INTEGRABILITY, GROWTH OF CONFORMAL MAPS, AND SUPERPOSITION OPERATORS

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ABSTRACT. We review some recent developments that relate the following questions: (1) the geometry of image domains and growth of univalent functions in classical spaces of analytic functions and (2) the characterization of nonlinear superposition operators between such spaces.

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INTRODUCTION

This brief set of lecture notes is an expanded version of the talk “Integrability of derivatives, geometry of domains, and growth of analytic functions in some classical spaces” given at the international two-day conference
“Complex, Harmonic and Functional Analysis” held in Thessaloniki on December 12th and 13th of 2003. It summarizes the results from several recent papers, including our works with several coauthors [BFV], [DGV], and [AMV], as well as some obtained by other authors.

The main focus is on some classical spaces of analytic functions such as the Bergman spaces $A^p$ and analytic Besov spaces $B^p$ (including the Dirichlet and Bloch spaces). The growth and integrability properties of univalent functions in such spaces are related to the images of the unit disk by these functions. This is used to characterize the (nonlinear) superposition operators between such spaces. Thus, the interplay between the simple geometric properties of certain simply connected domains on the one hand and the growth properties of the corresponding Riemann maps on the other hand, is the common thread and the main theme of this entire research project.

The results due to other authors and those that are considered standard knowledge are denoted by capital block letters, while the results in whose original proof this author has participated (and which have already been published elsewhere, or will appear soon) are numbered. Throughout the text we omit many technical details while referring the reader to the original papers. Instead, we stress the ideas behind the concepts involved. However, we do provide several minor or relatively simple details that are typically left out in condensed research papers, as well as some that were not commented in these papers. It is, thus, our hope that this survey will be suitable for a graduate student or a non-specialist in these questions in order to initiate them into this specific topic.

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1. Background

In this section we review the basic concepts and collect the essential facts that will be needed later. We begin by fixing some notation.

1.1. Notation. Throughout the paper, $\mathbb{C}$ will denote the complex plane and $\mathbb{D}$ the unit disk: $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$. We will write $\mathcal{H}(\mathbb{D})$ for the
algebra of all analytic (holomorphic) functions in $\mathbb{D}$. The notation $K \in \mathbb{D}$ will mean that $K$ is a compact subset of the unit disk.

We will often need estimates on the distance of a point $w$ in a domain $\Omega$ to the boundary $\partial \Omega$. Let us agree to write

$$d_{\Omega}(w) = \text{dist}(w, \partial \Omega) = \min\{|w - z| : z \in \partial \Omega\}.$$

Similarly, for a compact subset $Q$ of $\Omega$ (e.g., a closed square), let us also write $d_{\Omega}(Q)$ to denote the distance from the set $Q$ to $\partial \Omega$.

It will often be needed to compare two quantities asymptotically (without being concerned about the exact value of the constants). For two positive functions $u$ and $v$ we will use the notation $u \approx v$ to denote that $m u \leq b \leq M u$ for some fixed positive constants $m$ and $M$. Similarly, the notation $u \lesssim v$ will mean that $u \leq C v$ for some positive constant $C$.

1.2. Bergman spaces. Let $dA$ denote Lebesgue area measure in the unit disk $\mathbb{D}$, normalized so that $A(\mathbb{D}) = 1$: $dA(z) = \pi^{-1} r dr d\theta$. If $0 < p < \infty$, the Bergman space $A^p$ is the set of all analytic functions $f$ in the unit disk $\mathbb{D}$ with finite $L^p(\mathbb{D}, dA)$ norm:

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p dA(z) = 2 \int_0^1 M_p^p(r, f) r dr < \infty,$$

where

$$M_p(r, f) = \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}$$

are the standard integral means over the circle of radius $r$ centered at the origin.

Note that $\|f\|_{A^p}$ is a true norm if and only if $1 \leq p < \infty$ and, in this case, $A^p$ is a Banach space. When $0 < p < 1$, $A^p$ is still a complete space with respect to the translation-invariant metric defined by $d_p(f, g) = \|f - g\|_{A^p}$.

In what follows, we will essentially need only two basic facts about Bergman space functions. The first one is a standard example of a family of functions in such spaces.

**Lemma A.** The function $f_c$ given by $f_c(z) = (1 - z)^{-c}$ belongs to $A^p$ if and only if $cp < 2$.

The statement is easily verified by integrating in polar coordinates centered at the point $z = 1$ rather than at the origin.

Another relevant fact is the growth of the functions in Bergman spaces. There is a more precise statement but for our purpose it suffices to know just the correct maximum order of growth.

**Lemma B.** For every $f$ in $A^p$ and every $z$ in $\mathbb{D}$, we have

$$|f(z)| \leq \frac{\|f\|_{A^p}}{|1 - |z||^{2/p}}.$$
The proof follows in a straightforward manner by applying the sub-mean value property to the subharmonic function $|f|^p$ on a smaller disk of radius $1 - |z|$ centered at $z$.

A detailed account of the theory of Bergman spaces can be found in the recent books by Hedenmalm, Korenblum, and Zhu [HKZ] and by Duren and Schuster [DS], as well as in various earlier texts and survey papers.

1.3. **Univalent functions.** A function $f$ is *univalent* in $\mathbb{D}$ if $f \in \mathcal{H}(\mathbb{D})$ and is one-to-one. We will denote by $\mathcal{U}$ the set of all such functions. Clearly, whenever $f \in \mathcal{U}$, the domain $f(\mathbb{D})$ is simply connected. For different topics in the theory of univalent functions the reader may consult the monographs by Duren [D2] or Pommerenke [P], for example.

Here is another basic property of univalent functions in the disk that will be fundamental for us. (See Subsection 1.1 for the meaning of $\simeq$ and $d_{\Omega}(w)$.)

**Lemma C.** If $f \in \mathcal{U}$ then $(1 - |z|^2)|f'(z)| \simeq d_{f(\mathbb{D})}(f(z))$ for all $z \in \mathbb{D}$.

The proof of this standard result is based on the Schwarz-Pick lemma on the one hand and the Koebe one-quarter theorem on the other hand ([P], Section 1.3).

1.4. **The Bloch space.** By definition, $f \in \mathcal{B}$ if $f \in \mathcal{H}(\mathbb{D})$ and

$$
\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.
$$

A modification of the Schwarz-Pick lemma shows that $\mathcal{H}_\infty \subset \mathcal{B}$. The inclusion is strict. Actually, there are univalent functions, for example, $\log \frac{1 + z}{1 - z}$ that belong to $\mathcal{B} \setminus \mathcal{H}_\infty$.

Let $f \in \mathcal{B}$. Integration of the derivative $f'$ from the origin to $z$ along a line segment leads to the following basic growth estimate for arbitrary Bloch function $f$.

**Lemma D.** Whenever $f \in \mathcal{B}$ and $z \in \mathbb{D}$, we have

$$
|f(z) - f(0)| \leq \frac{1}{2} \log \frac{1 + |z|}{1 - |z|} \cdot \|f\|_{\mathcal{B}}.
$$

In particular, convergence in the Bloch space implies uniform convergence on any $K \subset \mathbb{D}$.

Note that the function $\log \frac{1 + z}{1 - z}$ mentioned above achieves maximum possible growth along the entire radius $[0, 1)$.

One of the basic papers on the subject of Bloch spaces is that of Anderson, Clunie, and Pommerenke [ACP]; see also Danikas’ lecture notes [Da] or Chapter 5 of Zhu’s book [Z1].
It is clear from the basic estimate that a univalent function $f$ belongs to $\mathcal{B}$ if and only if

$$\sup_{z \in \mathbb{D}} d_{f(\mathbb{D})}(f(z)) < \infty,$$

i.e. if and only if $f(\mathbb{D})$ does not contain arbitrarily large disks.

1.5. The Dirichlet space and analytic Besov spaces. The Dirichlet space $\mathcal{D}$ is the set of all analytic functions $f$ in $\mathbb{D}$ with the finite Dirichlet integral (i.e., such that $f' \in A^2$). It is a Hilbert space when equipped with the norm

$$\|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'|^2 dA.$$

It is actually possible to construct a scale of spaces of $L^p$ type that includes both $\mathcal{D}$ and $\mathcal{B}$. This is done by integrating $(1 - |z|^2)|f'(z)|$ with respect to the hyperbolic area element $dA(z)/(1 - |z|^2)^2$. Thus, when $1 < p < \infty$, a function $f \in H(\mathbb{D})$ is said to belong to the analytic (diagonal) Besov space $B^p$ if and only if

$$s_p(f)^p = \int_{\mathbb{D}} |f'(z)|^p(1 - |z|^2)^{-p-2}dA(z) < \infty.$$

The above seminorm $s_p$ is invariant under the conformal automorphisms of the disk: $s_p(f \circ \varphi) = s_p(f)$, for every disk automorphism $\varphi$. A true norm is usually given by the formula

$$\|f\|_{B^p}^p = |f(0)|^p + s_p(f)^p.$$

It is clear that $B^2 = \mathcal{D}$, the Dirichlet space. An important property of analytic Besov spaces is the following: $B^p \subset B^q$, whenever $1 < p < q \leq \infty$. Also, we may interpret the Bloch space as a limit case: $B^\infty = \mathcal{B}$. There is a way of defining the space $B^1$ as well but this would require more work and some further considerations, so we will avoid this in the present article.

In view of their conformal invariance, $B^p$ spaces are closely related with hyperbolic metric. Furthermore, they represent the range of the Bergman projection when acting on $L^p$ spaces with respect to the hyperbolic area measure. They also relate naturally with the membership in Schatten class of Hankel operators, etc. Thus, it is clear that they are very natural and important objects to study.

Note that the space $B^p$ is also invariant under translations $\tau(z) = z + a$ and dilations $d_r(z) = rz$, $0 < r < 1$; that is, if $f \in B^p$ then also $f \circ \tau \in B^p$ and $f \circ d_r \in B^p$.

Besides the invariance properties of $B^p$ spaces, we will only use the following statement in the sequel.
Lemma E. Whenever \( f \in B^p \), \( 1 < p < \infty \), and \( z \in \mathbb{D} \), we have
\[
|f(z) - f(0)| \leq C \left( \log \frac{1 + |z|}{1 - |z|} \right)^{1 - 1/p} \cdot \|f\|_{B^p}.
\]
In particular, convergence in \( B^p \) norm implies uniform convergence on any \( K \subset \mathbb{D} \).

A statement of this type can be found in Zhu’s semi-expository paper [Z2], for example. In general, [Z2] and Chapter 5 of [Z1] are useful references on analytic Besov spaces. One of the most influential papers on the subject was [AFP].

1.6. The Whitney decomposition of a planar domain. It is well known that the structure of open sets in \( \mathbb{R}^n \) \((n > 1)\) and, in particular, in \( \mathbb{C} \), is different from that of open sets on the real line. Namely, every open set in the plane can be represented as a countable union of closed dyadic squares (that is, squares of side length \( 2^k \) each, where \( k \in \mathbb{Z} \), and with sides parallel to the coordinate axes) with pairwise disjoint interiors.

The Whitney decomposition theorem refines this further by stating that we can simultaneously control the size of these squares and their distances to the boundary, as well as their total number for each possible size. A precise formulation is as follows. The details of proofs can be found in [S], Chapter VI.

Lemma F. An arbitrary non-empty open set \( \Omega \) in the plane, not the plane itself, can be represented as \( \Omega = \bigcup_{n=1}^{\infty} Q_n \), where \( Q_n \) are closed dyadic squares with pairwise disjoint interiors, and with the property that
\[
diam Q_n \leq d_{\Omega}(Q_n) \leq 4 \cdot diam Q_n,
\]
for all \( n \geq 1 \).

Moreover, it can also be verified that the sizes of any two neighboring squares (i.e., those that share a boundary segment, not just one point) are comparable (within a factor of 4). Thus, each square has at most 16 neighbors (squares that share with it a line segment), not counting the four squares that share only a vertex with it.

1.7. Entire functions. Two basic types of entire functions will be of interest to us here: the polynomials and the functions of finite positive order.

We recall the Cauchy estimates, a standard generalization of Liouville’s theorem.

Lemma G. Given a non-negative number \( a \) and an entire function \( \varphi \), then
\[
|\varphi(w)| \leq M|w|^a, \quad \text{for sufficiently large } |w|
\]
if and only if \( \varphi \) is a polynomial of degree at most \( [a] \), the greatest integer part of \( a \).
The proof is a standard exercise involving the Taylor series of \( \varphi \).

Given a non-constant entire function \( \varphi \), its \textit{order} \( \rho \) is determined by

\[
\rho = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r},
\]

where \( M(r) = \max\{|\varphi(z)| : |z| = r\} \). We will always work with one \( \varphi \) at a time and will therefore suppress any reference to \( \varphi \) in \( M(r) \) and \( \rho \) in order not to burden the notation. The \textit{type} \( \sigma \) of an entire function \( \varphi \) of order \( \rho \) \((0 < \rho < \infty)\) is given by

\[
(2) \quad \sigma = \limsup_{r \to \infty} \frac{\log M(r)}{r^\rho}.
\]

The possibilities \( \sigma = 0 \) and \( \sigma = \infty \) are not excluded.

The simplest example of a function of integer order \( \rho \) and finite positive type \( \sigma \) is \( e^{\sigma z^\rho} \). More involved examples (including the case of fractional orders) can be constructed in the form of power series or as integrals of certain complex functions.

Some very basic material on entire functions can be found in Chapter VIII of [T]. The first couple of chapters of the classical monograph [B] contain plenty of information.

2. \textbf{Bloch domains and univalent }\( B^p \)\textbf{ domains}

2.1. \textbf{Bloch domains.} A planar domain \( \Omega \) is said to be a \textit{Bloch domain} if and only if every \( f \in H(D) \) with the property \( f(D) \subset \Omega \) must belong to \( B \). (Note that \( f(D) = \Omega \) is not required and no special properties of \( \Omega \) are being assumed either.) The following result is considered a “folk knowledge”.

\textsc{Theorem H.} \textit{A planar domain is a Bloch domain if and only if it does not contain arbitrarily large disks.}

It should be remarked that similar result exist for other conformally invariant spaces of analytic functions, such as \( BMOA \) or \( Q_p \) but we will not discuss them here.

2.2. \textbf{Image domains under univalent Besov functions.} Among the simply connected domains, those that are images of the disk under univalent maps in \( B^p \) can be characterized in a very convenient way. This was observed recently by Walsh [W] and in [BFV].

\textsc{Proposition I.} \textit{Let }\( 1 < p < \infty \). \textit{A domain }\( \Omega \) \textit{has the property that every }\( f \in U \text{ such that } f(D) = \Omega \) \textit{belongs to }\( B^p \) \textit{if and only if }\( \Omega \) \textit{is simply connected and }\int_\Omega d\nu^p = \int_\Omega dA < \infty \).
Proof. Since the Jacobian of the change of variable \( w = f(z) \) is \( |f'(z)|^2 \) and 
\( (1 - |z|^2)|f'(z)| \asymp d_\Omega(f(z)) \) in view of Lemma C, we readily see that
\[
\int_\Omega |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) \asymp \int_\Omega d_\Omega(w)^{p-2} dA(w),
\]
which proves the statement. \( \square \)

This simple but extremely useful result allows us to construct simply connected domains whose Riemann maps lie in the desired \( B^p \) space. The following example is from [DGV].

**Proposition 1.** Let \( (a_n) \) be infinite sequence of positive numbers. Consider a sequence of open squares \( Q_n \) with side lengths \( a_n \) respectively and whose one side lies on the positive part of the \( x \)-axis so that \( Q_n \cap Q_{n+1} \) is a vertical segment for all \( n \). Define \( \Omega \) to be the interior of the union \( \bigcup_{n=1}^{\infty} Q_n \). Let \( F \) be a univalent map of the unit disk onto \( \Omega \). Then \( F \in B^p \) if and only if
\[
\sum_{n=1}^{\infty} a_n < \infty.
\]
Thus, there are univalent functions in every \( B^p \) and \( B^p \neq \bigcap_{q>p} B^q \).

**Proof.** We give a proof only when \( 2 \leq p < \infty \); the case \( 1 < p < 2 \) is handled analogously but with all inequalities reversed. For each point in \( Q_n \) we have \( d_\Omega(w) \leq a_n/2 \), whence
\[
\int_\Omega d_\Omega(w)^{p-2} dA(w) \lesssim \sum_{n=1}^{\infty} \int_{Q_n} a_n^{p-2} dA(w) \lesssim \sum_{n=1}^{\infty} a_n^p.
\]
For the reverse estimate, consider the square \( Q'_n \) concentric with \( Q_n \) but half the size. Every point \( w \) in this square satisfies \( d_\Omega(w) \geq a_n/4 \); hence
\[
\int_\Omega d_\Omega(w)^{p-2} dA(w) \gtrsim \sum_{n=1}^{\infty} \int_{Q'_n} a_n^{p-2} dA(w) \gtrsim \sum_{n=1}^{\infty} a_n^p,
\]
which proves the desired inequality. \( \square \)

Walsh used Proposition I to give a couple of interesting examples which were subsequently improved in our paper with Donaire and Girela [DGV].

Observe first that a univalent map onto a domain of finite area necessarily belongs to the Dirichlet space. But does it have to belong to any smaller \( B^p \) space (\( p < 2 \))? Walsh [W] answered this in the negative by giving the following example.

**Proposition J.** There exists a domain \( \Omega \) of finite area such that no univalent map \( f : \mathbb{D} \to \Omega \) can belong to \( \bigcup_{p<2} B^p \).

**Proof.** Such a domain \( \Omega \) can be constructed by making countably many slits in the unit square
\[
Q = \{ x + iy : 0 < x < 1, 0 < y < 1 \}.
\]
as follows: delete a vertical line segment of height $\frac{1}{n+1}$ at the base point $k/2^n$, $k = 1, 3, \ldots , 2^n - 1$, for every $n \geq 1$. Then every point $w = x + iy$ in $\Omega$ with the imaginary part 
\[ \frac{1}{n+2} < y < \frac{1}{n+1}, \quad n = 1, 2, \ldots \]
also satisfies $d_\Omega(w) \leq 1/2^{n+1}$. Hence, whenever $p < 2$, we have
\[ \int_{\Omega} d_\Omega(w)^{p-2} dA(w) \gtrsim \sum_{n=1}^{\infty} \frac{2(2-p)n}{(n+1)(n+2)} = \infty, \]
showing that a univalent map of $D$ onto $\Omega$ cannot belong to $A^p$, according to Proposition I.

Clearly, the domain from Proposition J is far from being a Jordan domain, for many of its boundary points are not simple. The following example was given in [DGV].

**Proposition 2.** There exists a Jordan domain $\Omega$ of finite area such that no univalent map $f : D \to \Omega$ can belong to $\cup_{p<2} B^p$.

**Proof.** Such a domain can be obtained by gluing to a vertical strip countably many “combs”, each finer and of smaller area than the previous one. To put it in precise terms, let
\[ \alpha_n = n^{-2}, \quad n \geq 1, \quad \beta_0 = 0, \quad \beta_n = \sum_{k=1}^{n} \alpha_k, \quad n \geq 1, \quad \beta = \sum_{n=1}^{\infty} \alpha_n = \frac{\pi^2}{6}. \]
Consider the domains
\[ R_n = \{ x + iy : 2\beta_{n-1} < y < 2\beta_{n-1} + \alpha_n, \quad 0 \leq x < \alpha_n \}, \quad n \geq 1. \]
\[ T_{n,k} = \{ x + iy : 2\beta_{n-1} < y < 2\beta_{n-1} + \alpha_n, \quad \frac{k}{2^n} \alpha_n \leq x < \frac{k}{2^n} \alpha_n + \frac{\alpha_n}{2n+1} \}, \]
for $n = 1, 2, 3, \ldots$ and $0 \leq k \leq 2^n - 1$. Define
\[ S_n = R_n \setminus \left( \bigcup_{k=0}^{2^n-1} T_{n,k} \right) \]
and finally
\[ \Omega = \left( \bigcup_{n=1}^{\infty} S_n \right) \cup \{ x + iy : -1 < x < 0, \quad 0 < y < 2\beta \}. \]
It is easy to see that $\Omega$ is a Jordan domain. By applying a reasoning similar to that of Proposition J, we arrive at the same conclusion: $\int_{\Omega} d_\Omega^{p-2} dA = \infty$ when $p < 2$. We omit the details here. □
It is clear that a univalent map of the disk onto a simply connected domain whose complement has finite area cannot belong to $D$. Can it belong to any larger $B^p$ space ($p > 2$)? Walsh [W] gave the following counterexample.

**Proposition K.** There exists a simply connected domain $\Omega$ whose complement has zero area, yet no univalent map of $D$ onto $\Omega$ belongs to $\cap_{p>2}B^p$.

**Proof.** The construction consists in deleting from the plane countably many vertical half-lines, each of them from a point with certain special rational coordinates to infinity. We refer the reader to [W] for the details. □

The following improvement was found in [DGV].

**Proposition 3.** There exists a simply connected domain $\Omega$ whose complement is a single Jordan arc, yet a univalent map of $D$ onto $\Omega$ does not belong to $\cap_{p>2}B^p$.

**Proof.** The domain is constructed by deleting from the plane a spiral-like curve that wanders off from the origin to the point at infinity. More precisely, let

$$r_0 = 0, r_n = \sum_{k=1}^{n} \frac{1}{k}, n \geq 1,$$

and consider the following sequence of Jordan arcs in the form of letter “C”:

$$C_n = \{ z : |z| = r_n, |\text{arg } z| > \frac{r_{n+1} - r_n}{2} = \frac{1}{2(n+1)} \}$$

and the following sequences of “upper” line segments:

$$U_n = [r_{2n-1}e^{i(r_{2n}-r_{2n-1})/2}, r_{2n}e^{i(r_{2n+1}-r_{2n})/2}]$$

and “lower” segments

$$L_n = [r_{2n}e^{-i(r_{2n+1}-r_{2n})/2}, r_{2n+1}e^{-i(r_{2n+2}-r_{2n+1})/2}].$$

Next, join the upper edges of $C_1$ and $C_2$ by $U_1$, as well as those of $C_3$ and $C_4$ by $U_2$, etc. Connect also the lower edges of $C_2$ and $C_3$ by $L_1$. Continuing this process yields as a result the simple arc

$$\Gamma = (\bigcup_{n=1}^{\infty} C_n) \cup (\bigcup_{n=1}^{\infty} U_n) \cup (\bigcup_{n=1}^{\infty} L_n)$$

that connects the origin with the point at infinity since $r_n \to \infty$ as $n \to \infty$. Hence the domain $\Omega = \mathbb{C} \setminus \Gamma$ is simply connected. A procedure similar to the one applied before allows us to estimate the integral $\int_{\Omega} d\Omega^{p-2}dA$ from above by a convergent series whenever $2 < p < \infty$. Again, we refer the reader to [DGV] for specific details of such estimates. □
2.3. Univalent $B^p$ domains. One can easily give a natural definition of a $B^p$ domain. By analogy with an unpublished argument due to Hedenmalm for $Q_p$ domains, the following was shown in [DGV].

**Proposition 4.** When $1 < p < \infty$, $B^p$ domains do not exist.

**Proof.** Suppose there exist such a domain $\Omega$. Then $\Omega$ contains some open disk. Since $B^p$ is invariant under translations and dilations, without loss of generality we may assume that $\mathbb{D} \subset \Omega$. Then every analytic function that maps the disk into itself must belong to $B^p$. In particular, every infinite Blaschke product does. This contradicts a theorem of H.O. Kim [K] which states that the only Blaschke products in $B^p$ are the finite ones. Hence there does not exist a $B^p$ domain. □

Even though the $B^p$ domains do not exist, there is a reasonable middle ground between this phenomenon and Proposition I. It is convenient and natural to define the following notion. We will say that $\Omega$ is a univalent $B^p$ domain if every univalent function $f$ in $\mathbb{D}$ such that $f(\mathbb{D}) \subset \Omega$ must have the property $f \in B^p$. Note that we are not requiring that $\Omega$ be simply connected, nor that $f$ be onto (as in the criterion of Walsh).

The following lemma might be interesting in itself. We remark that this is neither the only nor the most efficient way of constructing the domain $\Delta$ mentioned below, but it is perhaps the easiest one to write down.

**Lemma 5.** Whenever $p \geq 2$, every domain $\Omega$ contains a simply connected domain $\Delta$ such that

$$\int_{\Omega} d_{B^p}^{-2}dA \asymp \int_{\Delta} d_{B^p}^{-2}dA$$

Thus, for example, every domain $\Omega$ of finite area has a simply connected domain $\Delta$ whose area is at least a fixed positive constant times the area of $\Omega$.

**Proof.** Let $(Q_n)$ be the Whitney decomposition of $\Omega$ as in Lemma F. We can partition this collection of squares into countably many generations according to the following scheme. Choose one square as a pivot square; it will be the only member of generation 0. Generation 1 will be formed by all the neighbors of this square (recall that a neighbor of a square $Q_m$ is any other square $Q_n$ such that $Q_m \cap Q_n$ is a line segment). Inductively, define the generation $n + 1$ as the neighbors of the squares of the $n$-th generation already defined. Using a compactness argument and the crucial property that $diam Q_n \asymp d_{\Omega}(Q_n)$, it can be shown that no square will be omitted in this process, i.e., every square $Q_n$ belongs to some generation.

Consider the squares $C_n$, concentric with $Q_n$ but half the size of those. Construct a new domain $\Delta$ as the union of all squares $C_n$ and countably many thin tubes. By a “thin tube” we mean an $\varepsilon$-thickening (with a suitably chosen small $\varepsilon$) of the line segment from the center of square of generation $n$
to the center of a square of generation $n + 1$. Any “crossing of tubes” must be avoided. There are only finitely many pairs to be connected in each step, so this can be done safely. We may think of the process of constructing our domain as of a “growing tree” where the “tubes” behave like “branches” stemming out of nodes (squares $C_n$). A drawing can be helpful. Where connections between squares of two adjacent generations cannot be made because the crossing of tubes is unavoidable, leave some squares without connecting (“some branches of the tree end”). The domain $\Delta$ obtained this way is simply connected and satisfies

$$
\int_{\Delta} d_{\Delta}^{p-2}dA \geq \sum_n \int_{C_n} d_{\Delta}^{p-2}dA
\times \sum_n \int_{C_n} \text{diam}(C_n)^{p-2}dA
= \sum_n \int_{C_n} \text{diam}(C_n)^p
\times \sum_n \int_{Q_n} \text{diam}(Q_n)^p
\times \sum_n \int_{Q_n} \text{diam}(Q_n)^{p-2}dA
\times \sum_n \int_{Q_n} d_{\Omega}^{p-2}dA
\times \int_{\Omega} d_{\Omega}^{p-2}dA.
$$

by the properties of the Whitney decomposition listed in Lemma F. □

Here is one of the principal results of [DGV]. Note that this statement “interpolates” between the trivial case $p = 2$ (the univalent $D$ domains clearly being the ones of finite area) and the expected result for the Bloch space: the supremum of the radii of all disks contained in the domain must be finite.

**Theorem 6.** Univalent $B^p$ domains exist if and only if $2 \leq p \leq \infty$. If this is the case, then a domain $\Omega$ is such a domain if and only if $\int_{\Omega} d_{\Omega}^{p-2}dA < \infty$.

**Proof.** To show that there are no univalent $B^p$ domains when $1 < p < 2$, suppose that $\Omega$ is such a domain. Then $\Omega$ contains some disk $D_0$. Now, by Walsh’s example (Proposition J) there is a bounded function $f \in \mathcal{U}$ such that $f \notin B^p$. We can find complex constants $\alpha$ and $\beta$ so that if $g = \alpha + \beta f$ then $g(\mathbb{D}) \subset D_0 \subset \Omega$. Then $g \in \mathcal{U}$ and $g(\mathbb{D}) \subset \Omega$ but $g \notin B^p$, a contradiction.
Now for the main part: the description of univalent $B_p$ domains when $2 \leq p < \infty$. The proof is similar but easier for the Bloch space (case $p = \infty$) and will be omitted here. We remark that it does not require Whitney squares - one can comfortably work with disks.

The easy implication goes as follows. Let $\int_\Omega d_\Omega^{p-2}dA < \infty$ and let $f$ be an arbitrary univalent map such that $f(D) = D \subset \Omega$. It is easy to see that $d_D(w) \leq d_\Omega(w)$ for all $w$ in $D$. Taking into account that $p \geq 2$, we have

$$\int_D d_D^{p-2}dA \leq \int_D d_\Omega^{p-2}dA \leq \int_\Omega d_\Omega^{p-2}dA < \infty,$$

so by Proposition I we know that $f \in B_p$. This shows that $\Omega$ is a univalent $B_p$ domain.

We finally prove the difficult part. The key point is, of course, Lemma 5. We may reason as follows. Suppose $\Omega$ is a univalent $B_p$ domain. We have to show that $\int_\Omega d_\Omega^{p-2}dA < \infty$. Assume the contrary: $\int_\Omega d_\Omega^{p-2}dA = \infty$. By Lemma 5, $\Omega$ contains a simply connected domain $\Delta$ such that we also have $\int_\Delta d_\Delta^{p-2}dA = \infty$. Then by Proposition I there exists a univalent map $f$ of $\mathbb{D}$ onto $\Delta$ such that $f \not\in B_p$. Since $\Delta \subset \Omega$, this would mean that $\Omega$ is not a univalent $B_p$ domain. This contradiction completes the proof. \hfill $\square$

Somewhat similar constructions of domains like our domain $\Delta$ were employed earlier by Peter W. Jones, e.g. in [J1] and [J2], as we found out when the publishing process for [DGV] was already at an advanced stage.

It is clear from the definition that a subdomain of a univalent $B_p$ domain is also a univalent $B_p$ domain. An example of such a domain that is not even finitely connected is obtained by deleting the centers of all squares $Q_n$ in the domain from Proposition 1.

3. Superposition operators

Trivially, if $f \in A^p$ and $n \leq [p/q]$, then $f^n \in A^q$. It follows immediately that $P \circ f \in A^q$, for any polynomial $P$ of degree $\leq [p/q]$. Are such polynomials the only entire functions for which the statement is true? The answer is yes, as found by Cámara and Giménez [CG] in 1994. Similar phenomena are also easily observed in Hardy spaces $H^p$ (see Cámara’s survey [C]).

In general, given two metric function spaces $X$ and $Y$, where $X, Y \subset \mathcal{H}(\mathbb{D})$, we will say that $\varphi$ acts by superposition from $X$ into $Y$ if $\varphi \circ f \in Y$ for all $f$ in $X$. If this is the case, we say that $\varphi$ defines the superposition operator

$$S_\varphi : X \to Y, \quad S_\varphi(f) = \varphi \circ f.$$

It follows easily that $\varphi$ is entire if $X$ contains the linear functions. This is so because the identity function belongs to $X$ in this case, hence $\varphi \in Y$ and, in particular, $\varphi \in \mathcal{H}(\mathbb{D})$. By applying $\varphi$ to a linear function, it is then
easily seen that \( \varphi \) is analytic in any open disk and hence entire. Also (when \( X \) and \( Y \) are linear), \( S_{\varphi} \) is linear if and only if \( \varphi(z) = cz \), \( c = \text{const.} \).

We will still say \( S_{\varphi} \) is a \textit{bounded operator} if it maps bounded sets into bounded sets, even when no linearity is involved. For example, it is rather obvious that \( \varphi(z) = z^n \) and, analogously, any polynomial of degree \( n \leq [p/q] \) induces a bounded superposition operator from \( A^p \) into \( A^q \) (just inspect carefully the obvious estimates used in proving that the operator acts from one space into the other).

\textbf{Question.} Let \( X \) and \( Y \) be two metric spaces of functions in \( \mathcal{H}(\mathbb{D}) \) that contain the linear functions.

1. For which (entire functions) \( \varphi \) do we have \( S_{\varphi}(X) \subset Y \)?
2. If \( S_{\varphi}(X) \subset Y \), when is \( S_{\varphi} \) a bounded operator?

Knowing the answers to the above questions seems like a very natural way of comparing “how much faster” the functions in one space grow than those in the other. Superpositions between various spaces of real functions have been studied extensively (see Appel’s and Zabrejko’s monograph [AZ] for the developments up to about 1990). The question on when one function acts between two spaces also comes up often in certain topics of harmonic analysis, functional analysis, or function algebras. However, such a study between classical spaces of analytic functions and in the terms formulated above has begun, surprisingly enough, only quite recently.

In this section we present a small sample of results from [CG], [BFV], and [AMV], hoping that they will give the reader the flavor of this recent line of research.

3.1. \textbf{Superpositions acting between Bergman spaces.} The first result we review here concerns the superposition operators between two Bergman spaces. As already mentioned, it was proved in [CG]. The statement is intuitively quite reasonable to expect but its proof still requires a certain amount of work. The method employed in [CG] is the most natural one: it consists in choosing the right “test functions” in \( A^p \) and making a clever use of the standard Cauchy estimates for entire functions. It should be worth observing that several details in the original proof can be simplified. The result is a somewhat shorter proof we present below that involves choosing only one test function instead of several such functions.

\textbf{Theorem L.} Let \( 0 < p, q < \infty \) and let \( \varphi \) be entire. Then \( S_{\varphi} \) maps \( A^p \) into \( A^q \) if and only if \( \varphi \) is a polynomial whose degree is at most \( [p/q] \). If this is the case, then \( S_{\varphi} \) is actually a bounded operator.

\textit{Proof.} We have already checked the sufficiency of the condition on degree \( \leq [p/q] \) at the beginning of the section, so we need only verify the necessity. To this end, suppose \( \varphi \) is not a polynomial of degree \( \leq p/q \). Then we
can find $\varepsilon > 0$ such that $[p/q + \varepsilon] < [p/q] + 1$. According to the Cauchy estimates (Lemma G), there exists an infinite sequence of points $(w_n)$ in the plane with the property that

$$|\varphi(w_n)| > n|w_n|^{p/q+\varepsilon}, \text{ for all } n.$$  

At least one of the 8 octants

$$\{z : \pi n/4 \leq \arg z < \pi(n + 1)/4\}, \quad n = 0, 1, \ldots, 7,$$

contains infinitely many points $w_n$. Without loss of generality, we may assume it is the first octant. The reason for this is the following: $\varphi$ is a polynomial of degree $N$ if and only if the function given by $\psi(w) = \varphi(\lambda w)$ is also such, whenever $|\lambda| = 1$, so that rotations are allowed.

By the Bolzano-Weierstrass theorem, the sequence $(\arg w_n)$ will have a convergent subsequence. We may choose a further subsequence, denoted again $(w_n)$, so that the arguments $\arg w_n$ decrease to zero. Again, this is a consequence of the fact that $\varphi$ is a polynomial of degree $N$ if and only if the function given by $\psi(w) = \overline{\varphi(w)}$ is also such, so we are allowed to use reflections across the real axis. Another rotation can be used after this to take the sequence back to the first octant.

Now choose

$$f(z) = \left(\frac{1+z}{1-z}\right)^\alpha, \quad \alpha \frac{p}{q} + \varepsilon = \frac{2}{q}.$$ 

Since $0 < \alpha < 2/p$, it follows from Lemma A that $f \in A^p$.

Let us first consider the case when $p \geq 1$. Then $f \in U$ and maps the unit disk onto an angle with vertex at the origin and of aperture $< 2\pi$. Note also that it maps any Stolz angle in $D$ with vertex at $z = 1$ symmetric with respect to the real axis onto a symmetric angle of opening smaller than $\pi$ centered at the origin. In particular, the pre-image of the first octant is contained in a Stolz angle.

Let $z_n$ be points in the disk such that $f(z_n) = w_n$. The choice of $z_n$ is unique because $f$ is univalent. Moreover, by the mapping properties of $f$, all $z_n$ will belong to a Stolz angle. Then $1 - |z_n| > c|1 - z_n|$ for some $c \in (0, 1)$. By deleting again finitely many terms, we may also assume that $\Re z_n \geq 0$ for all $n$ (hence $|1 + z_n| \geq 1$). By our choice of $\varepsilon$ we have

$$\frac{\|\varphi \circ f\|_{A^q}}{(1 - |z_n|)^{2/q}} \geq |\varphi(f(z_n))| = |\varphi(w_n)| > n|w_n|^{p/q+\varepsilon}$$

$$= n \left(\frac{|1 + z_n|}{|1 - z_n|}\right)^{\alpha(p/q+\varepsilon)} \geq n \left(\frac{c}{1 - |z_n|}\right)^{\alpha(p/q+\varepsilon)}$$

$$\geq \frac{M n}{(1 - |z_n|)^{2/q}}.$$ 

In view of Lemma B this contradicts the assumption that $\varphi \circ f \in A^q$. 

The case when $0 < p < 1$ is easily taken care of by multiplying both $p$ and $q$ by a sufficiently large positive integer $N$ so as to have $Np \geq 1$ and $Nq \geq 1$, thus reducing the problem to the case already considered. □

Even when the operator $S_\varphi$ is not linear, one can understand the concepts of “continuous” and “locally Lipschitz” in the usual terms. The following statement was also proved in [CG].

**Theorem M.** When $\varphi$ acts by superposition from $A^p$ into $A^q$ as in the conditions of Theorem L, the operator $S_\varphi$ is continuous and also Lipschitz at every point of $A^p$.

We do not give a proof here but instead refer the reader to the original paper [CG]. Interesting results regarding superpositions from Bergman space into the Nevanlinna area class can also be found there. We also remind the reader that further details regarding Hardy and related spaces can be found in the survey [C].

### 3.2. Superpositions between Besov spaces

Our next result is technically more complicated as it requires controlling the derivative of the function in the initial space instead of the function itself. Also, the test function should be chosen as a univalent map in $B^p$ onto a certain domain whose exact shape is known only roughly but not completely. Thus, one needs maps similar to the ones constructed in Subsection 2.2 only slightly more general. The following class of examples was given in [BFV].

**Proposition 7.** Let $1 < p < \infty$, $(w_n)$ be a sequence of complex numbers, and let $(r_n)$ and $(h_n)$ be sequences of positive numbers with the following properties:

(a) $0 \leq \arg w_n < \pi/4$ and $|w_n| \leq |w_{n+1}|/2$, $n \in \mathbb{N}$;
(b) $r_n < |w_n|/4$ and $|h_n| < \min\{r_n, r_{n+1}\}/3$, $n \in \mathbb{N}$.

Let $D_n = D(w_n, r_n)$ and let $R_n$ be the rectangle whose longer symmetry axis is the segment $[w_n, w_{n+1}]$ and whose shorter side has length $2h_n$. Then the domain $\Omega = \bigcup_{n=1}^{\infty} (D_n \cup R_n)$ is simply connected and, if $f$ is a Riemann map of $\mathbb{D}$ onto $\Omega$, then $f \in B^p$ if and only if

$$\sum_{n=1}^{\infty} r_n^p + \sum_{n=1}^{\infty} |w_{n+1} - w_n| h_n^{p-1} < \infty.$$

The proof resembles that of Proposition 1.

Armed with this new tool, we are now ready to give a characterization of all entire maps that transform one Besov space into another (or into the Bloch space) via superposition. Intuitively, it is clear that $B^p$ is smaller than $B^q$ when $p < q$, but “not much smaller”. How should this be expressed in terms of the superposition operators acting from one space into another? The answer is fairly simple to state. This result is also from [BFV].
Theorem 8. Let $1 < p, q \leq \infty$, where $B^\infty = B$. Then we have the following conclusions.

(a) If $p \leq q$, then $S_\varphi : B^p \to B^q$ if and only if $\varphi$ is a linear function.
(b) If $p > q$, then $S_\varphi : B^p \to B^q$ if and only if a constant function.

Proof. The reasoning we are about to use applies equally to (a) and (b). We will only distinguish between these two cases at the end of the proof.

We first show that $\varphi$ must be linear in either case. Assume the contrary: $\varphi'$ is not identically constant, and let $r_n = 2^{-n}$. In view of Liouville’s theorem, $\varphi'$ is unbounded, so we can select inductively a sequence $(w_n)$ of complex numbers so that $|w_1| > 2$ and

$$|w_{n+1}| \geq 2|w_n|, \quad |\varphi'(w_n)| \geq r_n^2$$

for all $n$. As in the proof of Theorem L, at least one of the eight basic octants contains infinitely many points $w_n$. By a rotation if necessary, we may therefore assume that $0 \leq \arg w_n < \pi/4$, and so Proposition 7 is applicable. Define

$$h_n = 2^{-n-2}|w_{n+1} - w_n|^{-1/(p-1)}.$$ 

Let $\Omega$ be the domain defined in Proposition 7 using the sequences $(w_n)$, $(r_n)$, and $(h_n)$ as data and let $f : \mathbb{D} \to \Omega$ be a univalent map of $\mathbb{D}$ onto $\Omega$.

By Proposition 7, we know that $F \in B^p$. Let $f(z_n) = w_n$. It is easily seen that $|z_n| \to 1$ as $n \to \infty$. Applying Lemma C, we obtain

$$|\varphi'(w_n)| |f'(z_n)| (1 - |z_n|) \asymp |\varphi'(w_n)| d_\Omega(w_n) \geq C/r_n \to \infty,$$

which tells us that $\varphi \circ f \notin B$, a contradiction. This tells us that $S_\varphi(B^p) \subseteq B^q$ implies that $\varphi$ is linear, independently of the values of $p$ and $q$.

(a) Since $B^p \subseteq B^q$ when $p \leq q$, it is clear that every linear function acts by superposition from $B^p$ into $B^q$.

(b) We already know that $\varphi$ has to be linear but it cannot contain the $z$-term (otherwise it would easily follow that $B^p \subseteq B^q$, which is not the case. Thus, $\varphi \equiv \text{const.}$ \qed

The following statement was not recorded in [BFV] but is rather easy to prove.

Proposition 9. Let $1 < p, q \leq \infty$. If $S_\varphi$ acts from $B^p$ into $B^q$, it is also a bounded operator.

Proof. In the case (b) of Theorem 8, the range $S_\varphi(B^p)$ is a singleton, hence the superposition operator is trivially bounded.

In the case (a), we have $\varphi(z) = az + b$, where $a, b \in \mathbb{C}$. The key point consists in observing that the injection map from $B^p$ into $B^q$ (allowing the possibility of $B^\infty = B$) is a bounded linear operator. This is a consequence of the Closed Graph Theorem. Namely, by Lemma E the convergence in $B^p$ implies uniform convergence on compact subsets of $\mathbb{D}$; this also applies
to the Bloch space by Lemma D. Thus, assuming that \( f_n \to f \) in \( B^p \) and \( f_n \to g \) in \( B^q \), \( p \leq q \), we deduce that \( f_n \Rightarrow f \), as well as to \( g \), on all \( K \subset \mathbb{D} \), whence \( f \equiv g \). This shows that the injection map from \( B^p \) into \( B^q \) (possibly \( B \)) has closed graph and is therefore a bounded operator: \( \| f \|_{B^q} \leq C \| f \|_{B^p} \).

Now if \( \| f \|_{B^p} \leq M \) then

\[
\| S_\varphi(f) \|_{B^q} \leq |a| \| f \|_{B^q} + |b| \leq C M |a| + |b| ,
\]

showing that \( S_\varphi \) maps bounded sets into bounded sets and is, thus, a bounded operator from \( B^p \) into \( B^q \).

As was also the case with [DGV], due to limitations in space we were only able to give here a glimpse of results obtained in [BFV]. There are further theorems there regarding superpositions between other spaces of Dirichlet type. In certain cases, such results require slightly more involved examples of conformal maps as well as certain inequalities of Trudinger-Moser type due to Chang and Marshall. However, these results will not be presented in the present paper.

3.3. Superpositions from \( B \) to \( A^p \). Here we have a different situation, in the sense that any Bergman space is “much bigger” than the Bloch space; in other words, Bergman functions grow much faster than Bloch functions.

Since Bloch functions grow at most as \( \log \frac{1}{1-|z|} \) and \( A^p \) functions grow at most as \( (1-|z|)^{2/p} \), we may ask whether any function like \( \varphi(z) = e^{cz} \), \( c \neq 0 \), will still have the property

\[
f \in B \Rightarrow \varphi \circ f \in A^p .
\]

One immediately notices that, even though \( f_c(z) = c \log \frac{1}{1-z} \) is a Bloch function, the function

\[
e^{f_c(z)} = \frac{1}{(1-z)^c}
\]

will not belong to the Bergman space \( A^p \) if we choose \( c \geq 2/p \), according to Lemma A. This should lead to the understanding that entire functions of order one and finite positive type will not serve for mapping \( B \) into \( A^p \) by superposition. However, this is precisely where the cut occurs. The following result is proved in a forthcoming paper with Álvarez and Márquez [AMV].

**Theorem 10.** Let \( 0 < p < \infty \) and let \( \varphi \) be entire. Then the following statements are equivalent:

(a) \( S_\varphi : B \to A^p \);

(b) \( S_\varphi \) maps \( B \) boundedly into \( A^p \);

(c) \( \varphi \) either has order less than one, or order one and type zero.

We only mention a few key points of the proof here since it will be published elsewhere. The assumption (c) gives a simple estimate on the growth
of $\varphi \circ f$, which after some minimum work yields that (c) implies (b). It is plain that (b) implies (a), so we need only prove that (a) implies (c). This is the crucial part and it suffices to consider only the harder case when

$$\limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} = 1 \quad \text{and} \quad \lim_{r \to \infty} \frac{\log M(r)}{r} = 0,$$

that is, when $\varphi$ has order equal to one and type zero.

The strategy is similar to the ones employed in Theorem L and Theorem 8: assuming $\varphi$ can grow faster than it should, identify an infinite sequence $(w_n)$ on which the function $\varphi$ grows at this fast rate and then construct a univalent mapping that belongs to the Bloch space and interpolates these values $w_n$. The particularly interesting feature of the proof is that the mapping in question transforms the unit disk onto a “long, slowly winding road”, i.e., onto an infinite strip of fixed width (a union of countably many “cigar”-shaped domains) from the origin to the point at infinity. The fixed width not only assures the membership of such a map in the Bloch space but also guarantees its maximum growth along a curve that ends on the unit circle; the properties of hyperbolic metric are fundamental for getting all the requirements that are needed. A contradiction with the growth estimate for Bergman functions (Lemma B) ends the proof once again.

The three examples given here (Theorem L, Theorem 8, and Theorem 10) should not mislead the reader to expect that whenever $S_\varphi$ acts from one space to another it should also be bounded. There are examples that show that this is clearly not the case. We invite the reader to compare some of the results from [BFV] to those obtained in the forthcoming paper [V] in order to discover such an example. In [V] various relationships between the Dirichlet space, on the one hand, and Hardy and Bergman spaces, on the other hand, are explored and some new observations are revealed.

**References**


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