + c_{m+1}\\ &\equiv \delta_{m+1} + x_m + c_{m+1} - 1 \pmod{g}.\end{aligned}$$

Then we have to prove that $x_m + c_{m+1} = 1$.

a) If $x_m = 1$ then $z_{m-1} \leq \delta_m - 1$, so $y_m = \delta_m - z_{m-1} - 1$. Since

$$\Delta_m \equiv y_m + z_{m-1} + c_m \equiv \delta_m + c_m - 1 \pmod{g},$$

and we have assumed that $\Delta_m = \delta_m$, we conclude that $c_m \equiv 1 \pmod{g}$, so $c_m = 1$ (because $|c_m - 1| \leq 2 < g$). Thus,

$$c_{m+1} = (y_m + z_{m-1} + c_m - \delta_m)/g = (c_m - 1)/g = 0,$$

and then $x_m + c_{m+1} = 1$.

b) If $x_m = 0$, then $z_{m-1} \geq \delta_m$, so $y_m = g + \delta_m - z_{m-1} - 1$. The argument is similar to the one before except that now we get

$$c_{m+1} = (y_m + z_{m-1} + c_m - \delta_m)/g = (g + c_m - 1)/g = 1,$$

and again $x_m + c_{m+1} = 1$.

In any case, we have that $x_m + c_{m+1} = 1$, and then $\Delta_{m+1} = \delta_{m+1}$.

ii) $1 \leq i \leq m - 3$ (these cases are vacuous for $m = 3$):

$$\begin{aligned}\Delta_{m+i+1} &\equiv x_{m-i} + y_{m-i-1} + z_{m-i-1-2} + c_{m+i+1}\\ &\equiv \delta_{m+i+1} + x_{m-i} + c_{m+i+1} - 1 \pmod{g}.\end{aligned}$$

We have to prove that $x_{m-i} + c_{m+i+1} = 1$.

a) If $x_{m-i} = 1$, then $z_{m-i-1} \leq \delta_{m+i} - 1$, so $y_{m-i} = \delta_{m+i} - z_{m-i-1} - 1$. Since

$$\begin{aligned}\Delta_{m+i} &\equiv x_{m-i+1} + y_{m-i} + z_{m-i-1} + c_{m+i}\\ &\equiv x_{m-i+1} + \delta_{m+i} - 1 + c_{m+i} \pmod{g},\end{aligned}$$

and we have assumed that $\Delta_{m+i} = \delta_{m+i}$, we conclude that

$$x_{m-i+1} + c_{m+i} - 1 \equiv 0 \pmod{g},$$

therefore $x_{m-i+1} + c_{m+i} - 1 = 0$ (because $|x_{m-i+1} + c_{m+i} - 1| \leq 2$). Thus,

$$\begin{aligned}c_{m+i+1} &= (x_{m-i+1} + y_{m-i} + z_{m-i-1} + c_{m+i} - \delta_{m+i})/g\\ &= (x_{m-i+1} - 1 + c_{m+i})/g = 0,\end{aligned}$$

and $x_{m-i} + c_{m+i+1} = 1$.

b) If $x_{m-i} = 0$, then $z_{m-i-1} \geq \delta_{m+i}$, so $y_{m-i} = g + \delta_{m+i} - z_{m-i-1} - 1$. The argument is similar to one before except that now we get

$$\begin{aligned} c_{m+i+1} &= (x_{m-i+1} + y_{m-i} + z_{m-i-1} + c_{m+i} - \delta_{m+i})/g \\ &= (g + x_{m-i+1} - 1 + c_{m+i})/g = 1, \end{aligned}$$

and again $x_{m-i} + c_{m+i+1} = 1$.

In any case, we have that $x_{m-i} + c_{m+i+1} = 1$ and then $\Delta_{m+i+1} = \delta_{m+i+1}$.

iii) $i = m - 2$. We have

$$\Delta_{2m-1} \equiv x_2 + y_1 + c_{2m-1} \pmod{g}.$$

We distinguish two cases:

a) If $z_1 \leq \delta_{2m-2} - 1$, then $y_2 = \delta_{2m-2} - z_1 - 1$ and

$$\Delta_{2m-1} \equiv x_2 + y_1 + c_{2m-1} \equiv \delta_{2m-1} + c_{2m-1} \pmod{g}.$$

Since

$$\Delta_{2m-2} \equiv x_3 + y_2 + z_1 + c_{2m-2} \equiv x_3 + \delta_{2m-2} - 1 + c_{2m-2} \pmod{g},$$

and we have assumed that $\Delta_{2m-2} = \delta_{2m-2}$, we get $x_3 - 1 + c_{2m-2} = 0$ (because $|x_3 - 1 + c_{2m-2}| \leq 2$). Thus,

$$c_{2m-1} = (x_3 + y_2 + z_1 + c_{2m-2} - \delta_{2m-2})/g = 0,$$

and we have $\Delta_{2m-1} = \delta_{2m-1}$.

b) If $z_1 \geq \delta_{2m-2}$, then $y_2 = g + \delta_{2m-2} - z_1 - 1$ and

$$\Delta_{2m-1} \equiv x_2 + y_1 + c_{2m-1} \equiv \delta_{2m-1} + c_{2m-1} - 1 \pmod{g}.$$

We repeat the same argument as in case a) except that now

$$c_{2m-1} = (x_3 + y_2 + z_1 + c_{2m-2} - \delta_{2m-2})/g = 1,$$

and again $\Delta_{2m-1} = \delta_{2m-1}$.

iii) $i = m - 1$. We can check in the classification in types that if $\Delta_{2m-1} = \delta_{2m-1}$, then $\Delta_{2m} = \delta_{2m}$. In other words, that we have $c_{2m} = 0$ for the types A1 and A2 and we have $c_{2m} = 1$ for the types A3, A4, A5 and A6.

□

Proposition 2.2 shows that if $\Delta_m = \delta_m$ then $\Delta_i = \delta_i$ for all $i = 0, \ldots, 2m$ and then the three palindromes we have obtained do the job.

The problem appears when $\Delta_m \neq \delta_m$ and this occurs when $c_m \neq 1$. When this happens, we need to make an adjustment to our *temporary* configuration.

Notice that for $m \geq 3$ we have

$$\Delta_m \equiv \delta_m + c_m - 1 \pmod{g},$$

and that c_m takes the value $0, 1$ or 2.

All the possible situations are considered in the cases below:

I.1 $\boldsymbol{c_m = 1}$. In this case $\Delta_m = \delta_m$ and there is nothing to change. The temporary configuration is simply the final configuration since in all columns the sums of the digits including the carries yield the digits of n.

I.2 $\boldsymbol{c_m = 0}$. In this case we need to increment by one unit the digit we obtain in the column $m+1$. We can do this by changing the value of $x_{m+1} = 0$ to $x_{m+1} = 1$.

δ_m	δ_{m-1}
0	$*$
y_m	y_m
$*$	z_m

$\longrightarrow$

δ_m	δ_{m-1}
1	$*$
y_m	y_m
$*$	z_m

Notice that we have modified the central digit of the first palindrome, so the new first row is also a palindrome. Notice also that now we obtain the correct digit in column $m+1$ and also in all remaining columns.

I.3 $\boldsymbol{c_m = 2}$. In this case, we have that $y_m \neq 0$ (otherwise $c_m \neq 2$). We distinguish two cases:

I.3.i) $\boldsymbol{z_m \neq g - 1}$.

δ_m	δ_{m-1}
$*$	$*$
y_m	y_m
$*$	z_m

$\longrightarrow$

δ_m	δ_{m-1}
$*$	$*$
$y_m - 1$	$y_m - 1$
$*$	$z_m + 1$

I.3.ii) $\boldsymbol{z_m = g - 1}$.

δ_m	δ_{m-1}
0	$*$
y_m	y_m
$*$	$g - 1$

$\longrightarrow$

δ_m	δ_{m-1}
1	$*$
$y_m - 1$	$y_m - 1$
$*$	0

Observe that in every adjustment step we have been successful in increasing or decreasing the digit that was obtained in the column $m+1$ when $c_m = 0$ or 2, without altering the digits from the previous column. Notice also that in every adjustement we always modify the central digits of the temporary palindromes such that the new ones are also palindromes. Once we have realized these adjustments, the digit we get in the column $m+1$ is δ_m, the correct digit, and Proposition 2.2 proves that all the digits are correct.

2.6. **The three palindromes and an example.** We end this subsection by illustrating the application of Algorithm I to an example. Let n be the positive integer giving the first 21 decimal digits of π:

$$n = 314159265358979323846.$$

We see that n is of type A1, therefore the configuration after Step 1 is the following :

3	1	4	1	5	9	2	6	5	3	5	8	9	7	9	3	2	3	8	4	6
2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	2
	9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	9
		5	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	5

Thus n is a normal integer and we can apply Algorithm I.

Since $z_1 \geq \delta_{2m-2}$, Step 2 starts defining

$$\begin{aligned} x_2 &= D(\delta_{2m-1} - y_1 - 1) = D(1 - 9 - 1) = 1, \\ y_2 &= D(\delta_{2m-2} - z_1 - 1) = D(4 - 5 - 1) = 8, \\ z_2 &= D(\delta_1 - x_2 - y_2 - c_1) = D(4 - 1 - 8 - 1) = 4, \\ c_2 &= (x_2 + y_2 + z_2 + c_1 - \delta_1)/10 = 1, \end{aligned}$$

and the configuration after Step 2 is

3	1	4	1	5	9	2	6	5	3	5	8	9	7	9	3	2	3	8	4	6
2	**1**	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	**1**	2
	9	**8**	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	**8**	9
		5	**4**	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	**4**	5

and after Step 3 is

3	1	4	1	5	9	2	6	5	3	5	8	9	7	9	3	2	3	8	4	6
2	1	**0**	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	**0**	1	2
	9	8	**6**	.	.	.	.	.	.	.	.	.	.	.	.	.	.	**6**	8	9
		5	4	**1**	.	.	.	.	.	.	.	.	.	.	.	.	.	**1**	4	5

Continuing with the algorithm we get to the temporary configuration:

3	1	4	1	5	9	2	6	5	3	5	8	9	7	9	3	2	3	8	4	6
2	1	0	1	0	0	1	0	0	1	0	1	0	0	1	0	0	1	0	1	2
	9	8	6	3	9	9	2	9	4	0	0	4	9	2	9	9	3	6	8	9
		5	4	1	9	2	3	5	8	4	7	4	8	5	3	2	9	1	4	5

Since $c_m = 0$, we need to apply Adjustment I.2 and obtain the final configuration:

n	3	1	4	1	5	9	2	6	5	3	5	8	9	7	9	3	2	3	8	4	6
p_1	2	1	0	1	0	0	1	0	0	1	**1**	1	0	0	1	0	0	1	0	1	2
p_2		9	8	6	3	9	9	2	9	4	0	0	4	9	2	9	9	3	6	8	9
p_3			5	4	1	9	2	3	5	8	4	7	4	8	5	3	2	9	1	4	5

3. The remaining cases

3.1. **Algorithm II.** The algorithm only differs in the subindices of the δ_i's (because now $l = 2m$ is even) and in the adjustment step, which is slightly more complicated to describe because of the many cases to be considered. The cases $m \leq 2$ correspond to the *small cases*. For $m \geq 3$, we proceed in the following steps:

Step 1: We choose x_1, y_1, z_1 according to the configurations described in Section 2.2. Define $c_1 = (x_1 + y_1 + z_1)/g$, which is the carry of the column 1.

Step 2: Define the digits

$$\begin{aligned} x_2 &= \begin{cases} D(\delta_{2m-2} - y_1) & \text{if} \quad z_1 \leq \delta_{2m-3} - 1; \\ D(\delta_{2m-2} - y_1 - 1) & \text{if} \quad z_1 \geq \delta_{2m-3}; \end{cases} \\ y_2 &= D(\delta_{2m-3} - z_1 - 1); \\ z_2 &= D(\delta_1 - x_2 - y_2 - c_1); \\ c_2 &= (x_2 + y_2 + z_2 + c_1 - \delta_1)/g \quad \text{(the carry from column } 2). \end{aligned}$$

Step i, $3 \leq i \leq m-1$ (these steps are vacuos for $m = 3$): Define the digits

$$\begin{aligned} x_i &= \begin{cases} 1 & \text{if} \quad z_{i-1} \leq \delta_{2m-i-1} - 1; \\ 0 & \text{if} \quad z_{i-1} \geq \delta_{2m-i-1}; \end{cases} \\ y_i &= D(\delta_{2m-i-1} - z_{i-1} - 1); \\ z_i &= D(\delta_{i-1} - x_i - y_i - c_{i-1}); \\ c_i &= (x_i + y_i + z_i + c_{i-1} - \delta_{i-1})/g \quad \text{(the carry from column } i). \end{aligned}$$

Step m**:** Define the digits

$$\begin{aligned} x_m &= 0. \\ y_m &= D(\delta_{m-1} - z_{m-1} - c_{m-1}). \end{aligned}$$

The temporary configuration is:

δ_{2m-1}	δ_{2m-2}	δ_{2m-3}	*	*	*	δ_m	δ_{m-1}	δ_{m-2}	*	*	*	*	*	δ_1	δ_0
x_1	.	.	.	.	.	0	0	x_{m-1}	.	.	.	.	.	.	x_1
	y_1	.	.	.	.	y_{m-1}	y_m	y_{m-1}	.	.	.	.	.	.	y_1
		z_1	.	.	.	.	z_{m-1}	z_{m-1}	.	.	.	.	.	.	z_1

or

1	δ_{2m-1}	δ_{2m-2}	δ_{2m-3}	*	*	*	δ_m	δ_{m-1}	δ_{m-2}	*	*	*	*	*	δ_1	δ_0
	x_1	.	.	.	.	.	0	0	x_{m-1}	.	.	.	.	.	.	x_1
		y_1	.	.	.	.	y_{m-1}	y_m	y_{m-1}	.	.	.	.	.	.	y_1
			z_1	.	.	.	.	z_{m-1}	z_{m-1}	.	.	.	.	.	.	z_1

with $\delta_{m-1} \neq 0$ and $\delta_m \neq 0$.

Proposition 3.1. *Let $g \geq 5$ and $m \geq 3$. We have that $\Delta_i = \delta_i$ for all $0 \leq i \leq m-1$. Furhthermore, for any $0 \leq i \leq m-2$, if $\Delta_{m+i} = \delta_{m+i}$, then $\Delta_{m+i+1} = \delta_{m+i+1}$.*

Proof. The proof is similar to the proof of Proposition 2.2. We only give the details for $i = 0$, which is the only case somewhat different.

Assume that $\Delta_m = \delta_m$. In other words, that $(y_{m-1} + z_{m-2} + c_m - \delta_m)/g$ is an integer. We have

$$\Delta_{m+1} \equiv x_{m-1} + y_{m-2} + z_{m-3} + c_{m+1} \equiv x_{m-1} + \delta_{m+1} - 1 + c_{m+1} \pmod g.$$

If $x_{m-1} = 0$, then $z_{m-2} \geq \delta_m$ and $y_{m-1} = g + \delta_m - z_{m-2} - 1$. Thus,

$$c_{m+1} = (y_{m-1} + z_{m-2} + c_m - \delta_m)/g = (g + c_m - 1)/g = 1$$

because c_{m+1} is an integer and $|c_m - 1| \leq 1 < g$.

If $x_{m-1} = 1$, then $z_{m-2} \leq \delta_m - 1$ and $y_{m-1} = \delta_m - z_{m-2} - 1$. Thus,

$$c_{m+1} = (y_{m-1} + z_{m-2} + c_m - \delta_m)/g = (c_m - 1)/g = 0$$

because c_{m+1} is an integer and $|c_m - 1| \leq 1 < g$.

In any case, we have that $x_{m-1} + c_{m+1} = 1$, so $\Delta_{m+1} \equiv \delta_{m+1}$. □

The above proposition implies that if $\Delta_m = \delta_m$, then $\Delta_i = \delta_i$ for all $i = 0, \ldots, 2m-1$.

Adjustment step: Notice that $\Delta_m \equiv \delta_m + c_m - 1 \pmod g$. Thus, we make the adjustment according this observation.

II.1 $\boldsymbol{c_m = 1}$. We do nothing and the temporary configuration becomes the final one.

II.2 $\boldsymbol{c_m = 0}$. We distinguish the following cases:

II.2.i) $\boldsymbol{y_m \neq 0}$.

δ_m	δ_{m-1}
0	0
$*$	y_m
$*$	$*$

$\longrightarrow$

δ_m	δ_{m-1}
1	1
$*$	$y_m - 1$
$*$	$*$

II.2.ii) $\boldsymbol{y_m = 0}$.

II.2.ii.a) $\boldsymbol{y_{m-1} \neq 0,\ z_{m-1} \neq g - 1}$.

δ_m	δ_{m-1}	δ_{m-2}
0	0	$*$
y_{m-1}	0	y_{m-1}
$*$	z_{m-1}	z_{m-1}

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}
1	1	$*$
$y_{m-1} - 1$	$g - 2$	$y_{m-1} - 1$
$*$	$z_{m-1} + 1$	$z_{m-1} + 1$

II.2.ii.b) $\boldsymbol{y_{m-1} \neq 0,\ z_{m-1} = g - 1}$.

δ_m	δ_{m-1}	δ_{m-2}
0	0	$*$
y_{m-1}	0	y_{m-1}
$*$	$g - 1$	$g - 1$

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}
2	2	$*$
$y_{m-1} - 1$	$g - 2$	$y_{m-1} - 1$
$*$	0	0

II.2.ii.c) $\boldsymbol{y_{m-1} = 0}$. In this case, we have that $z_{m-1} \neq 0$. Otherwise we would have that $\delta_{m-1} = 0$ (because $c_{m-1} = 0$), which is not allowed.

δ_m	δ_{m-1}	δ_{m-2}
0	0	$*$
0	0	0
$*$	z_{m-1}	z_{m-1}

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}
0	0	$*$
1	1	1
$*$	$z_{m-1} - 1$	$z_{m-1} - 1$

II.3 $\boldsymbol{c_m = 2}$. In this case it is clear that $z_{m-1} = y_m = g - 1$ (otherwise $c_m \neq 2$). Note also that if $y_{m-1} = 0$, then $c_{m-1} \neq 2$ and then $c_m \neq 2$.

δ_m	δ_{m-1}	δ_{m-2}
0	0	$*$
y_{m-1}	$g - 1$	y_{m-1}
$*$	$g - 1$	$g - 1$

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}
1	1	$*$
$y_{m-1} - 1$	$g - 2$	$y_{m-1} - 1$
$*$	0	0

Let us illustrate this algorithm with an example. We consider the positive integer representing the first 22 decimal digits of e:

$$n = 2718281828459045235360.$$

First let us note that since $\delta_{10} \neq 0$ and $\delta_{11} \neq 0$, then n is a *normal integer*. In addition n is of type A1. Therefore the initial configuration is:

2	7	1	8	2	8	1	8	2	8	4	5	9	0	4	5	2	3	5	3	6	0
1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1
	8	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	8
		1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	1

Applying the algorithm II we get to the temporary configuration:

2	7	1	8	2	8	1	8	2	8	4	5	9	0	4	5	2	3	5	3	6	0
1	8	0	0	1	0	1	0	0	1	0	0	1	0	0	1	0	1	0	0	8	1
	8	9	9	9	2	1	7	4	8	2	0	2	8	4	7	1	2	9	9	9	8
		1	8	2	5	9	0	7	9	1	5	5	1	9	7	0	9	5	2	8	1

Observe that the digit in column 12 is not correct (we get a 3 instead of a 4 for the sum). This is because $c_{11} = 0$, therefore we have to apply the adjustment step. Since $y_{11} = 0$, $y_{10} \neq 0$ and $z_{10} \neq 0$, the adjustment step is that described in II.2.ii.a):

n	2	7	1	8	2	8	1	8	2	8	4	5	9	0	4	5	2	3	5	3	6	0
p_1	1	8	0	0	1	0	1	0	0	1	**1**	**1**	1	0	0	1	0	1	0	0	8	1
p_2		8	9	9	9	2	1	7	4	8	**1**	**8**	**1**	8	4	7	1	2	9	9	9	8
p_3			1	8	2	5	9	0	7	9	1	**6**	**6**	1	9	7	0	9	5	2	8	1

3.2. **Algorithm III.** The cases $m \leq 2$ correspond to the *small cases*. For $m \geq 3$, we proceed in the following steps:

Step 1: We choose x_1, y_1, z_1 according to the configurations described in Section 2.2. Define $c_1 = (1 + y_1 + z_1)/g$, which is the carry of the column 1.

Step 2: Define the digits

$$\begin{aligned}
x_2 &= \begin{cases} D(\delta_{2m-2} - y_1) & \text{if} \quad z_1 \leq \delta_{2m-3} - 1; \\ D(\delta_{2m-2} - y_1 - 1) & \text{if} \quad z_1 \geq \delta_{2m-3}; \end{cases} \\
y_2 &= D(\delta_{2m-3} - z_1 - 1); \\
z_2 &= D(\delta_1 - x_1 - y_2 - c_1); \\
c_2 &= (x_1 + y_2 + z_2 + c_1 - \delta_1)/g \quad \text{(the carry from column } 2).
\end{aligned}$$

Step $i,\ 3 \le i \le m-1$**:** (these steps are vacuos for $m = 3$). Define the digits

$$
\begin{aligned}
x_i &= \begin{cases} 1 & \text{if} \quad z_{i-1} \le \delta_{2m-i-1} - 1; \\ 0 & \text{if} \quad z_{i-1} \ge \delta_{2m-i-1}; \end{cases} \\
y_i &= D(\delta_{2m-i-1} - z_{i-1} - 1); \\
z_i &= D(\delta_{i-1} - x_{i-1} - y_i - c_{i-1}); \\
c_i &= (x_{i-1} + y_i + z_i + c_{i-1} - \delta_{i-1})/g \quad \text{(the carry from column } i).
\end{aligned}
$$

Step m**:** Define the digits

$$
\begin{aligned}
x_m &= 0. \\
y_m &= D(\delta_{m-1} - z_{m-1} - x_{m-1} - c_{m-1}).
\end{aligned}
$$

The temporary configuration is:

1	δ_{2m-1}	δ_{2m-2}	$*$	$*$	$*$	δ_m	δ_{m-1}	δ_{m-2}	$*$	$*$	$*$	$*$	$*$	δ_1	δ_0
1	x_1	.	.	.	x_{m-1}	0	x_{m-1}	x_{m-2}	.	.	.	.	.	x_1	1
		y_1	.	.	.	y_{m-1}	y_m	y_{m-1}	.	.	.	.	.	.	y_1
			z_1	.	.	.	z_{m-1}	z_{m-1}	.	.	.	.	.	.	z_1

We omit the proof of the following proposition because it is similar to the Proposition 2.2 of Algorithm I.

Proposition 3.2. *Let* $g \ge 5$ *and* $m \ge 3$*. We have that* $\Delta_i = \delta_i$ *for all* $0 \le i \le m-2$*. Furthermore, for any* $-1 \le i \le m-2$*, if* $\Delta_{m+i} = \delta_{m+i}$*, then* $\Delta_{m+i+1} = \delta_{m+i+1}$*.*

Again, the above proposition gives that if $\Delta_m = \delta_m$, then $\Delta_i = \delta_i$ for $i = 0, \dots, 2m-1$.

Adjustment step: Notice that $\Delta_m \equiv \delta_m + c_m - 1 \pmod g$. According this observation we distinghish the following cases:

III.1 $\boldsymbol{c_m = 1}$. We do nothing and the temporary configuration becomes the final one.

III.2 $\boldsymbol{c_m = 0}$.

δ_m	δ_{m-1}
0	$*$
$*$	$*$
$*$	$*$

$\longrightarrow$

δ_m	δ_{m-1}
1	$*$
$*$	$*$
$*$	$*$

III.3 $\boldsymbol{c_{m-1} = 2}$. Notice that $y_m \ne 0$ (otherwise $c_m \ne 2$). This is clear for $m \ge 4$ because x_{m-1} takes the values 0 or 1. It also holds for $m = 3$ because x_2 takes the values $0, 1, 2$ or 3 for integers of type B when $g \ge 5$ and then $x_2 \le g-2$.

III.3.i) $\boldsymbol{y_{m-1} \neq 0, \quad z_{m-1} \neq g-1}$.

δ_m	δ_{m-1}	δ_{m-2}
0	$*$	$*$
y_{m-1}	y_m	y_{m-1}
$*$	z_{m-1}	z_{m-1}

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}
0	$*$	$*$
$y_{m-1}-1$	y_m-1	$y_{m-1}-1$
$*$	$z_{m-1}+1$	$z_{m-1}+1$

III.3.ii) $\boldsymbol{y_{m-1} \neq 0, \quad z_{m-1} = g-1}$.

δ_m	δ_{m-1}	δ_{m-2}
0	$*$	$*$
y_{m-1}	y_m	y_{m-1}
$*$	$g-1$	$g-1$

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}
1	$*$	$*$
$y_{m-1}-1$	y_m	$y_{m-1}-1$
$*$	0	0

III.3.iii) $\boldsymbol{y_{m-1} = 0, \quad z_{m-1} \neq g-1}$. In this case $x_{m-1} \neq 0$.

δ_{m+1}	δ_m	δ_{m-1}	δ_{m-2}
x_{m-1}	0	x_{m-1}	$*$
$*$	0	y_m	0
$*$	$*$	z_{m-1}	z_{m-1}

$\longrightarrow$

δ_{m+1}	δ_m	δ_{m-1}	δ_{m-2}
$x_{m-1}-1$	0	$x_{m-1}-1$	$*$
$*$	$g-1$	y_m-1	$g-1$
$*$	$*$	$z_{m-1}+1$	$z_{m-1}+1$

III.3.iv) $\boldsymbol{y_{m-1} = 0, \quad z_{m-1} = g-1}$. In this case $x_{m-1} \neq 0$.

δ_{m+1}	δ_m	δ_{m-1}	δ_{m-2}
x_{m-1}	0	x_{m-1}	$*$
$*$	0	y_m	0
$*$	$*$	$g-1$	$g-1$

$\longrightarrow$

δ_{m+1}	δ_m	δ_{m-1}	δ_{m-2}
$x_{m-1}-1$	1	$x_{m-1}-1$	$*$
$*$	$g-1$	y_m	$g-1$
$*$	$*$	0	0

Example: Let us illustrate this algorithm with an example. We consider the positive integer representing the first 21 decimal digits of $\zeta(3)$:

$$n = 120205690315959428539.$$

First let us note that n is a *normal integer* because the number of digits is odd. In addition n is of type B3. Therefore the initial configuration is:

1	2	0	2	0	5	6	9	0	3	1	5	9	5	9	4	2	8	5	3	9
1	**1**	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	**1**	**1**
		9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	**9**
			9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	**9**

Applying the algorithm III we get to the temporary configuration. Since $c_{10} = 1$ we do not need any adjustment step and the temporary configuration is also the final configuration.

n	1	2	0	2	0	5	6	9	0	3	1	5	9	5	9	4	2	8	5	3	9
p_1	1	1	0	0	1	0	1	0	0	1	0	1	0	0	1	0	1	0	0	1	1
p_2			9	2	0	0	7	4	0	5	0	5	0	5	0	4	7	0	0	2	9
p_3				9	9	4	8	4	9	7	0	9	9	0	7	9	4	8	4	9	9

3.3. **Algorithm IV.** The cases $m \leq 3$ correspond to the *small cases*. For $m \geq 4$, we proceed in the following steps:

Step 1: We choose x_1, y_1, z_1 according to the configurations described in Section 2.2. Define $c_1 = (1 + y_1 + z_1)/g$, which is the carry of the column 1.

Step 2: Define the digits

$$
\begin{aligned}
x_2 &= \begin{cases} D(\delta_{2m-3} - y_1) & \text{if} \quad z_1 \leq \delta_{2m-4} - 1; \\ D(\delta_{2m-3} - y_1 - 1) & \text{if} \quad z_1 \geq \delta_{2m-4}; \end{cases} \\
y_2 &= D(\delta_{2m-4} - z_1 - 1); \\
z_2 &= D(\delta_1 - x_1 - y_2 - c_1); \\
c_2 &= (x_1 + y_2 + z_2 + c_1 - \delta_1)/g \quad \text{(the carry from column } 2).
\end{aligned}
$$

Step i, $3 \leq i \leq m-2$**:** Define the digits

$$
\begin{aligned}
x_i &= \begin{cases} 1 & \text{if} \quad z_{i-1} \leq \delta_{2m-i-2} - 1; \\ 0 & \text{if} \quad z_{i-1} \geq \delta_{2m-i-2}; \end{cases} \\
y_i &= D(\delta_{2m-i-2} - z_{i-1} - 1); \\
z_i &= D(\delta_{i-1} - x_{i-1} - y_i - c_{i-1}); \\
c_i &= (x_{i-1} + y_i + z_i + c_{i-1} - \delta_{i-1})/g \quad \text{(the carry from column } i).
\end{aligned}
$$

Step $i = m-1$**:** Define the digits

$$
\begin{aligned}
x_{m-1} &= \begin{cases} 1 & \text{if} \quad z_{m-2} \leq \delta_{m-1} - 1; \\ 0 & \text{if} \quad z_{m-2} \geq \delta_{m-1}; \end{cases} \\
y_{m-1} &= D(\delta_{m-1} - z_{m-2} - 1) \\
z_{m-1} &= D(\delta_{m-2} - x_{m-2} - y_{m-1} - c_{m-2}).
\end{aligned}
$$

The temporary configuration is:

1	δ_{2m-2}	δ_{2m-3}	*	*	*	δ_m	δ_{m-1}	δ_{m-2}	*	*	*	*	*	δ_1	δ_0
1	x_1	.	.	.	x_{m-2}	x_{m-1}	x_{m-1}	x_{m-2}	.	.	.	.	.	x_1	1
		y_1	.	.	.	y_{m-2}	y_{m-1}	y_{m-1}	y_{m-2}	.	.	.	.	.	y_1
			z_1	.	.	.	z_{m-2}	z_{m-1}	z_{m-2}	.	.	.	.	.	z_1

Proposition 3.3. *Let $g \geq 5$ and $m \geq 4$. We have that $\Delta_i = \delta_i$ for all $0 \leq i \leq m-2$. Furhthermore, for any $-1 \leq i \leq m-3$, if $\Delta_{m+i} = \delta_{m+i}$, then $\Delta_{m+i+1} = \delta_{m+i+1}$.*

Proof. The first statement of the proposition is clear. For the second one, we consider firs the case $i = -1$. Assuming that $\Delta_{m-1} = \delta_{m-1}$ we have to prove that $\Delta_m = \delta_m$. Indeed

$$\Delta_m \equiv x_{m-1} + y_{m-2} + z_{m-3} + c_m \equiv \delta_m + x_{m-1} + c_m - 1 \pmod{g}.$$

If $x_{m-1} = 1$ then $z_{m-2} \leq \delta_{m-1} - 1$ and $y_{m-1} = \delta_{m-1} - z_{m-2} - 1$. On the other hand, since $\Delta_{m-1} \equiv \delta_{m-1} + x_{m-1} + c_{m-1} - 1 \pmod{g}$ and $\Delta_{m-1} = \delta_{m-1}$, we have that $x_{m-1} + c_{m-1} = 1$. Thus, $c_{m-1} = 0$. Finally

$$c_m = (x_{m-1} + y_{m-1} + z_{m-2} + c_{m-1} - \delta_{m-1})/g = 0.$$

If $x_{m-1} = 0$, then $z_{m-2} \geq \delta_{m-1}$ and $y_{m-1} = g + \delta_{m-1} - z_{m-2} - 1$. On the other hand, since $\Delta_{m-1} \equiv \delta_{m-1} + x_{m-1} + c_{m-1} - 1 \pmod{g}$ and $\Delta_{m-1} = \delta_{m-1}$, we have that $x_{m-1} + c_{m-1} = 1$. Thus $c_{m-1} = 1$. Finally

$$c_m = (x_{m-1} + y_{m-1} + z_{m-2} + c_{m-1} - \delta_{m-1})/g = 1.$$

In any case we have that $x_{m-1} + c_m = 1$ and then we conclude that $\Delta_m = \delta_m$.

We omit the proof of the proposition for the other cases because they are similar to the case $i = -1$. □

The above proposition gives that if $\Delta_{m-1} = \delta_{m-1}$ then $\Delta_i = \delta_i$ for all $i = 0, \ldots, 2m-2$.

The adjustment step of this algorithm is more complicated than the previous ones.

Adjustment step: Assume that $m \geq 4$. Notice that in this algorithm we have that

$$\Delta_{m-1} \equiv \delta_{m-1} + x_{m-1} + c_{m-1} - 1 \pmod{g}.$$

IV.1 $\boldsymbol{x_{m-1} + c_{m-1} = 1}$. We do nothing and the temporary configuration becomes the final one.

IV.2 $\boldsymbol{x_{m-1} + c_{m-1} = 0,\ y_{m-1} \neq g - 1}$. Then $x_{m-1} = c_{m-1} = 0$. If $y_{m-1} = 0$, then $z_{m-2} \equiv \delta_{m-2} - 1 \pmod{g}$, thus $z_{m-1} \leq \delta_{m-2} - 1$, so $x_{m-1} = 1$ unless $\delta_{m-1} = 0$, which is not allowed. Thus, $y_{m-1} \neq 0$.

IV.2.i) $\boldsymbol{z_{m-1} \neq 0}$.

δ_{m-1}	δ_{m-2}
$*$	$*$
y_{m-1}	y_{m-1}
$*$	z_{m-1}

$\longrightarrow$

δ_{m-1}	δ_{m-2}
$*$	$*$
$y_{m-1}+1$	$y_{m-1}+1$
$*$	$z_{m-1}-1$

IV.2.ii) $\boldsymbol{z_{m-1} = 0,\ y_{m-2} \neq 0}$.

IV.2.ii.a) $\boldsymbol{y_{m-1} \neq 1\ z_{m-2} \neq g-1}$.

δ_m	δ_{m-1}	δ_{m-2}	$*$
0	0	$*$	$*$
y_{m-2}	y_{m-1}	y_{m-1}	y_{m-2}
$*$	z_{m-2}	0	z_{m-2}

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
$y_{m-2}-1$	$y_{m-1}-1$	$y_{m-1}-1$	$y_{m-2}-1$
$*$	$z_{m-2}+1$	1	$z_{m-2}+1$

IV.2.ii.b) $\boldsymbol{y_{m-1} \neq 1\ z_{m-2} = g-1}$.

δ_m	δ_{m-1}	δ_{m-2}	$*$
0	0	$*$	$*$
y_{m-2}	y_{m-1}	y_{m-1}	y_{m-2}
$*$	$g-1$	0	$g-1$

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}	$*$
2	2	$*$	$*$
$y_{m-2}-1$	$y_{m-1}-2$	$y_{m-1}-2$	$y_{m-2}-1$
$*$	0	3	0

IV.2.ii.c) $\boldsymbol{y_{m-1} = 1,\ z_{m-2} \neq g-1}$.

δ_m	δ_{m-1}	δ_{m-2}	$*$
0	0	$*$	$*$
y_{m-2}	1	1	y_{m-2}
$*$	z_{m-2}	0	z_{m-2}

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
$y_{m-2}-1$	0	0	$y_{m-2}-1$
$*$	$z_{m-2}+1$	1	$z_{m-2}+1$

IV.2.ii.d) $\boldsymbol{y_{m-1} = 1,\ z_{m-2} = g-1}$.

δ_m	δ_{m-1}	δ_{m-2}	$*$
0	0	$*$	$*$
y_{m-2}	1	1	y_{m-2}
$*$	$g-1$	0	$g-1$

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
$y_{m-2}-1$	$g-1$	$g-1$	$y_{m-2}-1$
$*$	0	3	0

IV.2.iii) $\boldsymbol{z_{m-1} = 0,\ y_{m-2} = 0}$. Notice that $y_{m-2} \equiv \delta_m - z_{m-3} - 1 \pmod{g}$. Since $y_{m-2} = 0$ and $\delta_m \neq 0$, we have that $z_{m-3} \leq \delta_m - 1$ and then $x_{m-2} \neq 0$ (even when $m = 4$).

IV.2.iii.a) $\boldsymbol{z_{m-2} \neq g-1}$. It follows that $y_{m-1} \neq 0$. Otherwise we would have $\delta_{m-1} = 0$, which is not allowed.

$*$	δ_m	δ_{m-1}	δ_{m-2}	$*$
x_{m-2}	0	0	x_{m-2}	$*$
$*$	0	y_{m-1}	y_{m-1}	0
$*$	$*$	z_{m-2}	0	z_{m-2}

$\longrightarrow$

$*$	δ_m	δ_{m-1}	δ_{m-2}	$*$
$x_{m-2}-1$	1	1	$x_{m-2}-1$	$*$
$*$	$g-1$	$y_{m-1}-1$	$y_{m-1}-1$	$g-1$
$*$	$*$	$z_{m-2}+1$	1	$z_{m-2}+1$

IV.2.iii.b) $\boldsymbol{z_{m-2} = g-1,\ y_{m-1} \neq 1}$.

$*$	δ_m	δ_{m-1}	δ_{m-2}	$*$
x_{m-2}	0	0	x_{m-2}	$*$
$*$	0	y_{m-1}	y_{m-1}	0
$*$	$*$	$g-1$	0	$g-1$

$\longrightarrow$

$*$	δ_m	δ_{m-1}	δ_{m-2}	$*$
$x_{m-2}-1$	2	2	$x_{m-2}-1$	$*$
$*$	$g-1$	$y_{m-1}-2$	$y_{m-1}-2$	$g-1$
$*$	$*$	0	3	0

IV.2.iii.c) $\boldsymbol{z_{m-2} = g-1,\ y_{m-1} = 1}$.

$*$	δ_m	δ_{m-1}	δ_{m-2}	$*$
x_{m-2}	0	0	x_{m-2}	$*$
$*$	0	1	1	0
$*$	$*$	$g-1$	0	$g-1$

$\longrightarrow$

$*$	δ_m	δ_{m-1}	δ_{m-2}	$*$
$x_{m-2}-1$	2	2	$x_{m-2}-1$	$*$
$*$	$g-2$	$g-3$	$g-3$	$g-2$
$*$	$*$	1	5	1

IV.3 $\boldsymbol{x_{m-1} + c_{m-1} = 0}$, $\boldsymbol{y_{m-1} = g-1}$. Since $c_{m-1} = 0$, it follows that $x_{m-2} = z_{m-1} = 0$. Notice that if $y_{m-2} = 0$, then $\delta_m = 0$ (otherwise $z_{m-3} = \delta_m - 1$ and then $x_{m-2} \neq 0$), which is not allowed.

IV.3.i) $\boldsymbol{z_{m-2} \neq g-1}$.

δ_m	δ_{m-1}	δ_{m-2}	$*$
0	0	$*$	$*$
y_{m-2}	$g-1$	$g-1$	y_{m-2}
$*$	z_{m-2}	0	z_{m-2}

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
$y_{m-2}-1$	$g-2$	$g-2$	$y_{m-2}-1$
$*$	$z_{m-2}+1$	1	$z_{m-2}+1$

IV.3.ii) $\boldsymbol{z_{m-2} = g-1}$.

δ_m	δ_{m-1}	δ_{m-2}	$*$
0	0	$*$	$*$
y_{m-2}	$g-1$	$g-1$	y_{m-2}
$*$	$g-1$	0	$g-1$

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}	$*$
2	2	$*$	$*$
$y_{m-2}-1$	$g-3$	$g-3$	$y_{m-2}-1$
$*$	0	3	0

IV.4 $\boldsymbol{x_{m-1} + c_{m-1} = 2,\ x_{m-1} = 0,\ c_{m-1} = 2}$. If $y_{m-1} = 0$, then $z_{m-2} = g-1$ and then $\delta_{m-1} \neq 0$. So, $y_{m-1} \neq 0$.

IV.4.i) $\boldsymbol{z_{m-1} \neq g-1}$.

δ_{m-1}	δ_{m-2}
$*$	$*$
y_{m-1}	y_{m-1}
z_{m-2}	z_{m-1}

$\longrightarrow$

δ_{m-1}	δ_{m-2}
$*$	$*$
$y_{m-1}-1$	$y_{m-1}-1$
z_{m-2}	$z_{m-1}+1$

IV.4.ii) $\boldsymbol{z_{m-1} = g-1,\ z_{m-2} \neq g-1}$. Notice that $y_{m-1} \neq 1$. Otherwise $c_{m-1} \neq 2$ (even when $m=4$)

IV.4.ii.a) $\boldsymbol{y_{m-2} \neq 0}$.

δ_m	δ_{m-1}	δ_{m-2}	$*$
0	0	$*$	$*$
y_{m-2}	y_{m-1}	y_{m-1}	y_{m-2}
$*$	z_{m-2}	$g-1$	z_{m-2}

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
$y_{m-2}-1$	$y_{m-1}-2$	$y_{m-1}-2$	$y_{m-2}-1$
$*$	$z_{m-2}+1$	1	$z_{m-2}+1$

IV.4.ii.b) $\boldsymbol{y_{m-2} = 0}$. As in case IV.2.iii), we have that $x_{m-2} \neq 0$.

$*$	δ_m	δ_{m-1}	δ_{m-2}	$*$
x_{m-2}	0	0	x_{m-2}	$*$
$*$	0	y_{m-1}	y_{m-1}	0
$*$	$*$	z_{m-2}	$g-1$	z_{m-2}

$\longrightarrow$

$*$	δ_m	δ_{m-1}	δ_{m-2}	$*$
$x_{m-2}-1$	1	1	$x_{m-2}-1$	$*$
$*$	$g-1$	$y_{m-1}-2$	$y_{m-1}-2$	$g-1$
$*$	$*$	$z_{m-2}+1$	1	$z_{m-2}+1$

IV.5 $\boldsymbol{x_{m-1}+c_{m-1} = 2,\ x_{m-1} = 1,\ c_{m-1} = 1}$. In particular, it follows that $z_{m-2} \neq g-1$ (otherwise we would have $x_{m-1} = 0$).

IV.5.i) $\boldsymbol{z_{m-1} \neq g-1,\ y_{m-1} \neq 0}$.

δ_{m-1}	δ_{m-2}
$*$	$*$
y_{m-1}	y_{m-1}
$*$	z_{m-1}

$\longrightarrow$

δ_{m-1}	δ_{m-2}
$*$	$*$
$y_{m-1}-1$	$y_{m-1}-1$
$*$	$z_{m-1}+1$

IV.5.ii) $\boldsymbol{z_{m-1} \neq g-1,\ y_{m-1} = 0}$.

$*$	δ_{m-1}	δ_{m-2}
1	1	$*$
$*$	0	0
$*$	$*$	z_{m-1}

$\longrightarrow$

$*$	δ_{m-1}	δ_{m-2}
0	0	$*$
$*$	$g-1$	$g-1$
$*$	$*$	$z_{m-1}+1$

IV.5.iii) $\boldsymbol{z_{m-1} = g-1,\ z_{m-2} \neq 0}$.

IV.5.iii.a) $\boldsymbol{y_{m-2}, y_{m-1} \neq g-1}$.

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
y_{m-2}	y_{m-1}	y_{m-1}	y_{m-2}
$*$	z_{m-2}	$g-1$	z_{m-2}

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}	$*$
0	0	$*$	$*$
$y_{m-2}+1$	$y_{m-1}+1$	$y_{m-1}+1$	$y_{m-2}+1$
$*$	$z_{m-2}-1$	$g-2$	$z_{m-2}-1$

IV.5.iii.b) $\boldsymbol{y_{m-2} = g-1,\ y_{m-1} \neq 0, 1}$.

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
$g-1$	y_{m-1}	y_{m-1}	$g-1$
$*$	z_{m-2}	$g-1$	z_{m-2}

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}	$*$
2	2	$*$	$*$
$g-2$	$y_{m-1}-2$	$y_{m-1}-2$	$g-2$
$*$	$z_{m-2}+1$	1	$z_{m-2}+1$

IV.5.iii.c) $\boldsymbol{y_{m-2} = g-1,\ y_{m-1} = 0}$.

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
$g-1$	0	0	$g-1$
	z_{m-2}	$g-1$	z_{m-2}

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
$g-2$	$g-2$	$g-2$	$g-2$
.	$z_{m-2}+1$	1	$z_{m-2}+1$

IV.5.iii.d) $\boldsymbol{y_{m-2} = g-1,\ y_{m-1} = 1}$.

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
$g-1$	1	1	$g-1$
$*$	z_{m-2}	$g-1$	z_{m-2}

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
$g-2$	$g-1$	$g-1$	$g-2$
$*$	$z_{m-2}+1$	1	$z_{m-2}+1$

IV.5.iii.e) $\boldsymbol{y_{m-1} = g-1,\ y_{m-2} \neq g-1}$.

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
y_{m-2}	$g-1$	$g-1$	y_{m-2}
$*$	z_{m-2}	$g-1$	z_{m-2}

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
$y_{m-2}+1$	0	0	$y_{m-2}+1$
$*$	$z_{m-2}-1$	$g-2$	$z_{m-2}-1$

IV.5.iv) $\boldsymbol{z_{m-1} = g-1,\ z_{m-2} = 0,\ y_{m-2} \neq 0}$.

IV.5.iv.a) $\boldsymbol{y_{m-1} \neq 0, 1}$.

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
y_{m-2}	y_{m-1}	y_{m-1}	y_{m-2}
$*$	0	$g-1$	0

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}	$*$
2	2	$*$	$*$
$y_{m-2}-1$	$y_{m-1}-2$	$y_{m-1}-2$	$y_{m-2}-1$
$*$	1	1	1

IV.5.iv.b) $\boldsymbol{y_{m-1}=0}$.

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
y_{m-2}	0	0	y_{m-2}
$*$	0	$g-1$	0

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
$y_{m-2}-1$	$g-2$	$g-2$	$y_{m-2}-1$
	1	1	1

IV.5.iv.c) $\boldsymbol{y_{m-1}=1}$.

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	.	.
y_{m-2}	1	1	y_{m-2}
.	0	$g-1$	0

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
$y_{m-2}-1$	$g-1$	$g-1$	$y_{m-2}-1$
$*$	1	1	1

IV.5.v) $\boldsymbol{z_{m-1}=g-1,\ z_{m-2}=0,\ y_{m-2}=0}$. If $x_{m-2}=0$, then $\delta_m=0$, which is not allowed. Thus, $x_{m-2}\neq 0$ (even when $m=4$).

IV.5.v.a) $\boldsymbol{y_{m-1}\neq 0,1}$. As in case IV.2.iii), we have that $x_{m-2}\neq 0$.

$*$	δ_m	δ_{m-1}	δ_{m-2}	$*$
x_{m-2}	1	1	x_{m-2}	$*$
$*$	0	y_{m-1}	y_{m-1}	0
$*$	$*$	0	$g-1$	0

$\longrightarrow$

$*$	δ_m	δ_{m-1}	δ_{m-2}	$*$
$x_{m-2}-1$	2	2	$x_{m-2}-1$	$*$
	$g-1$	$y_{m-1}-2$	$y_{m-1}-2$	$g-1$
.	.	1	1	1

IV.5.v.b) $\boldsymbol{y_{m-1}=0}$.

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
0	0	0	0
	0	$g-1$	0

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
$g-2$	$g-3$	$g-3$	$g-2$
$*$	2	1	2

IV.4.v.c) $\boldsymbol{y_{m-1}=1}$.

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
0	1	1	0
$*$	0	$g-1$	0

$\longrightarrow$

δ_m	δ_{m-1}	δ_{m-2}	$*$
1	1	$*$	$*$
$g-2$	$g-2$	$g-2$	$g-2$
	2	1	2

IV.6 $\boldsymbol{x_{m-1}+c_{m-1}=3}$. Then $x_{m-1}=1$ and $c_{m-1}=2$. We always have that $x_{m-2}\leq 3$ (even when $m=4$). It follows that $y_{m-1}\geq 1$ and $z_{m-1}=g-1$ (otherwise

$z_{m-1} + y_{m-1} + x_{m-2} + c_{m-2} \leq g - 1 + 4 + 2 \leq 2g - 1$ and then $c_{m-1} \neq 2$).

δ_{m-1}	δ_{m-2}
$*$	$*$
y_{m-1}	y_{m-1}
$*$	$g-1$

$\longrightarrow$

δ_{m-1}	δ_{m-2}
$*$	$*$
$y_{m-1}-1$	$y_{m-1}-1$
$*$	0

3.4. **Algorithm V.** We recall that in this case the associated palindrome p_1 of n has $2m$ digits and that $\delta_{m-1} = 0$ or $\delta_m = 0$. First we consider the integer

$$n' = n - s, \qquad \text{where} \qquad s = g^m + g^{m-1}.$$

If $\delta'_{m-1} \neq 0$ and $\delta'_m \neq 0$, we keep n'. Otherwise we consider the integer $n' = n - 2s$. It is easy to check that one of $n' = n - s$ or $n' = n - 2s$ satisfies that $\delta'_{m-1} \neq 0$ and $\delta'_m \neq 0$.

We distinguish two cases:

i) The associated palindrome p'_1 of n' has also $2m$ digits (this is the typical situation).

We apply Algorithms II or IV according the type of n'. Then $n' = p'_1 + p'_2 + p'_3$ and so

$$n = n' + s = (p'_1 + s) + p'_2 + p'_3.$$

Notice that $p'_1 + s$ is also a palindrome because we are adding 1 or 2 to the two central digits of p'_1. Note that if we have applied Algorithm II, then the central digits are x'_m and x'_m, which are 0 or 1 for $m \geq 3$. Note also that if we have applied Algorithm IV, then the central digits are x'_{m-1} and x'_{m-1}, which are 0 or 1 for $m \geq 4$. Hence, in all the cases the value of the two central digits is at most 3, which are legal digits for $g \geq 5$ (indeed, even for $g \geq 4$).

ii) The associated palindrome p'_1 of n' has $2m - 1$ digits.

This is only possible if n is of the form $n = 104\ldots$ and $n' = 103\ldots$. In this special situation, we consider n' as of type B1 or B2 and apply the Algorithm IV to n' (instead of Algorithm I). Notice that the configuration of the starting point in B1 and B2 is also valid when $\delta_{l-3} = 3$. Then the palindrome p'_1 we get in this way has $2m$ digits and, as above, we have

$$n = n' + s = (p'_1 + s) + p'_2 + p'_3.$$

Example: We finish with one example which shows how to apply Algorithms IV and V. Let n be the positive integer giving the first 20 digits of $\sqrt[3]{2}$:

$$n = 12267420107203532444.$$

The number n is a special number because it has an even number of digits, 20, $m = 10$ and $\delta_m = 0$. Thus, we apply Algorithm V and consider $n' = n - s$, where $s = 10^{10} + 10^9$.

Note that $n' = 12267420096203532444$, which is a normal number because $\delta'_m \neq 0$ and $\delta'_{m-1} \neq 0$.

We observe that n' is of type B.3, so we apply Algorithm IV to n'. The initial configuration is

1	2	2	6	7	4	2	0	0	9	6	2	0	3	5	3	2	4	4	4
1	**1**	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	**1**	**1**
		9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	**9**
			4	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	**4**

The temporary configuration is

1	2	2	6	7	4	2	0	0	9	6	2	0	3	5	3	2	4	4	4
1	1	3	1	0	0	0	0	1	1	1	1	0	0	0	0	1	3	1	1
		9	1	5	7	8	5	0	6	1	1	6	0	5	8	7	5	1	9
			4	1	6	3	4	9	2	4	9	4	2	9	4	3	6	1	4

Note that we need an adjustment becuase the digit in column 10 is not correct. The reason is that $x_9 + c_9 = 2$. Looking at the central digits, we must follow the Adjustment Step IV.5.iii.a):

n'	1	2	2	6	7	4	2	0	0	9	6	2	0	3	5	3	2	4	4	4
p'_1	1	1	3	1	0	0	0	0	1	**0**	**0**	1	0	0	0	0	1	3	1	1
p'_2			9	1	5	7	8	5	0	**7**	**2**	**2**	**7**	0	5	8	7	5	1	9
p'_3				4	1	6	3	4	9	2	**3**	**8**	**3**	2	9	4	3	6	1	4

Finally, we add $s = 10^{10} + 10^9$ to n' to obtain a representation of n as a sum of three palindromes.

n	1	2	2	6	7	4	2	0	1	0	7	2	0	3	5	3	2	4	4	4
p_1	1	1	3	1	0	0	0	0	1	**1**	**1**	1	0	0	0	0	1	3	1	1
p_2			9	1	5	7	8	5	0	7	2	2	7	0	5	8	7	5	1	9
p_3				4	1	6	3	4	9	2	3	8	3	2	9	4	3	6	1	4

4. Small integers

Proposition 4.1. *All positive integers with less than seven digits are the sum of three palindromes in base $g \geq 5$.*

Proof. The proof is a consequence of the Lemmas 4.2, 4.3, 4.4, 4.5 and 4.6. □

Lemma 4.2. *All positive integers with two digits are the sum of two palindromes in base $g \geq 5$, except those of the form $n = (\delta + 1)\delta$, $1 \leq \delta \leq g - 2$, which are sum of three palindromes.*

Proof. Let $n = \delta_1\delta_0$.

$\delta_1 \le \delta_0$

δ_1	δ_0
δ_1	δ_1
	$\delta_0 - \delta_1$

$\delta_1 \le \delta_0$

δ_1	δ_0
$\delta_1 - 1$	$\delta_1 - 1$
	$g + \delta_0 - \delta_1 + 1$

$\delta_1 = \delta_0 + 1$

$\delta_0 + 1$	δ_0
δ_0	δ_0
	$g - 1$
	1

□

Lemma 4.3. *All positive integers with three digits are the sum of two palindromes in base $g \ge 5$, except $n = 201$ which is sum of three palindromes.*

Proof. Let $n = \delta_2\delta_1\delta_0$.

$\delta_2 \le \delta_0$

δ_2	δ_1	δ_0
δ_2	δ_1	δ_2
		$\delta_0 - \delta_2$

$\delta_2 \ge \delta_0{+}1,\ \delta_1 \ne 0$

δ_2	δ_1	δ_0
δ_2	$\delta_1 - 1$	δ_2
		$g + \delta_0 - \delta_2$

$\delta_2 \ge \delta_0{+}1,\ \delta_1 = 0,\ D(\delta_2{-}\delta_0{-}1) \ne 0$

δ_2	δ_1	δ_0
$\delta_2 - 1$	$g - 1$	$\delta_2 - 1$
		$g + \delta_0 - \delta_2 + 1$

If $\delta_2 \ge \delta_0 + 1$, $\delta_1 = 0$, and $D(\delta_2 - \delta_0 - 1) = 0$, we have that $\delta_0 \equiv \delta_2 - 1 \pmod g$ and we distinguish the following cases:

$\delta_2 \ge 3$

δ_2	0	$\delta_2 - 1$
$\delta_2 - 2$	$g - 1$	$\delta_2 - 2$
1	1	1

$\delta_2 = 2$

2	0	1
1	0	1
	$g - 1$	$g - 1$
		1

$\delta_2 = 1$

1	0	0
	$g - 1$	$g - 1$
		1

□

Lemma 4.4. *All positive integers with four digits are the sum of three palindromes in base $g \ge 5$.*

Proof. Let $n = \delta_3\delta_2\delta_1\delta_0$.

i) $n \ge \delta_3 00\delta_3$, and n is not of the form $n = \delta_3 00\delta_3 {+} m$ with $m = 201$, or $m = (\delta{+}1)\delta$ with $\delta \ge 1$. Then $n - \delta_3 00\delta_3$ is the sum of two palindromes p_1, p_2 and

$$n = \delta_3 00\delta_3 + p_1 + p_2.$$

ii) $n = \delta_3 00\delta_3 + 201$.

$\delta_3 \ne 1, g - 1$ $\qquad$ $\delta_3 = 1$ $\qquad$ $\delta_3 = g - 1$

δ_3	2	0	δ_3+1
δ_3-1	$g-1$	$g-1$	δ_3-1
	2	1	2

1	2	0	2
1	1	1	1
		$g-2$	$g-2$
			3

$g-1$	2	1	0
$g-1$	1	1	$g-1$
		$g-2$	$g-2$
			3

iii) $n = \delta_3 00 \delta_3 + (\delta+1)\delta,\ 1 \le \delta \le g-2:$

a) $\delta_3 + \delta = \delta_0$,

$\delta_3 \neq 1$

δ_3	0	$\delta+1$	δ_0
δ_3-1	$g-2$	$g-2$	δ_3-1
	1	$\delta+2$	1
			δ

$\delta_3 = 1$

1	0	$\delta+1$	$\delta+1$
	$g-1$	$g-1$	$g-1$
		$\delta+1$	$\delta+1$
			1

b) $\delta_3 + \delta = g + \delta_0$ with $0 \le \delta_0 \le g-1$:

δ_3	0	$\delta+1$	δ_0
δ_3	0	0	δ_3
		δ	δ

iv) $n \le \delta_3 00(\delta_3-1)$ and $\delta_3 \neq 1$. Then:

δ_3	0	0	δ_0
δ_3-1	$g-1$	$g-1$	δ_3-1
			$g+\delta_0-\delta_3$
			1

v) $n \le \delta_3 00(\delta_3-1)$ and $\delta_3 = 1$. Then:

1	0	0	0
	$g-1$	$g-1$	$g-1$
			1

□

Lemma 4.5. *All positive integers with five digits are the sum of three palindromes in base $g \ge 5$.*

Proof. If $\delta_4 \neq 1$, then n is of type A and we apply Algorithm I, which works for $m = 2$.

Thus, we assume that $\delta_4 = 1$. Let $n = 1\delta_3\delta_2\delta_1\delta_0$.

i) $n \ge 1\delta_3 0 \delta_3 1$ and n is not of the form $n = 1\delta_3 0\delta_3 1 + m$ with $m = 201$, or $m = (\delta+1)\delta$ with $\delta \ge 1$. By Propositions 4.2 and 4.3, $n - 1\delta_3 0 \delta_3 1$ is the sum of two palindromes p_1, p_2 and then

$$n = 1\delta_3 0 \delta_3 1 + p_1 + p_2.$$

ii) $n = 1\delta_3 0\delta_3 1 + 201$:

1	δ_3	2	δ_3	2
1	δ_3	1	δ_3	1
		1	0	1

iii) $n = 1\delta_3 0 \delta_3 1 + (\delta+1)\delta, \quad 1 \le \delta \le g-2,\ \delta_3 \neq 0:$

a) $\delta + 1 + \delta_3 \le g-1:$

1	δ_3	0	$\delta_3+\delta+1$	$\delta+1$
1	δ_3-1	1	δ_3-1	1
		$g-1$	$\delta+1$	$g-1$
				$\delta+1$

b) $\delta_3 + 1 + \delta = g + \delta_1$ with $0 \le \delta_1 \le g-1$:

1	δ_3	1	δ_1	$\delta+1$
1	δ_3-1	0	δ_3-1	1
		$g-1$	$\delta+1$	$g-1$
				$\delta+1$

iv) $n = 1\delta_3 0 \delta_3 1 + (\delta+1)\delta, \quad 1 \le \delta \le g-2,\ \delta_3 = 0:$

1	0	0	$\delta+1$	$\delta+1$
	$g-1$	$g-1$	$g-1$	$g-1$
			$\delta+1$	$\delta+1$
				1

v) $n \le 1\delta_3 0 \delta_3 0$ and $\delta_3 = 0$. Then:

1	0	0	0	0
	$g-1$	$g-1$	$g-1$	$g-1$
				1

vi) $n \le 1\delta_3 0 \delta_3 0$ and $\delta_3 \neq 0$ with $n = 1(\delta_3-1)(g-1)(\delta_3-1)1 + m$ with $m \neq 201$ and $m \neq (\delta+1)\delta,\ 1 \le \delta \le g-2$. Propositions 4.2 and 4.3 imply that m is sum of two palindromes p_1, p_2 and then

$$n = 1(\delta_3-1)(g-1)(\delta_3-1)1 + p_1 + p_2.$$

vii) $n = 1(\delta_3-1)(g-1)(\delta_3-1)1 + 201,\ \delta_3 \neq 0:$

1	δ_3	1	δ_3-1	2
1	δ_3-1	$g-2$	δ_3-1	1
		2	$g-1$	2
				$g-1$

viii) $n = 1(\delta_3-1)(g-1)(\delta_3-1)1 + (\delta+1)\delta,\ \delta_3 \neq 0,\ \delta_3 + \delta \le g-1:$

1	δ_3-1	$g-1$	$\delta_3+\delta$	$\delta+1$
1	δ_3-1	$g-2$	δ_3-1	1
		1	$\delta+1$	1
				$\delta-1$

viii) $n = 1(\delta_3 - 1)(g-1)(\delta_3 - 1)1 + (\delta+1)\delta,\ \delta_3 \neq 0,\ \delta_3 + \delta = g + \delta_1,\ 0 \leq \delta_1 \leq g-1:$

1	$\delta_3 - 1$	$g-1$	$\delta_3 + \delta$	$\delta + 1$
1	$\delta_3 - 1$	$g-3$	$\delta_3 - 1$	1
		1	$\delta + 1$	1
				$\delta - 1$

□

Lemma 4.6. *All positive integers with six digits are the sum of three palindromes in base $g \geq 5$.*

Proof. First, we consider the case $\delta_5 \neq 1$.

We apply Algorithm II for $m = 3$ with some exceptions. Note that Algorithm II was applied to *normal numbers*. It was only used in the Adjustment Step II.2.ii.c), where we assumed that $\delta_2 \neq 0$ and then that $z_2 \neq 0$ in that step. Thus, to apply Algorithm II when n is not a *normal number*, we have to account also for the possibility $z_2 = 0$ in the Step II.2.ii.c). This is the temporary configuration in Step II.2.ii.c) ($c_2 = 0,\ y_3 = y_2 = 0$) with $z_2 = 0$.

δ_5	δ_4	δ_3	δ_2	δ_1	δ_0
x_1	x_2	0	0	x_2	x_1
	y_1	0	0	0	y_1
		z_1	0	0	z_1

If $x_2 \neq 0$, then the adjustment step is the following:

δ_5	δ_4	δ_3	δ_2	δ_1	δ_0
x_1	x_2	0	0	x_2	x_1
	y_1	0	0	0	y_1
		z_1	0	0	z_1

$\longrightarrow$

δ_5	δ_4	δ_3	δ_2	δ_1	δ_0
x_1	$x_2 - 1$	$g-1$	$g-1$	$x_2 - 1$	x_1
	y_1	1	1	1	y_1
		z_1	0	0	z_1

If $x_2 = 0$, we distinghish several cases:

i) $x_1 = 1$. It follows that $\delta_5 = 1$ (which is not allowed), unless $y_1 = z_1 = g-1$. The adjustment step is the following:

δ_5	δ_4	δ_3	δ_2	δ_1	δ_0
1	0	0	0	0	1
	$g-1$	0	0	0	$g-1$
		$g-1$	0	0	$g-1$

$\longrightarrow$

δ_5	δ_4	δ_3	δ_2	δ_1	δ_0
2	0	0	0	0	2
				1	1
					$g-4$

ii) $x_1 \neq 1,\ y_1 \neq g-1$. The adjustment step is the following:

δ_5	δ_4	δ_3	δ_2	δ_1	δ_0
x_1	0	0	0	0	x_1
	y_1	0	0	0	y_1
		z_1	0	0	z_1

$\longrightarrow$

δ_5	δ_4	δ_3	δ_2	δ_1	δ_0
x_1-1	$g-1$	0	0	$g-1$	x_1-1
	y_1+1	0	$g-1$	0	y_1+1
		z_1	1	1	z_1

iii) $x_1 \neq 1,\ z_1 \neq g-1$. The adjustment step is the following:

δ_5	δ_4	δ_3	δ_2	δ_1	δ_0
x_1	0	0	0	0	x_1
	y_1	0	0	0	y_1
		z_1	0	0	z_1

$\longrightarrow$

δ_5	δ_4	δ_3	δ_2	δ_1	δ_0
x_1-1	$g-1$	$g-2$	$g-2$	$g-1$	x_1-1
	y_1	1	1	1	y_1
		z_1+1	0	0	z_1+1

iv) $x_1 \neq 1, g-1,\ z_1 = x_1 = g-1$. The adjustment step is the following:

δ_5	δ_4	δ_3	δ_2	δ_1	δ_0
x_1	0	0	0	0	x_1
	$g-1$	0	0	0	$g-1$
		$g-1$	0	0	$g-1$

$\longrightarrow$

δ_5	δ_4	δ_3	δ_2	δ_1	δ_0
x_1+1	0	0	0	0	x_1+1
				1	1
					$g-4$

v) $x_1 = y_1 = z_1 = g-1$. Note that in this case we have that

$$\delta_5\delta_4\delta_3\delta_2\delta_1\delta_0 = (g-1)0000(g-1) + (g-1)000(g-1) + (g-1)00(g-1) + 1000$$

but we can check easily that this number has 7 digits.

Secondly, we consider the case $\delta_5 = 1$.

i) $z_1 = D(\delta_0 - \delta_4 + 1) \neq 0$ and $D(\delta_0 - \delta_4 + 2) \neq 0$.

1	δ_4	δ_3	δ_2	δ_1	δ_0
	x_1	x_2	x_3	x_2	x_1
	y_1	y_2	y_3	y_2	y_1
			z_1	z_2	z_1

We choose x_1, y_1 such that $1 \leq x_1, y_1 \leq g-1$ and $x_1 + y_1 = g + \delta_4 - 1$. This is possible because $2 \leq g + \delta_4 - 1 \leq 2g - 2$.

We choose x_2, y_2 such that $0 \leq x_2, y_2 \leq g-1$ and $x_2 + y_2 = g + \delta_3 - 1$. This is possible because $0 \leq g + \delta_4 - 1 \leq 2g - 2$. We also define $z_2 = D(\delta_1 - x_2 - y_2 - c_1)$.

We choose x_3, y_3 such that $0 \leq x_3, y_3 \leq g-1$ and $x_3 + y_3 = g + \delta_2 - c_2 - z_1$. This is possible because, as $z_1 \neq 0$, we have that $g + \delta_2 - c_2 - z_1 \leq 2g - 2$, and since $D(\delta_0 - \delta_4 + 2) \neq 0$, we have $z_1 \neq g-1$ and therefore

$$g + \delta_2 - c_2 - z_1 \geq g + 0 - 2 - (g-2) = 0.$$

ii) $D(\delta_0 - \delta_4 + 2) = 0,\ \delta_2 \neq 0.$

1	δ_4	δ_3	δ_2	δ_1	δ_0
	x_1	x_2	x_3	x_2	x_1
	y_1	y_2	y_3	y_2	y_1
			z_1	z_2	z_1

We choose x_1, y_1 such that $1 \leq x_1, y_1 \leq g-1$ y $x_1 + y_1 = g + \delta_4 - 1$.
We choose x_2, y_2 such that $0 \leq x_2, y_2 \leq g-1$ y $x_2 + y_2 = g + \delta_3 - 1$.
We choose x_3, y_3 such that $0 \leq x_3, y_3 \leq g-1$ y $x_3 + y_3 = g + \delta_2 - c_2 - z_1$.
All such choices are possible by the same argument as in i) except that now we have to justify in a different way that $g + \delta_2 - c_2 - z_1 \geq 0$. But this is clear because $g + \delta_2 - c_2 - z_1 \geq g + 1 - 2 - (g-1) = 0$.

iii) $D(\delta_0 - \delta_4 + 2) = 0,\ \delta_2 = 0.$

a) $\delta_4 = 0$. Then $\delta_0 = g - 2$.

1	0	δ_3	0	δ_1	$g-2$
	$g-2$	x_2	x_3	x_2	$g-2$
	1	y_2	y_3	y_2	1
		$g-1$	z_2	z_2	$g-1$

We choose x_2, y_2 such that $0 \leq x_2, y_2 \leq g-1$ and $x_2 + y_2 = \delta_3$.
We choose x_3, y_3 such that $0 \leq x_3, y_3 \leq g-1$ and $x_3 + y_3 = g - c_2 - z_2$.
Observe that $c_2 = (x_2 + y_2 + z_2 + c_1 - \delta_1)/g \leq (g - 1 + g - 1 + 1)/g < 2$. Thus, $c_2 \neq 2$ and $g - c_2 - z_2 \geq g - 1 - (g-1) \geq 0$, therefore we can choose such x_3 and y_3.

b) $\delta_4 = 1$. Then $\delta_0 = g - 1$.

1	1	δ_3	0	δ_1	$g-1$
	$g-1$	x_2	x_3	x_2	$g-1$
	1	y_2	y_3	y_2	1
		$g-1$	z_2	z_2	$g-1$

The choices for the x_i's are identical to the ones from case a).

c) $\delta_4 = 2$. Then $\delta_0 = 0$.

1	2	δ_3	0	δ_1	0
	$g-1$	x_2	x_3	x_2	$g-1$
	2	y_2	y_3	y_2	2
		$g-1$	z_2	z_2	$g-1$

We choose x_2, y_2 such that $0 \leq x_2, y_2 \leq g-1$ y $x_2 + y_2 = \delta_3$.
We choose x_3, y_3 such that $0 \leq x_3, y_3 \leq g-1$ y $x_3 + y_3 = g - c_2 - z_2$.
If $c_2 \neq 2$, then we can make such a choice for x_3 and y_3.
However, if $c_2 = 2$, then $x_2 + y_2 = z_2 = g-1$ and $\delta_1 = 0$ and $\delta_3 = g-1$. In this special case, we have:

1	2	$g-1$	0	0	0
1	2	$g-2$	$g-2$	2	1
			1	$g-3$	1
					$g-2$

iii) $D(\delta_0 - \delta_4 + 1) = 0, \ \delta_3 \neq 0:$

1	δ_4	δ_3	δ_2	δ_1	δ_0
	x_1	x_2	x_3	x_2	x_1
	y_1	y_2	y_3	y_2	y_1
			z_1	z_2	z_1

We choose x_1, y_1 such that $1 \leq x_1, y_1 \leq g-1$ and $x_1 + y_1 = g + \delta_4$. This is possible because $\delta_4 \leq 2 \leq g-2$. On the other hand, $z_1 = g-1$.

We choose x_2, y_2 such that $0 \leq x_2, y_2 \leq g-1$ and $x_2 + y_2 = \delta_3 - 1$.

We choose x_3, y_3 such that $0 \leq x_3, y_3 \leq g-1$ and

$$x_3 + y_3 = g + \delta_2 - c_2 - z_1 = 1 + \delta_2 - c_2.$$

This is possible because $c_2 \leq 1$. Indeed,

$$c_2 = (x_2 + y_2 + z_2 + c_1 - \delta_1)/g \leq (\delta_3 - 1 + g - 1 + 2)/g < 2.$$

iv) $D(\delta_0 - \delta_4 + 1) = 0, \ \delta_3 = 0$.

a) $\delta_4 = 0$. Then $\delta_0 = g-1$.
If $\delta_2 \neq 0$, then $n - 100001 = \delta_2\delta_1(g-2)$ is a sum of two palindromes.
If $\delta_2 = 0$ and $\delta_1 \neq 0, g-1$, then $n - 100001 = (\delta_1 - 1)(g-1)$ is also a sum of two palindromes.
If $\delta_2 = 0$ and $\delta_1 = 0$, then

1	0	0	0	0	$g-1$
1	0	0	0	0	1
					$g-2$

If $\delta_2 = 0$ and $\delta_1 = g-1$, then

1	0	0	0	$g-1$	$g-1$
	$g-1$	0	1	0	$g-1$
		$g-1$	$g-2$	$g-2$	$g-1$
			1	0	1

b) $\delta_4 = 1$. Then $\delta_0 = 0$.
If $\delta_2 \geq 2$ or if $\delta_2 = 1$ and $\delta_1 \neq 0, 1$ then $n - 110011$ has three digits, its last digit is $g-1$, therefore it can be written as a sum of two palindromes.
If $\delta_2 = 1$ and $\delta_1 = 0$, then

1	1	0	1	0	0
1	0	$g-1$	$g-1$	0	1
			1	$g-1$	1
					$g-2$

If $\delta_2 = 1$ and $\delta_1 = 1$, then

1	1	0	1	1	0
1	1	0	0	1	1
				$g-1$	$g-1$

If $\delta_2 = 0$ and $\delta_1 \geq 2$, then

1	1	0	0	δ_1	0
1	1	0	0	1	1
				$\delta_1 - 2$	$\delta_1 - 2$
					$g - \delta_1 + 1$

If $\delta_2 = 0$ and $\delta_1 = 1$, then

1	1	0	0	1	0
1	0	0	0	0	1
	1	0	0	0	1
					$g-2$

If $\delta_2 = 0$ and $\delta_1 = 0$ then

1	1	0	0	0	0
1	0	0	0	0	1
		$g-1$	$g-1$	$g-1$	$g-1$

c) $\delta_4 = 2$. Then $\delta_0 = 1$.
If $\delta_2 \geq 2$ or if $\delta_2 = 1$ and $\delta_1 \neq 0, 1$, then $n - 120021$ has three digits, its last digit is $g-1$, therefore can be written as a sum of two palindromes.

If $\delta_2 = 1$ and $\delta_1 = 0$, then

1	2	0	1	0	0
1	1	$g-1$	$g-1$	1	1
			1	$g-2$	1
					$g-2$

If $\delta_2 = 1$ and $\delta_1 = 1$, then

1	2	0	1	1	0
1	1	$g-1$	$g-1$	1	1
			1	$g-1$	1
					$g-2$

If $\delta_2 = 0$ and $\delta_1 \geq 3$, then

1	2	0	0	δ_1	0
1	2	0	0	2	1
				$\delta_1 - 3$	$\delta_1 - 3$
					$g - \delta_1 + 2$

If $\delta_2 = 0$ and $\delta_1 = 2$, then

1	2	0	0	2	0
1	1	$g-1$	$g-1$	1	1
			1	0	1
					$g-2$

If $\delta_2 = 0$ and $\delta_1 = 1$, then

1	2	0	0	1	0
1	0	0	0	0	1
	2	0	0	0	2
					$g-3$

If $\delta_2 = 0$ and $\delta_1 = 0$, then

1	2	0	0	0	0
1	1	$g-1$	$g-1$	1	1
				$g-2$	$g-2$
					1

□

5. The proofs of Theorems 1.3 and 1.4

5.1. **Proof of Theorem 1.3.** To get the lower bound we argue in the following way. Let P_l be the set of palindromes with l base g digits. Its cardinality is bounded by $g^{(l+1)/2}$. Let X be large and l be that positive integer such that $2g^l \le X < 2g^{l+1}$. It is clear that for all $r \ge 1$, $|P_l + P_{l-r}|$ is a lower bound for the number of positive integers less than or equal to X which are a sum of two base g palindromes. We use the relation

$$|P_l||P_{l-r}| = \sum_{n \in P_l + P_{l-r}} r(n) \le |P_l + P_{l-r}| \max_{n \in P_l + P_{l-r}} r(n).$$

Consider the representations of n of the form $n = x + y$ with $x \in P_l$ and $y \in P_{l-r}$. Assume that $l = 2mr + t$, with $0 \le t \le 2r - 1$.

If

$$x = x_1x_2 \dots x_2x_1 \quad \text{and} \quad y = y_1y_2 \dots y_2y_1$$

are the base g representations of x and y, then we group the digits in blocks of length r from the left to the right and we get left over with a middle block of length t:

$$x = \underline{x_1 \dots x_r} \cdots \underline{x_{2r(m-1)+1} \cdots x_{2rm}}\ \underline{x_{2rm+t} \cdots x_{2rm+1}}\ \underline{x_{2rm} \cdots x_{2r(m-1)+1}} \cdots \underline{x_r \dots x_1}.$$

If $X = x_1 \dots x_r$, we define $f(X) := x_r \dots x_1$. With this notation, x and y are represented as $X_i, Y_i, f(X_i),\ f(Y_i),\ \Delta_3$ of length r, while Δ_1, Δ_2 have lengh t:

$$\begin{array}{lccccccccccc} x = & X_1 & \cdots & \cdots & \cdots & X_m & \Delta_1 & f(X_m) & \cdots & \cdots & \cdots & f(X_1) \\ y = & & f(Y_1) & \cdots & \cdots & f(Y_{m-1}) & \Delta_2 & \Delta_3 & Y_{m-1} & \cdots & \cdots & Y_1 \end{array}$$

When we sum x and y, digit by digit, in every column we could get a carry or not. Let t_i for $i = 1, \dots, 2m$ be the carries in each column and let $\bar{t} = (t_1, \dots, t_{2m})$ be the vector of carries. We denote by $r_{\bar{t}}(n)$ the number of representations of n under the form $n = x+y$ with $x \in P_l,\ y \in P_{l-r}$ with a carries vector $\bar{t}$. Clearly,

$$r(n) = \sum_{\bar{t}} r_{\bar{t}}(n).$$

As in the case of x and y, we write n with the same length of the string of digits as x.

$$\begin{array}{lccccccccccccc} n = & \delta_{2m} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \delta_0 \\ \hline \\ x = & X_1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & f(X_1) \\ y = & & f(Y_1) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & Y_1 \end{array}$$

Let us see that $X_i, Y_i, \Delta_1, \Delta_2, \Delta_3$ are all determined by δ_i and by the vector $\bar{t}$.

In fact, X_1 is determined by δ_{2m} are t_{2m}. We then put $f(X_1)$, which in turn determines Y_1. If the carry in the first column does not coincide with t_1, then $r_{\bar{t}}(n) = 0$. If it does, then we put $f(Y_1)$ in its appropriate position. We then determine X_2 using

δ_{2m-1} and t_{2m-1}. Again if the carry in the second column does not correspond with t_2, then $r_{\bar{t}}(n) = 0$; otherwise, we keep on determining X_i, Y_i and Δ_3. If one of these determinations is not compatible with the corresponding t_i's then $r_{\bar{t}}(n) = 0$. In the last step, we have to determine who is Δ_1. Since Δ_1 is a palindrome itself and has length t, there are at most g^r possibilities for it. Once we made up our mind about Δ_1, the value of Δ_2 is determined. So, $r_{\bar{t}}(n) \le g^r$ and therefore $r(n) \le 2^m g^r$.

Hence,

$$\begin{aligned} |P_l + P_{l-r}| &\ge g^{l+1-r/2} 2^{-m} g^{-r} \\ &\ge (X/2) g^{-3r/2} 2^{-m} \\ &\ge (X/2) g^{-3r/2} 2^{-l/(2r)} \\ &\ge (X/2) g^{-\frac{1}{2}(3r + \frac{l \log g}{r \log 2})}. \end{aligned}$$

Taking $r = \lfloor \sqrt{l(\log g)/(3\log 2)} \rfloor$ and using the fact that $l \sim \log X / \log g$, we get

$$|P_l + P_{l-r}| \gg X g^{-\sqrt{3l \log g/\log 2}} \gg X e^{-c\sqrt{\log X}}.$$

5.2. **Proof of Theorem 1.4.** For $g \ge 3$, it is not hard to see that then the number

$$(g-1)(g-1) * * * \cdots * 0 (g-1) \tag{5.1}$$

is not a sum of two base g palindromes. Indeed, assume that the length of the above n is $l \ge 4$ and that $x = x_{l-1-r} \cdots x_0 \ge y = y_{l-1-s} \cdots y_0$ are base g palindromes whose sum is the above n, where $r,\ s$ are nonnegative integers. Since $x_0 + y_0 \le 2g - 2$ and the last digit of n is $g - 1$, there is no carry in the last position when summing x and y in base g, so $x_0 + y_0 = g - 1$ with $1 \le x_0, y_0 \le g - 2$. If both $r > 0$ and $s > 0$ (so, the lengths of both x and y are smaller than l), then $n = x + y$ which has length l in base g should start with 1, which is not the case. If $r = 0$ but $s > 0$, then $x_0 = g - 2$ and $y_0 = 1$. Since $y_{l-2} = 1$ or 0 according to whether $s = 1$ or $s \ge 2$, respectively, and since there is a carry in the position $l - 2$ when adding x with y, we conclude that $x_{l-2} = g - 2$ or $g - 1$. But then $g + 1 \ge x_{l-2} + y_{l-2} + 1 \ge g + (g - 1) = 2g - 1$, where the last inequality follows from the fact that the digit in the position $l-2$ of n is $g-1$, and the above string of inequalities is impossible. Hence, $r = s = 0$. Now looking at x_1 and y_1, we get that $x_1 + y_1 = 0$ or g. Looking now at the left, we conclude that $x_{l-2} + y_{l-2} = x_1 + y_1 = 0$ or g, so in the position $l - 2$ of the digits of n we should have either the digit 0 or 1 according to whether there is no carry coming from the sum of digits of x and y from the position $l-3$, or of there is one such carry, respectively, and both these numbers are smaller than the corresponding digit $g - 1$ of n, which is the final contradiction.

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Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM) and
Departamento de Matemáticas, Universidad Autónoma de Madrid
28049, Madrid, España

E-mail address: franciscojavier.cilleruelo@uam.es

School of Mathematics, University of the Witwatersrand
P. O. Box Wits 2050, South Africa

E-mail address: Florian.Luca@wits.ac.za