$B_2[g]$ SETS AND A CONJECTURE OF SCHINZEL AND SCHMIDT

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ABSTRACT. We obtain a new lower bound for $F(g,n)$, the largest cardinality of a $B_2[g]$ set in $\{1,\dots,n\}$. More precisely we prove that $\liminf_{n\to\infty}\frac{F(g,n)}{\sqrt{gn}} \ge \frac{2}{\sqrt{\pi}} - \varepsilon_g$ where $\varepsilon_g \to 0$ when $g\to\infty$. We also relate this problem to a kind of continuous version introduced by Schinzel and Schmidt

1. INTRODUCTION

A set of integers $\mathcal{A}$ is called a $B_2[g]$ set if every integer n has at most g representations $n = a + a'$, with $a \le a'$ and $a, a' \in \mathcal{A}$. We write $r_{\mathcal{A}}(n)$ for the number of such representations.

A major problem in additive number theory is the study of the behaviour of $F(g,n)$, the largest cardinality of a $B_2[g]$ set in $\{1,\dots,n\}$.

It is a well known result on Sidon sets that $F(1,n) \sim n^{1/2}$, but the asymptotic behavior of $F(g,n)$ is an open problem for $g \ge 2$. The trivial counting argument gives $F(g,n) \le 2\sqrt{gn}$ and it is not too difficult to show (see section 2) that $F(g,n) \gtrsim \sqrt{gn}$.

Then, we define

$$\beta(g) = \liminf_{n\to\infty} \frac{F(g,n)}{\sqrt{gn}} \le \limsup_{n\to\infty} \frac{F(g,n)}{\sqrt{gn}} = \alpha(g).$$

In the last years some progress has been done, improving the easier estimates $1 \le \beta(g) \le \alpha(g) \le 2$. We list below the successive results obtained by several authors including the improvement obtained in this work.

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$\alpha(g)$ ≤ 2 (trivial)
≤ 1.864 (J. Cilleruelo - I. Ruzsa - C. Trujillo, [1])
≤ 1.844 (B. Green, [2])
≤ 1.839 (G. Martin - K. O'Bryant, [5])
≤ 1.789 (G. Yu, [9])

$\beta(g)$ ≥ 1 (M. Kolountzakis, [3])
$\gtrsim 1.060$ (J. Cilleruelo - I. Ruzsa - C. Trujillo, [1])
$\gtrsim 1.122$ (G. Martin - K. O'Bryant, [4])
$\gtrsim 2/\sqrt{\pi} = 1.128...$ (Theorem 1.2)

The aim of this work is not only to provide an improvement on the lower bound for $\beta(g)$ but also to relate this problem with the one posed by Schinzel and Schmidt [7] which can be seen as the continuous version of this problem.

We define the Schinzel-Schmidt's constant S as the number

$$S = \sup_{f \in \mathcal{F}} \frac{1}{|f * f|_\infty} \tag{1}$$

where $f * f(x) = \int f(t)f(x-t)\ dt$ and $\mathcal{F} = \{f : \ f \geq 0,\ \mathrm{sop}(f) \subseteq [0,1],\ |f|_1 = 1\}$. We use the notation $|g|_1 = \int_0^1 g(x)\ dx$ and $|g|_\infty = \sup_x g(x)$.

Remark 1.1. *In fact they define* $S = \sup_{f \in \widetilde{\mathcal{F}}} |f|_1^2/|f*f|_\infty$ *with* $\widetilde{\mathcal{F}} = \{f : f \geq 0,\ f \not\equiv 0,\ sop(f) \subseteq [0,1],\ f \in L_1[0,1]\}$*, but we can assume that* $|f|_1 = 1$ *because* $|f|_1^2/|f*f|_\infty$ *is invariant under dilates of* f*.*

It is easy to see that $1 \leq S \leq 2$ but Schinzel and Schmidt proved in [7] that $4/\pi \leq S \leq 1.7373$. The witness for the lower found is the function $f(x) = \frac{1}{2\sqrt{x}} \in \mathcal{F}$. Indeed they conjecture that $S = 4/\pi$. Our main theorem relate $\alpha(g)$ and $\beta(g)$ to S.

Theorem 1. $\sqrt{S} \leq \liminf_{g\to\infty} \beta(g) \leq \limsup_{g\to\infty} \alpha(g) \leq \sqrt{2S}$.

Corollary 1.2. $\beta(g) \geq 2/\sqrt{\pi} - \varepsilon_g$*, where* $\varepsilon_g \to 0$ *when* $g \to \infty$*.*

We conjecture that $\lim_{g\to\infty} \beta(g) = \lim_{g\to\infty} \alpha(g) = 2/\sqrt{\pi}$.

2. Constructions for the lower bounds

It is convenient to introduce the following definitions:

Definition 1. *We say that* $\mathcal{A}$ *is a* $B_2^*[g]$ *set if any integer* n *has at most* g *representations* $n = a + a'$ *with* $a, a' \in \mathcal{A}$*. We write* $r_{\mathcal{A}}^*(n)$ *for the number of such representations.*

Definition 2. *We say that* $\mathcal{A}$ *is a Sidon set* $\pmod m$ *if* $a+a' \equiv a''+a'''$ $\pmod m \implies \{a,a'\}=\{a'',a'''\}$.

All the lower bounds for $\beta(g)$ are obtained from the next lemma (see [1]).

Lemma 1. *If* $\mathcal{A}=\{0=a_1<\ldots<a_k\}$ *is a* $B_2^*[g]$ *set and* $\mathcal{C}\subseteq[1,m]$ *is a Sidon set* $\pmod m$*, then* $\mathcal{B}=\cup_{i=1}^k(\mathcal{C}+ma_i)$ *is a* $B_2[g]$ *set in* $[1,m(a_k+1)]$ *with* $k|\mathcal{C}|$ *elements.*

Remark 2.1. *The lemma says that the way of obtaining* $B_2[g]$ *sets is "pasting properly" (with a dilation of a* $B_2^*[g]$ *set) copies of a Sidon set* $\pmod m$.

Proof. To prove that B is a $B_2[g]$ set, suppose that we have

$$b_{1,1}+b_{2,1}=\cdots=b_{1,g+1}+b_{2,g+1} \tag{2}$$

for some $b_{1,j},b_{2,j}\in\mathcal{B}$. We can write each $b_{i,j}=c_{i,j}+ma_{i,j}$ in only one way with $c_{i,j}\in\mathcal{C}$ and $a_{i,j}\in\mathcal{A}$. Let us order the elements $b_{i,j}$ of each sum in such a way that for any i,j we have $c_{1,j}\le c_{2,j}$, and when $c_{1,j}=c_{2,j}$ we order them so $a_{1,j}\le a_{2,j}$.

To see that $\mathcal{B}\in B_2[g]$ we have to see that there exist j and j' such that $b_{1,j}=b_{1,j'},\ b_{2,j}=b_{2,j'}$.

Considering the equalities (2) $\pmod m$ and because $\mathcal{C}$ is a Sidon set $\pmod m$ we obtain that $\{c_{1,1},c_{2,1}\}=\{c_{1,j},c_{2,j}\}$ for every $1\le j\le g+1$. Moreover, since we ordered the elements of the equalities in that way, we have $c_{1,1}=c_{1,j}$ and $c_{2,1}=c_{2,j}$ for every j.

Then, the equalities (2) imply these other equalities

$$a_{1,1}+a_{2,1}=a_{1,2}+a_{2,2}=\cdots=a_{1,g+1}+a_{2,g+1}. \tag{3}$$

And since $\mathcal{A}$ satisfies the $B_2^*[g]$ condition there exist j and j' such that $a_{1,j}=a_{1,j'}$ and $a_{2,j}=a_{2,j'}$.

Then, for these j and j' we have that $b_{1,j}=b_{1,j'}$ and $b_{2,j}=b_{2,j'}$. This proves that $\mathcal{B}\in B_2[g]$.

Finally, it is clear that $B\subset[1,\ldots,(a_k+1)m]$ and $|\mathcal{B}|=k|\mathcal{C}|$. $\square$

In order to apply lemma above in an efficient way, we have to take dense Sidon sets $\pmod m$. For example, for each prime p we consider $\mathcal{C}_p$ the Sidon set $\pmod m$ with $p-1$ elements and $m=p(p-1)$ discovered by Ruzsa (see [6]).

Given N, we write $(a_k+1)p_n(p_n-1)<N\le(a_k+1)p_{n+1}(p_{n+1}-1)$ for suitable consecutive primes p_n,p_{n+1}. Clearly

$$\frac{F(g,N)}{\sqrt{gN}}\ge\frac{|\mathcal{C}_{p_n}|k}{\sqrt{g(a_k+1)p_{n+1}(p_{n+1}-1)}}\ge\frac{k}{\sqrt{g(a_k+1)}}\cdot\frac{p_n-1}{p_{n+1}}.$$

Thus

$$\beta(g)=\liminf_{N\to\infty}\frac{F(g,N)}{\sqrt{gN}}\geq\frac{k}{\sqrt{g(a_k+1)}}\liminf_{n\to\infty}\frac{p_n-1}{p_{n+1}}.$$

Since $\liminf_{n\to\infty}\frac{p_n}{p_{n+1}}=1$ as a consequence of the prime number theorem, we get

$$\beta(g)\geq\frac{k}{\sqrt{g(a_k+1)}}. \tag{4}$$

So, in order to improve the lower bound for $\beta(g)$, we are looking for $\mathcal{A}=\{0=a_1<\ldots<a_k\}$ which satisfies the $B_2^*[g]$ condition and maximizes the quotient $\frac{k}{\sqrt{g(a_k+1)}}$.

The sets

(a) $\mathcal{A}=\{0,1,\ldots,g-1\}$
(b) $\mathcal{A}=\{0,1,\ldots,g-1\}\cup\{g+1,g+3,\ldots,g-1+2\lfloor g/2\rfloor\}$
(c) $\mathcal{A}=[0,\lfloor g/3\rfloor)\cup(g-\lfloor g/3\rfloor+2\cdot[0,\lfloor g/6\rfloor))$
$\cup[g,g+\lfloor g/3\rfloor)\cup(2g-\lfloor g/3\rfloor,3g-\lfloor g/3\rfloor]$

provide, respectively, the lower bounds

(a) $\beta(g)\geq 1$
(b) $\beta(g)\geq\frac{g+\lfloor g/2\rfloor}{\sqrt{g^2+2g\lfloor g/2\rfloor}}\geq\sqrt{\frac{9}{8}}-\varepsilon_g=1.060\ldots-\varepsilon_g$
(c) $\beta(g)\geq\frac{g+2\lfloor\frac{g}{3}\rfloor+\lfloor\frac{g}{6}\rfloor}{\sqrt{3g^2-g\lfloor\frac{g}{3}\rfloor+g}}\geq\sqrt{\frac{121}{96}}-\varepsilon_g=1.122\ldots-\varepsilon_g,$

cited in the introduction.

In the next section we will find a denser set $\mathcal{A}$.

3. Schinzel's conjecture

The convolution $f*f$ in the Schinzel-Schmidt's problem can be thought as the continuous version of the function $r^*_{\mathcal{A}}(n)$ and $|f*f|_\infty$ as the analogous of the maximum of $r^*_{\mathcal{A}}(n)$.

The idea is to take a function $f\in\mathcal{F}$ such that $1/|f*f|_\infty$ is close to S (see definition in formula (1)) and use f as a model to construct our set $\mathcal{A}$. We will do it using the probabilistic method.

An interesting result in [7] relates the constant S with the coefficients of squares of polynomials. We state that result in a more convenient way for our purposes.

Theorem 2. *For any $\varepsilon>0$, for any $n>n(\varepsilon)$, there exists a sequence of non negative real numbers $c_0,\ldots,c_{n-1}$ such that*

i) $\sum_{j=0}^{n-1}c_j=\sqrt{n}$.

ii) $c_j \le n^{-1/6}(1+\varepsilon)$ *for all* $j = 0, \dots, n-1$.
iii) $\sum_{j<m/2} c_j c_{m-j} \le \frac{1}{2S}(1+\varepsilon)$ *for any* $m = 0, \dots, n-1$.

Proof. We follow the ideas of the proof of assertion (iii) of theorem 1 in [7]. Let $f \in \mathcal{F}$ with $|f * f|_\infty$ close to $1/S$, say $|f * f|_\infty \le 1/S + 1/n$, and define for $j = 0, \dots, n-1$,

$$a_j = \frac{n}{2t} \int_{(j+1/2-t)/n}^{(j+1/2+t)/n} f(x)\ dx$$

where $t = \lceil 2n^{1/3} \rceil$. We have the following estimate

$$\begin{aligned}\left(\int_r^s f(x)\ dx\right)^2 &\le \iint\limits_{2r \le x+y \le 2s} f(x)f(y)\ dxdy \\ &= \int_{2r}^{2s} \left(\int f(x)f(z-x)\ dx\right)\ dz \\ &= \int_{2r}^{2s} f * f(z)\ dz \le 2(s-r)(1/S + 1/n) \le 4(s-r),\end{aligned}$$

where in the last inequality we have used the fact that $S \ge 1$ and $n \ge 1$.

In particular, we can deduce $a_j \le (2n/t)^{1/2}$. The core of the proof of theorem 1 (iii) in [7] consists of showing that $\sum_{j=0}^{n-1} a_j \ge n + o(n)$ and $\sum_{j=0}^{m} a_j a_{m-j} \le (1/S)(n + o(n))$ for all m. The details can be checked there.

Now we define $c_j = a_j \rho$ with $\rho = \frac{\sqrt{n}}{\sum_{j=0}^{n-1} a_j}$. Clearly $\rho \le (1/\sqrt{n})(1+o(1))$, so $c_j \le n^{-1/6}(1+o(1))$, $\sum_{j=0}^{n-1} c_j = \sqrt{n}$ and $\sum_{j=0}^{m} c_j c_{m-j} \le (1/S)(1+o(1))$. □

4. The proof

We will use in the proof an special case of Chernoff's inequality (see corollary 1.9 in [8]):

Proposition 4.1. *(Chernoff's inequality) Let* $X = t_1 + \cdots + t_n$ *where the* t_i *are independent boolean random variables. Then for any* $\delta > 0$

$$\mathbb{P}(|X - \mathbb{E}(X)| \ge \delta \mathbb{E}(X)) \le 2e^{-\min(\delta^2/4, \delta/2)\mathbb{E}(X)}. \tag{5}$$

Given $\varepsilon > 0$ and the c_j's defined in theorem 2, we consider the probability space of all the subsets $\mathcal{A} \subseteq \{0, 1, 2, \dots, n-1\}$ defined by $\mathbb{P}(j \in \mathcal{A}) = \lambda_n c_j$, where $\lambda_n = \lfloor n^{1/6}/(1+\varepsilon) \rfloor$ (observe that $c_j \lambda_n \le 1$ for n large enough).

Lemma 2. *With the conditions above, given $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$*

$$\mathbb{P}\big(|\mathcal{A}| \geq \lambda_n \sqrt{n}(1-\varepsilon)\big) > 0.9.$$

Proof. Since $|\mathcal{A}|$ is a sum of independent boolean variables and $\mathbb{E}(|\mathcal{A}|) = \sum_{j=0}^{n-1} \mathbb{P}(j \in \mathcal{A}) = \lambda_n \sqrt{n}$ we can apply Chernoff's lemma to deduce that

$$\mathbb{P}\Big(|\mathcal{A}| < \lambda_n \sqrt{n}(1-\varepsilon)\Big) \leq 2e^{-\min(\varepsilon^2/4, \varepsilon/2)\lambda_n\sqrt{n}} < 0.1$$

for n large enough. □

Lemma 3. *Again with the same conditions, given $0 < \varepsilon < 1$, there exists n_1 such that for all $n \geq n_1$*

$$r_{\mathcal{A}}^*(m) \leq \frac{\lambda_n^2}{S}(1+\varepsilon)^3 \quad \textit{for all } m$$

with probability > 0.9.

Proof. Since $r_{\mathcal{A}}^*(m) = \sum_{j=0}^{m} \mathbb{I}(j \in \mathcal{A})\mathbb{I}(m-j \in \mathcal{A})$ is a sum of boolean variables which are not independent, its convenient to define a new variable ${r_{\mathcal{A}}^*}'(m) = \frac{1}{2}r_{\mathcal{A}}^*(m) - \frac{1}{2}\mathbb{I}(m/2 \in \mathcal{A}) = \sum_{j<m/2} \mathbb{I}(j \in \mathcal{A})\mathbb{I}(m-j \in \mathcal{A})$. Now we can apply Chernoff's inequality to this variable.

We write μ_m for the expected value of ${r_{\mathcal{A}}^*}'(m)$. We observe that, from the independence of the indicator functions, $\mathbb{E}\big(\mathbb{I}(j \in \mathcal{A})\mathbb{I}(m-j \in \mathcal{A})\big) = \mathbb{P}(j \in \mathcal{A})\mathbb{P}(m-j \in \mathcal{A}) = \lambda_n^2 c_j c_{m-j}$ for every $j < m/2$ and so

$$\mu_m = \sum_{j<m/2} \mathbb{E}\big(\mathbb{I}(j \in \mathcal{A})\mathbb{I}(m-j \in \mathcal{A})\big) = \sum_{j<m/2} \lambda_n^2 c_j c_{m-j} \leq \frac{\lambda_n^2}{2S}(1+\varepsilon),$$

by theorem 2 iii).

- If $\mu_m \geq \frac{\lambda_n^2}{6S}(1+\varepsilon)$, we apply proposition 4.1 (observe that $\varepsilon < 2$ implies that $\varepsilon^2/4 \leq \varepsilon/2$) to obtain

$$\begin{aligned}
\mathbb{P}\left({r_{\mathcal{A}}^*}'(m) \geq \frac{\lambda_n^2}{2S}(1+\varepsilon)^2\right) &\leq \mathbb{P}\big({r_{\mathcal{A}}^*}'(m) \geq \mu_m(1+\varepsilon)\big) \\
&\leq 2\exp\left(-\frac{\mu_m \varepsilon^2}{4}\right) \\
&\leq 2\exp\left(-\frac{\lambda_n^2}{24S}(1+\varepsilon)\varepsilon^2\right).
\end{aligned}$$

- If $\mu_m = 0$ then ${r_{\mathcal{A}}^*}'(m) = 0$.

- If $0 < \mu_m < \frac{\lambda_n^2}{6S}(1+\varepsilon)$, for $\delta = \frac{\lambda_n^2}{\mu_m 2S}(1+\varepsilon)^2 - 1 \geq 2$ (now $\delta/2 \leq \delta^2/4$) we obtain

$$\begin{aligned}\mathbb{P}\left(r_{\mathcal{A}}^{*\prime}(m) \geq \frac{\lambda_n^2}{2S}(1+\varepsilon)^2\right) &= \mathbb{P}\big(r_{\mathcal{A}}^{*\prime}(m) \geq \mu_m(1+\delta)\big)\\ &\leq 2\exp(-\delta\mu_m/2)\\ &\leq 2\exp\left(-\frac{\lambda_n^2}{4S}(1+\varepsilon)^2 + \frac{\mu_m}{2}\right)\\ &\leq 2\exp\left(-\frac{\lambda_n^2}{4S}(1+\varepsilon)^2 + \frac{\lambda_n^2}{12S}(1+\varepsilon)\right)\\ &\leq 2\exp\left(-\frac{\lambda_n^2}{6S}(1+\varepsilon)^2\right).\end{aligned}$$

Then

$$\begin{aligned}&\mathbb{P}\left(r_{\mathcal{A}}^{*\prime}(m) \geq \frac{\lambda_n^2}{2S}(1+\varepsilon)^2 \text{ for some } m\right)\\ &\leq 2n\left(\exp\left(-\frac{\lambda_n^2}{24S}(1+\varepsilon)\varepsilon^2\right) + \exp\left(-\frac{\lambda_n^2}{6S}(1+\varepsilon)^2\right)\right) < 0.1\end{aligned}$$

for n large enough.

Because of the way we defined $r_{\mathcal{A}}^{*\prime}(m)$, this means

$$\mathbb{P}\left(r_{\mathcal{A}}^{*}(m) \geq \frac{\lambda_n^2}{S}(1+\varepsilon)^2 + \mathbb{I}(m/2 \in \mathcal{A}) \text{ for some } m\right) < 0.1.$$

So, finally,

$$\mathbb{P}\left(r_{\mathcal{A}}^{*}(m) \geq \frac{\lambda_n^2}{S}(1+\varepsilon)^3 \text{ for some } m\right) < 0.1$$

for n large enough.

□

Lemmas 1 and 2 imply that for any $0 < \varepsilon < 1$, for $n \geq n(\varepsilon) = \max(n_0, n_1)$ the probability that $|\mathcal{A}| \geq \lambda_n\sqrt{n}(1-\varepsilon)$ and $r_{\mathcal{A}}^*(m) \leq \frac{\lambda_n^2}{S}(1+\varepsilon)^3$ for all m is greater than 0.8. Finally we will consider any of these sets $\mathcal{A} \subset \{0, \ldots, n-1\}$ for a suitable n.

Write $g_\varepsilon = \lfloor \frac{\lambda_{n(\varepsilon)}^2}{S}(1+\varepsilon)^3 \rfloor$. For any $g \geq g_\varepsilon$ we take n such that $g = \lfloor \frac{\lambda_n^2}{S}(1+\varepsilon)^3 \rfloor$ (this is possible because $\frac{\lambda_n^2}{S}(1+\varepsilon)^3$ grows slower than n). Thus, for $g \geq g_\varepsilon$,

$$\beta(g) \geq \frac{|\mathcal{A}|}{g^{1/2}n^{1/2}} \geq \frac{\lambda_n\sqrt{n}(1-\varepsilon)}{(\lambda_n/\sqrt{S})(1+\varepsilon)^{3/2}n^{1/2}} = \sqrt{S}\frac{1-\varepsilon}{(1+\varepsilon)^{3/2}}$$

which completes the proof of the left inequality of theorem 1 since we can take ε arbitrary small.

For the right inequality of theorem 1, we can use the next theorem (assertion (ii) of theorem 1 in [7]):

Theorem 3. *Let S be the Schinzel-Schmidt's constant and $\mathcal{Q} = \{Q : Q \in \mathbb{R}_{\geq 0}[x],\ Q \not\equiv 0,\ \deg(Q) < n\}$. Then*

$$\frac{1}{n} \sup_{Q \in \mathcal{Q}} \frac{|Q^2(x)|_1}{|Q^2(x)|_\infty} \leq S,$$

where $|P|_1$ is the sum and $|P|_\infty$ the maximum of the coefficients of a polynomial, P.

Given a $B_2[g]$ set, $\mathcal{A} \subseteq \{0, \ldots, n-1\}$, we define the polynomial $Q_{\mathcal{A}}(x) = \sum_{a \in \mathcal{A}} x^a$, so $Q_{\mathcal{A}}^2(x) = \sum_n r_{\mathcal{A}}^*(n) x^n$. The theorem says that, in particular,

$$S \geq \frac{1}{n} \sup_{\mathcal{A} \subseteq \{0, \ldots, n-1\}} \frac{|\mathcal{A}|^2}{2g} = \frac{F^2(g,n)}{2gn},$$

and so $\dfrac{F(g,n)}{\sqrt{gn}} \leq \sqrt{2S}$.

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