VISIBLE LATTICE POINTS AND THE CHROMATIC ZETA FUNCTION OF A GRAPH

JAVIER CILLERUELO

Abstract. We study the probability that a random polygon of k vertices in the lattice $\{1,\dots,n\}^s$ does not contain more lattice points than the k vertices of the polygon. Then we introduce the chromatic zeta function of a graph to generalize this problem to other configurations induced by a given graph $\mathcal{H}$.

1. Introduction

Two distinct points X, Y of the s-dimensional integer lattice are said to be mutually visible if the line segment joining them contains no other lattice point. We denote this situation by $X \diamond Y$. It is well known [1] that if X, Y are lattice points taken at random uniformly in $[1,n]^s$ then $\mathbb{P}(X \diamond Y) \sim \zeta^{-1}(s)$ as $n \to \infty$, where $\zeta(s)$ is the classical Riemann zeta function. Since $X \diamond Y$ and $Y \diamond Z$ are independent events, then $\mathbb{P}(X \diamond Y \diamond Z) \sim \zeta^{-2}(s)$. What about $\mathbb{P}(X \diamond Y \diamond Z \diamond X)$? In other words, what is the probability that the three edges of a random triangle X, Y, Z contains no other lattice points than their vertices?

At first sight, we could expect that $\mathbb{P}(X \diamond Y \diamond Z \diamond X) \sim \zeta^{-3}(s)$ since apparently the three events, $X \diamond Y,\ Y \diamond Z,\ Z \diamond X$, are independent events. We prove that this intuition is not correct. In fact we obtain a more general result.

Theorem 1.1. *Let $s, k \geq 2$ positive integers. If $X^1, \dots, X^k$ are lattice points taken uniformly at random in $[1,n]^s$, we have*

$$\lim_{n\to\infty} \mathbb{P}(X^1 \diamond X^2 \diamond \cdots \diamond X^k \diamond X^1) = \zeta^{-k}(s) \prod_p \left(1 + \frac{(-1)^k}{(p^s-1)^{k-1}}\right).$$

The repetition of vertices X^i is allowed in Theorem 1.1. However, the probability of these degenerate cases tends to zero as $n \to \infty$, so we could have formulated Theorem 1.1 saying that $X^1, \dots, X^k$ are distinct lattice points.

It is interesting to note that the value of the limit in Theorem 1.1 is smaller than $\zeta^{-k}(s)$ when k is odd and greater than $\zeta^{-k}(s)$ when k is even. We do not understand the reason of this phenomenom.

The following version of Theorem 1.1 can be more illustrative. Take a lattice point X^1 at random. Then take a random lattice point X^2 visible from X^1, then take a random lattice point X^3 visible from X^2 and so on. What is the probability that X^k is visible from X^1? Corollary 1.1, which is a trivial consequence of Theorem 1.1, answers this question.

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Corollary 1.1. *Let $s, k \geq 2$ positive integers. If $X^1, \dots, X^k$ are lattice points taken uniformly at random in $[1,n]^s$ we have*

$$\lim_{n\to\infty} \mathbb{P}(X^k \diamond X^1 / X^1 \diamond X^2 \diamond \cdots \diamond X^k) = \zeta^{-1}(s) \prod_p \left(1 + \frac{(-1)^k}{(p^s-1)^{k-1}}\right).$$

Again, we see that $\mathbb{P}(X^k \diamond X^1 / X^1 \diamond X^2 \diamond \cdots \diamond X^k)$ is smaller than $\mathbb{P}(X^k \diamond X^1)$ when k is odd and greater when k is even. Theorem 1.1 can be extended to more general configurations.

Definition 1. *Given a graph $\mathcal{H}$ of order k we say that that a sequence of lattice points $(X^1, \dots, X^k)$ is $\mathcal{H}$-visible if $X^i \diamond X^j$ whenever $\{i,j\} \in E(\mathcal{H})$.*

Our main Theorem is the following.

Theorem 1.2. *Let $s, k \geq 2$ positive integers and $\mathcal{H}$ a graph of order k. If $X^1, \dots, X^k$ are lattice points taken uniformly at random in $[1,n]^s$ then we have*

$$\lim_{n\to\infty} \mathbb{P}((X^1, \cdots, X^k) \text{ is } \mathcal{H}\text{-visible}) = \zeta_{\mathcal{H}}^{-1}(s),$$

where $\zeta_{\mathcal{H}}(s)$ is the chromatic zeta function of $\mathcal{H}$ defined by

$$\zeta_{\mathcal{H}}(s) = \prod_p \left(\frac{P_{\mathcal{H}}(p^s)}{p^{ks}}\right)^{-1}$$

where $P_{\mathcal{H}}$ is the chromatic polynomial of $\mathcal{H}$.

If we consider the linear graph $\mathcal{H} = L_k$, with chromatic polynomial $P_{L_k}(x) = x(x-1)^{k-1}$, we have that $\zeta_{\mathcal{H}}(s) = \zeta^{k-1}(s)$ and we recover the classic result [1]:

$$\lim_{n\to\infty} \mathbb{P}(X^1 \diamond \cdots \diamond X^k : \ X^i \in [1,n]^s) = \zeta^{-(k-1)}(s).$$

Theorem 1.1 follows from Theorem 1.2 by taking the cycle of k vertices, $\mathcal{H} = C_k$, and observing that $P_{C_k}(x) = (x-1)^k + (-1)^k(x-1)$:

$$\zeta_{C_k}^{-1}(s) = \prod_p \left(\frac{(p^s-1)^k + (-1)^k(p^s-1)}{p^{ks}}\right) = \prod_p \left(1 - \frac{1}{p^s}\right)^k \prod_p \left(1 + \frac{(-1)^k}{(p^s-1)^{k-1}}\right).$$

David Rearick [2] considered a related problem. Given a set $S_m = \{X^1, \dots, X^m\}$ of m mutually visible lattice points, he studied the probability that a random lattice point in $[1,n]^s$ is visible from all the lattice points of S_m. He proved that

$$\lim_{n\to\infty} \mathbb{P}(X \in [1,n]^2 : \ X \diamond X^i, \ i = 1, \dots, m) = \prod_p \left(1 - \frac{m}{p^s}\right) \tag{1.1}$$

if $m < 2^s$ and 0 if $m \geq 2^s$. In particular (1.1) implies that if $m < 2^s$ and $X^1, \dots, X^{m+1}$ are taken uniformily at random in $[1,n]^s$, then

$$\lim_{n\to\infty} \mathbb{P}(X^1, \dots, X^{m+1} \text{ is } K_{m+1}\text{-visible}/X^1, \dots, X^m \text{ is } K_m\text{-visible}) = \prod_p \left(1 - \frac{m}{p^s}\right).$$

This result can be obtained easily from Theorem 1.2 considering the chromatic polynomials of the complete graphs, $P_{K_{m+1}}(x) = x(x-1)\cdots(x-m)$, $\quad P_{K_m}(x) = x(x-1)\cdots(x-m+1)$, and observing that

$$\frac{\zeta_{K_{m+1}}^{-1}(s)}{\zeta_{K_m}^{-1}(s)} = \prod_p \frac{P_{K_{m+1}}(p^s)}{p^{(m+1)s}} \prod_p \frac{p^{ms}}{P_{K_m}(p^s)} = \prod_p \frac{p^s - m}{p^s} = \prod_p \left(1 - \frac{m}{p^s}\right).$$

2. Proof of Theorem 1.2

Given two lattice points $X^i = (x_1^i, \dots, x_s^i)$ and $X^j = (x_1^j, \dots, x_s^j)$ we write $X^i \equiv X^j \pmod p$ if $x_r^i \equiv x_r^j \pmod p$ for all $r = 1, \dots, s$. We write $X^i \not\equiv X^j \pmod p$ otherwise.

Given a prime p, we say that $(X^1, \dots, X^k)$ is $\mathcal{H}_p$-visible if $X^i \not\equiv X^j \pmod p$ whenever $\{i, j\} \in E(\mathcal{H})$. The first observation is that

$$(X^1, \dots, X^k) \text{ is } \mathcal{H}\text{-visible} \iff (X^1, \dots, X^k) \text{ is } \mathcal{H}_p\text{-visible for any prime } p. \tag{2.1}$$

For any positive integer M and $n > M$ we have

$$\begin{aligned} &|\{X^1, \dots, X^k \in [1,n]^s : (X_1, \dots, x^k) \text{ is } \mathcal{H}_p\text{-visible for any } p\}| \\ = \ &|\{X^1, \dots, X^k \in [1,n]^s : (X^1, \dots, X^k) \text{ is } \mathcal{H}_p\text{-visible for any } p \le M\}| + O(|R|), \end{aligned} \tag{2.2}$$

where

$$R = \{X^1, \dots, X^k \in [1,n]^s : (X^1, \dots, X^k) \text{ is not } \mathcal{H}_p\text{-visible for some } p > M\}.$$

We split R in two sets: $R = R_1 \cup R_2$. The set R_1 contains those $(X^1, \dots, X^k)$ with $X^i = X^j$ for some $i \neq j$ and R_2 contains those with all X^i distinct.

Clearly,

$$|R_1| \le \binom{k}{2} n^{s(k-1)}. \tag{2.3}$$

On the other hand we observe that if $X^i \neq X^j$ then $X^i \not\equiv X^j \pmod p$ for $p \ge n$, so $(X^1, \dots, X^k)$ is always $\mathcal{H}_p$-visible when $p \ge n$ for those $(X^1, \dots, X^k)$ counted in R_2. Indeed, for a fixed $X^i = (x_1^i, \dots, x_s^i)$ the number of $X^j = (x_1^j, \dots, x_s^j) \in [1,n]^s$ such that $X^j \equiv X^i \pmod p$ is $(n/p + O(1))^s \ll n^s/p^s$ for $p < n$. Thus,

$$\begin{aligned} |R_2| &\le \sum_{M<p<n} |\{\text{distinct } X^1, \dots, X^k \in [1,n]^s : (X^1, \dots, X^k) \text{ is not } \mathcal{H}_p\text{-visible }\}| \\ &\le \sum_{M<p<n} |\{\text{distinct } X^1, \dots, X^k \in [1,n]^s : X^i \equiv X^j \pmod p \text{ for some } i \neq j\}\}| \\ &\le \sum_{M<p<n} \binom{k}{2} |\{\text{distinct } X^1, \dots, X^k \in [1,n]^s : X^1 \equiv X^2 \pmod p)\}\}| \\ &\ll \sum_{M<p<n} \frac{n^{ks}}{p^s} \end{aligned}$$

and we get the upper bound

$$|R_2| \ll n^{ks} M^{1-s}. \tag{2.4}$$

By (2.2), (2.3) and (2.4) we have

$$\begin{aligned} &|\{X^1, \dots, X^k \in [1,n]^s : (X^1, \dots, X^k) \text{ is } \mathcal{H}\text{-visible }\}| \\ = \ &|\{X^1, \dots, X^k \in [1,n]^s : (X^1, \dots, X^k) \text{ is } \mathcal{H}_p\text{-visible for any } p \le M\}| \\ &+O(n^{s(k-1)}) + O(n^{ks} M^{1-s}). \end{aligned} \tag{2.5}$$

The next step is to estimate the quantity

$$|\{X^1, \dots, X^k \in [1,n]^s : (X^1, \dots, X^k) \text{ is } \mathcal{H}_p\text{-visible for any } p \le M\}|. \tag{2.6}$$

A good coloration of a labeled graph $\mathcal{H}$ is an assigment of colours to the vertices such that two adjacents vertices do not share the same colour. The polynomal chromatic $P_{\mathcal{H}}(x)$ counts the number of good colorations of $\mathcal{H}$ using x colours.

For each p we assign to each vertex $X = (x_1, \dots, x_s)$ the p-colour $c_p(X)$ defined as the only vector $c_p(X) \in [0, p-1]^s$ such that $c_p(X) \equiv X \pmod p$.

We observe that $(X^1 \dots, X^k)$ is $\mathcal{H}_p$-visible if and only if there exists a good p-coloration $C_p = (c_p^1, \dots, c_p^k)$ of $\mathcal{H}$ such that $(c_p(X^1), \dots, c_p(X^k)) = C_p$.

Thus, $(X^1 \dots, X^k)$ is $\mathcal{H}_p$-visible for any $p \le M$ if and only if there exists a sequence of good colorations $(C_p)_{p \le M}$ such that $(c_p(X^1), \dots, c_p(X^k)) = C_p$ for all $p \le M$.

Since for each prime p there are p^s colours, the number of good p-colorations of $\mathcal{H}$ is $P_{\mathcal{H}}(p^s)$, where $P_{\mathcal{H}}$ is the chromatic polynomial of $\mathcal{H}$. Therefore, the number of sequences of good colorations $(C_p)_{p \le M}$ is

$$\prod_{p \le M} P_{\mathcal{H}}(p^s). \tag{2.7}$$

Thus we have

$$\begin{aligned} &|\{X^1, \dots, X^k \in [1,n]^s : \ (X^1, \dots, X^k) \text{ is } \mathcal{H}_p\text{-visible for any } p \le M\}| \\ = \ &\sum{}^{*} |\{X^1, \dots, X^k \in [1,n]^s : \ (c_p(X^1), \dots, c_p(X^k)) = C_p, \ p \le M\}| \end{aligned} \tag{2.8}$$

where the sum $\sum^*$ is extended over all sequences of good colorations $(C_p)_{p \le M}$ of the graph $\mathcal{H}$.

Given a sequence of colorations $(C_p)_{p \le M} = (c_p^1, \dots, c_p^k)_{p \le M}$ we have that

$$\begin{aligned} &|\{X^1, \dots, X^k \in [1,n]^s : \ c_p(X^i) = c_p^i, \ i = 1, \dots, k, \ \text{for all } p \le M\}| \\ = \ &\prod_{i=1}^{k} |\{X \in [1,n]^s : \ c_p(X) = c_p^i, \ \text{for all } p \le M\}|. \end{aligned} \tag{2.9}$$

Given the vectors $c_p^i = (c_{p1}^i, \dots, c_{ps}^i), \ p \le M$, the lattice points $X = (x_1, \dots, x_s)$ with $c_p(X) = c_p^i$ for all $p \le M$ will be those such that the congruences $x_r \equiv c_{pr}^i \pmod p$, $p \le M$ hold for any $r = 1, \dots, s$. By the Chinese Remainder Theorem these congruences are equivalent, for each $r = 1, \dots, s$, to the congruence $x_r \equiv a_r \pmod{\prod_{p \le M} p}$ for some a_r. The number of $x_r \le n$ satisfying each congruence is $\frac{n}{\prod_{p \le M} p} + O(1)$, so the number of $X \in [1,n]^s$ with $c_p(X) = c_p^i$ for all $p \le M$ is

$$\left(\frac{n}{\prod_{p \le M} p} + O(1) \right)^s.$$

Since this estimate does not depend on the values of c_p^i we have

$$\prod_{i=1}^{k} |\{X \in [1,n]^s : \ c_p(X) = c_p^i \text{ for all } p \le M\}| = \left(\frac{n}{\prod_{p \le M} p} + O(1) \right)^{sk}. \tag{2.10}$$

Summing up, as consequence of (2.8), (2.9), (2.10) and (2.7) we obtain

$$\begin{aligned} &|\{X^1, \dots, X^k \in [1,n]^s : \ (X^1, \dots, X^k) \text{ is } \mathcal{H}_p\text{-visible for any } p \le M\}| \\ = \ &\left(\frac{n}{\prod_{p \le M} p} + O(1) \right)^{sk} \times |\{\text{sequences of good colorations } (c_p^1, \dots, c_p^k), \ p \le M\}| \\ = \ &\left(\frac{n}{\prod_{p \le M} p} + O(1) \right)^{sk} \prod_{p \le M} P_{\mathcal{H}}(p^s) = n^{sk} \left(\prod_{p \le M} \frac{P_{\mathcal{H}}(p^s)}{p^{sk}} \right) \left(1 + O\left(\frac{\prod_{p \le M} p}{n} \right) \right)^{sk}. \end{aligned}$$

In terms of probability we have proved that

$$\begin{aligned}&\mathbb{P}\left(\{X^1,\dots,X^k\in[1,n]^s:\ (X^1,\dots,X^k)\text{ is }\mathcal{H}_p\text{-visible for any }p\le M\}\right)\\ =\ &\left(\prod_{p\le M}\frac{P_{\mathcal{H}}(p^s)}{p^{sk}}\right)\left(1+O\left(\frac{\prod_{p\le M}p}{n}\right)\right)^{sk}.\end{aligned}$$

Using (2.5) we have that

$$\begin{aligned}&\mathbb{P}\left(\{X^1,\dots,X^k\in[1,n]^s:\ (X^1,\dots,X^k)\text{ is }\mathcal{H}\text{-visible }\}\right)\\ =\ &\left(\prod_{p\le M}\frac{P_{\mathcal{H}}(p^s)}{p^{sk}}\right)\left(1+O\left(\frac{\prod_{p\le M}p}{n}\right)\right)^{sk}+O(n^{-s})+O(M^{1-s}).\end{aligned}$$

Taking the limit as $n\to\infty$ we get

$$\lim_{n\to\infty}\mathbb{P}\left(\{X^1,\dots,X^k\in[1,n]^s:\ ()X^1,\dots,X^k)\text{ is }\mathcal{H}\text{-visible }\}\right)=\prod_{p\le M}\frac{P_{\mathcal{H}}(p^s)}{p^{sk}}+O(M^{1-s}).$$

Finally, taking the limit as $M\to\infty$ we have

$$\lim_{n\to\infty}\mathbb{P}\left(\{X^1,\dots,X^k\in[1,n]^s:\ (X^1,\dots,X^k)\text{ is }\mathcal{H}\text{-visible }\}\right)=\prod_{p}\frac{P_{\mathcal{H}}(p^s)}{p^{sk}}=\zeta_{\mathcal{H}}^{-1}(s).$$

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Instituto de Ciencias Matemáticas (ICMAT) and Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain