THE CENTERED HARDY-LITTLEWOOD MAXIMAL OPERATOR IN HIGH DIMENSIONS

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Here, now.
**Definition**

Hardy-Littlewood maximal operator:

\[ Mf(x) := \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| \, dy. \]

- Variants:
  - Uncentered
  - Over different balls
  - Over more general sets
  - Using other measures
  - \( M\nu, \nu \) a measure
Useful because:

- $Mf$ is larger than $|f|$ but not much larger

- $Mf$ is more regular (lower semi-continuous) than $|f|$ (for $f$ measurable).
$Mf$ is larger than $|f|$ but not much larger.

Consider Lebesgue measure on $\mathbb{R}^d$.

- $|f| \leq Mf$ a.e.

- $M$ satisfies the strong type $(p, p)$ inequality: For $1 < p \leq \infty$,

$$\|Mf\|_p \leq C_p\|f\|_p.$$

- $\|Mf\|_\infty \leq \|f\|_\infty$, so $\|Mf\|_\infty = \|f\|_\infty$
• $M$ is unbounded on $L_1$

• Example: Take $f = 1_{[0,1]}$

• Hence $\lim_{\rho \downarrow 1} C_\rho = \infty$
• Boundedness properties depend on the operator.

• Example: Stein’s spherical maximal operator is bounded for $p > \frac{d}{d-1}, d \geq 3$ (Stein, 1976)); and for $d = 2$ (Bourgain, 1986).
**General question: How do the constants \( c_1, C_p \) behave?**

- For \( p = 1 \), \( M \) satisfies the weak type \((1, 1)\) inequality

\[
\sup_{\alpha > 0} \alpha \left| \{ Mf > \alpha \} \right| \leq c_1 \| f \|_1.
\]

- Dimension \( d = 1 \), Lebesgue measure. Centered maximal operator \( M \), best \( c_1 \approx 1.56 \) (the constant below).

\[
\left| \{ x : Mf(x) > \alpha \} \right| \leq \frac{11 + \sqrt{61}}{12\alpha} \| f \|_1
\]

Dimension $d = 1$, Lebesgue measure, Strong type $(p, p)$ inequality, $1 < p < \infty$:

$$\| Mf \|_p \leq C_p \| f \|_p.$$  

- Centered maximal operator $M_c$, best $C_p$ unknown.
Optimal bounds are hard to find. In general, we are happy if we find “good” bounds.
Behavior of the constants $c_{1,d}$, $C_{p,d}$ as the dimension $d \to \infty$

- Uncentered Hardy-Littlewood maximal operator:
- Euclidean balls, Lebesgue measure in $\mathbb{R}^d$
- $C_{p,d}$, $c_{1,d}$ grow exponentially in $d$. 
Strong type bounds, Centered maximal operator, Lebesgue measure in $\mathbb{R}^d$:

- Euclidean balls, fixed $p > 1$, $\sup_{d \geq 1} C_{p,d} < \infty$, (Stein, 1982).

  *Proof.* Use the boundedness of Stein’s spherical maximal function.

- Arbitrary balls, $p \geq 2$, $\sup_{d \geq 1} C_{p,d} < \infty$ (Bourgain, 1986)

  *Proof.* Put the ball in isotropic position. Use Fourier Analysis.
• Arbitrary balls, fixed $p > 3/2$, $\sup_{d \geq 1} C_{p,d} < \infty$ (Bourgain, 1986, Carbery, 1986).

Proof. Fourier Analysis.

• Recall: $\ell_q$ balls ($1 \leq q < \infty$) given by the norm

$$\|x\|_q := (|x_1|^q + |x_2|^q + \cdots + |x_d|^q)^{1/q}$$
• ℓ_q balls (1 ≤ q < ∞) fixed p > 1, sup_{d≥1} C_{p,d} < ∞, (D. Müller, 1990).

• B_d ball with unit volume. Müller proved: If minimal sections (through the origin) and maximal projections (over hyperplanes passing through the origin) are bounded independently of d, then sup_{d≥1} C_{p,d} < ∞.

Proof. Fourier Analysis.
• Left open: \( \ell_\infty \) balls (cubes with sides parallel to the axes), \( 1 < p \leq 3/2 \).

• Volume of minimal sections: \( 1 \).

• Volume of maximal projections: \( \sqrt{d} \).
• $\ell_\infty$ balls (cubes), uniform bounds also exist for fixed $1 < p \leq 3/2$:

$$\sup_{d \geq 1} C_{p,d} < \infty.$$  

(Bourgain, Math ArXiv, December 2012).
• Summing up: For all $p > 1$ and all $\ell_q$ balls ($1 \leq q \leq \infty$), there exist strong bounds uniform in the dimension.

• For all $p > 3/2$ and all balls, there exist strong bounds uniform in the dimension.
**Weak type (1, 1) bounds, Centered maximal operator, Lebesgue measure in \( \mathbb{R}^d \):**

- Vitali Covering Theorem, \( c_{1,d} \leq 3^d \), or \( c_{1,d} \leq 2^d \).
- Arbitrary balls, \( c_{1,d} \leq O(d \log d) \), covering lemma of “Vitali type” (Stein and Strömberg, 1983).
- Euclidean balls, \( c_{1,d} \leq O(d) \), maximal ergodic theorem for the heat semigroup. (Stein and Strömberg, 1983).
- Question: For Euclidean balls, is \( \sup_{d \geq 1} c_{1,d} < \infty \)? (Stein and Strömberg, 1983).
- Answer not known.
- My guess: NO.
• Reason: For $\ell_\infty$ balls (cubes), uniform bounds do NOT exist: $\lim_{d \to \infty} c_{1,d} = \infty$ (JMA, 2011).

**Proof.** Discretization + calculus + normal approximation to the binomial distribution.

• Rate of divergence: $c_{1,d} \geq \Theta(\log^{1-o(1)} d)$ (G. Aubrun, 2009, shortly after my paper).

**Proof.** Discretization (same example) + stochastic process (the Brownian bridge).
**GUESSES, INTUITIONS**

- Maximal functions associated to cubes and to euclidean balls (or more general balls) should behave roughly in the same way.

- So for Lebesgue measure on $\mathbb{R}^d$, $p > 1$, and arbitrary balls, we expect uniform bounds for $C_{p,d}$, but not for $c_{1,d}$.

- Good bounds seem to depend on “balls of the same radius have similar measures”, rather than on doubling.
NON-uniform bounds, Centered maximal operator, Euclidean balls, measures in $\mathbb{R}^d$:

- Finite measures defined via radial, radially decreasing bounded densities (example, standard gaussian):

- Exponential increase of $c_{1,d}$ with $d$ (JMA, 2007)

*Proof*. Discretization (1 delta). Use that balls centred far away from the origin have much smaller measure than balls of the same radius centred at the origin.
• Refinements of the same idea yield exponential increase for small values of $p > 1$

• For finite measures defined via radial, radially decreasing bounded densities (Criado, 2010, Ph. D. Thesis 2012 under Prof. F. Soria)

• For more general (radial, radially decreasing) measures, including some doubling measures (J. Pérez Lázaro-JMA, 2011) and some high values of $p$ in the latter case.

• For the standard gaussian measure and all $1 < p < \infty$, $C_{p,d}$ increases exponentially with $d$ (Criado-Sjögren, 2012)
Some doubling measures:

- \( t \in (0, 1) \), \( d\mu_{t,d} := \|x\|_2^{-td} \, dx \) on \( \mathbb{R}^d \).

- \( \mu_{t,d} \) is doubling.

- For all \( t \in (1/2, 1) \) and all \( 1 < p < \infty \), \( C_{p,d} \) increases exponentially with \( d \) (J. Pérez Lázaro-JMA, A. Criado, Ph. D. Thesis, independently)

**Proof.** Adaptation of the Criado-Sjögren argument for the gaussian measure.
Beyond $\mathbb{R}^d$:

- Volume in $d$-dimensional hyperbolic space, geodesic balls
- Volume is not doubling. In fact, hyperbolic space admits no doubling measures
- Balls of equal radius have equal volume
- $c_{1,d} \leq O(d \log d)$ (Li-Lohoué, 2012)

Proof. Semigroup methods

- $p > 1$, $\sup_{d \geq 1} C_{p,d} < \infty$ (Li, personal communication)

Proof. Semigroup methods
Centered Hardy-Littlewood maximal operator, Metric measure spaces

- Metric measure space: Separable metric space with a Radon measure.
- Naor and Tao (2010): the Stein-Strömberg $c_{1,d} \leq O(d \log d)$ bound holds for metric measure spaces that satisfy the “Strong Microdoubling Condition” (ex., Ahlfors-David regular spaces).
• Strong $d$-Microdoubling with constant $K$: For all $x$, all $r > 0$ and all $y \in B(x, r)$,

$$\mu B \left( y, \left( 1 + \frac{1}{d} \right) r \right) \leq K \mu B(x, r).$$

• Idea: Use ultrametric spaces, random martingales, Doob’s maximal inequality.

THANKS FOR LISTENING!