Hoja 2

1) Let $p$ be a prime number. What is the remainder of the division of the binomial coefficient $\binom{2p}{p}$ by $p$?

2) Compute the value of the integral

$$\int_0^1 \frac{2x^3 - 3x^2 + 3x - 1}{(x^2 + 1)(x^2 - 2x + 2)} \sin(\pi x) \, dx.$$ 

3) Prove the identity

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$ 

4) Let \(\{a_n\}_{n=1}^{\infty}\) be a strictly increasing sequence in \(\mathbb{Z}^+\). Prove that

$$\sum_{n=1}^{\infty} \frac{1}{\text{lcm}(a_n, a_{n+1})}$$

converges where lcm denotes the least common multiple.

5) Prove that

$$\sum_{n=1}^{\infty} \frac{2\|\sqrt{n}\| + 2^{-\|\sqrt{n}\|}}{2^n} = 3$$

where \(\|x\|\) stands for the closest integer to \(x\).